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# Analysis of DSm belief conditioning rules and extension of their applicability

Published in:
Florentin Smarandache & Jean Dezert (Editors)
**Advances and Applications of DSmT for Information Fusion**
(Collected works), Vol. III
American Research Press (ARP), Rehoboth, 2009
ISBN-10: 1-59973-073-1
ISBN-13: 978-1-59973-073-8
Chapter X, pp. 323 - 343

**Abstract:** *Analysis of belief conditioning rules (BCRs) is presented in this chapter. Some simplifications of formulas for BCRs are suggested. A comparison of BCRs with classic rules of conditioning is performed. Finally, definition domains and applicability of BCRs are extended. Full formal definition of the extended version of BCR12 is presented.*

The work was supported by ESF EUROCORES FP006 project ICC/08/E018 of the Grant Agency of the Czech Republic and the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications".

## 10.1 Introduction

This chapter is devoted to belief conditioning in DSmT. We have a (generalized) belief function (BF) given by a (generalized) basic belief assignment (bba) $m$ on a hyper-power set $D^\Theta$, which should be conditioned assuming a sure assumption that the truth is in a given $A$ where $A \in D^\Theta \setminus \emptyset$.

A long series of 31 belief conditioning rules (BCRs) was defined in DSm book vol. 2 [11]. One of the rules — BCR12 — was regarded to be a generalization of Shafer's (i.e., Dempster's) rule of conditioning in the free DSm model, the others are its alternatives. Several techniques are combinatorically combined to define BCRs there. A detailed analysis of all the 31 BCRs has just appeared in Technical Report [6]. The report presents also a comparison of all BCRs with Dempster's rule of conditioning and with its real generalization to DSm hyper-power sets. Based on the presented analysis, extended definitions of BCRs are introduced there. These definitions as much as possible enable to extend definition domains of the rules to increase their applicability to wider class of belief functions.

In this chapter we present the main results of the analysis, the idea of extended definitions, and the complete formal definition of the extended version of BCR12. This theoretical text provides also a series of examples illuminating its theoretical results.

This chapter follows Chapter 9 [12] from DSm book vol. 2 [11], thus it is recommended (but not necessary) to read [12] in advance. Also, the reader can find author's notation of DSmT in Chapter 3 of [11].

## 10.2 Brief preliminaries

As this is a chapter in the 3rd volume on DSmT we suppose that reader is already familiar with the basis of DSmT, otherwise Chapters 1 and 4 of the 1st volume and/or Chapter 3 of the 2nd volume or the brief introduction from DSmT homepage[1] are recommended. Hence we do not repeat all the general basic notions here, but only principal of those which are closely related to belief conditioning rules.

Conditioning of a basic belief assignment (bba) $m$ by a set $A$: Smarandache & Dezert assume in [12] that $A \in D^\Theta \setminus \emptyset$, this works in the free DSm model $\mathcal{M}^f$. Unfortunately there is no explicit mention of any hybrid DSm model in [12], assumptions about hybrid models are hidden there. Nevertheless when working with hybrid DSm models we have to explicitly say these assumptions. We assume that $A \not\equiv \emptyset$ and that all bbms are correctly defined on the hybrid DSm model which is used. Further all denominators of formulas are assumed to be non-zero, see corrigenda of Chapter 9 of [11].

Rules BCR2–11 use splitting of $\Theta$ as it is defined in DSm Book vol. 2 Chap. 9, i.e., $\Theta = D_1 \cup D_2 \cup D_3$, where $D_1 = \{X \mid \emptyset \neq X \in D^\Theta,\ X \subseteq A\}$, $D_2 = ((\Theta \setminus s(A)), \cap, \cup)$, $s(A)$ is the set of all elements of $\Theta$, which compose $A$, $D_3 = D^\Theta \setminus (D_1 \cup D_2 \cup \emptyset)$.

[1] www.gallup.unm.edu/∼smarandache/DSmT.htm.

Rules BCR12–31 use $D_2$ further splitted in two parts $D_{2D} = \{X | X \in D_2 \,\&\, X \cap A \equiv \emptyset\}$, $D_{2I} = \{X | X \in D_2 \,\&\, X \cap A \not\equiv \emptyset\}$, where bbms of focal elements for $D_{2I}$ are processed in the same way (BCR12-BCR21) or in a similar way (BCR22-BCR31) as those from $D_3$. Hence there is a new modified splitting of $\Theta$ into 3 disjoint parts: $\Theta = D_S \cup D_D \cup D_I = D_1 \cup D_{2D} \cup (D_{2I} \cup D_3)$, defined in [6]. The new splitting $\Theta = D_S \cup D_D \cup D_I$ (to $D_S = D_1$ ... non-empty subsets[2] of $A$, $D_D = D_{2D}$ ... sets disjunctive from $A$, $D_I = D_{2I} \cup D_3$ ... non-subsets of $A$, but intersecting $A$) is more intuitive. Nevertheless due to editors' wish, and to consistency with DSm book vol. 2 and chapters of this volume we use the original notation $\Theta = D_1 \cup (D_{2D} \cup D_{2I}) \cup D_3$ in this text.

Let $W \in D_3$, we say that $X \in D_1$ is the *k-largest*, $k \geq 1$, element from $D_1$ that is included in $W$, if $(\nexists Y \in D_1 \setminus \{X\})(X \subset Y, Y \subset W)$; depending on the model, there are $k \geq 1$ such elements, see [12], corrigenda of page 240. The same is used also for $W \in D_2$, such that $W \cap A \neq \emptyset$. For definitions of $k$-smallest, $k$-median and $k$-average elements from $D_1, D_2$ see [12].

## 10.3 Belief conditioning rule BCR1

*Belief Conditioning Rule no. 1* (*BCR1*) is defined for $X \in D_1$ by the formula[3]

$$m_{BCR1}(X|A) = m(X) + \frac{m(X)\sum_{Z\in D_2\cup D_3} m(Z)}{\sum_{Y\in D_1} m(Y)} = \frac{m(X)}{\sum_{Y\in D_1} m(Y)}.$$

Alternatively, we can write:

$$m_{BCR1}(X|A) = \frac{m(X)}{\sum_{Y\in D_1} m(Y)} = \frac{m(X)}{\sum_{Y\subseteq A} m(Y)} = \frac{m(X)}{Bel(A)}.$$

$m_{BCR1}(X|A) = 0$ for $X \in D^\Theta \setminus D_1$.

BCR1 is the simplest belief conditioning rule. This rule is a generalization of Belief Focusing Rule[4] defined in D-S theory.

The rule is not defined for Vacuous Belief Function ($VBF$) for which $m_{VBF}(\Theta) = 1$, it is further not defined e.g. for $m'_{BCR1}(X|\{\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4\})$, when $m'(\theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5) = 1$, etc. In general, $m_{BCR1}(X|A)$ is not defined whenever $Bel(A) = \sum_{Y\subseteq A} m(Y) = \sum_{Y\in D_1} m(Y) = 0$.

The rule is very sensitive with respect to $m(X)$ for $X \subseteq A$, on the other hand all bbms $m(Y)$ such that $Y \cap A \not\equiv \emptyset$ & $Y \not\subseteq A$ are completely ignored by BCR1, see

[2] More precisely subsets which are not equal to empty set in the case of hybrid DSm models.

[3] We have to put stress on the fact, that it is necessary to keep in mind, that definition of sets $D_1, D_2, D_3$, i.e. splitting of $D^\Theta$, depends on the conditioning set $A$, which is included in the formula through the set $D_1$.

[4] This rule was mentioned in [8], unfortunately, the author of this chapter does not know its original publication.

example 1. Bbm $m(Y)$, where $Y \cap A \not\equiv \emptyset$ may be even assigned to subset of $A$, which is disjoint from $Y$ (i.e., which has empty intersection[5] with $Y$), see example 2.

***Example 1.*** *Let us suppose* $\Theta = \{a, b, c\}$, *the free DSm model* $\mathcal{M}^f$, $m(a) = 0.001$, $m(b) = 0.004$, $m(a \cup c) = 0.800$, $m(b \cup c) = 0.195$ *and conditioning set* $A = \{a, b\} = a \cup b$. *We obtain* $m_{BCR1}(a|A) = 0.20, m_{BCR1}(b|A) = 0.80$, *regardless to large bbm* $m(a \cup c)$.
*Moreover if we significantly decrease the bbm of* $b \cup c \not\subseteq A$ *in favour of* $a \cup c \not\subseteq A$ *as it follows in* $m'$, *the resulting conditional bba* $m'_{BCR1}(X|A)$ *does not reflect it:* $m'(a) = 0.001, m'(b) = 0.004, m'(a \cup c) = 0.990, m'(b \cup c) = 0.005$, $m'_{BCR1}(a|A) = 0.20, m'_{BCR1}(b|A) = 0.80$.
*On the other side, if we slightly decrease the same bbm of* $b \cup c \not\subseteq A$ *in favour of* $a \subseteq A$ *then the conditioned bba is changed (ignoring size of* $m(X)$ *for* $X \not\subseteq A$ *again):* $m''(a) = 0.006, m''(b) = 0.004, m''(a \cup c) = 0.800, m''(b \cup c) = 0.190$, $m''_{BCR1}(a|A) = 0.60, m''_{BCR1}(b|A) = 0.40$.

***Example 2.*** *Let us suppose* $\Theta = \{a, b, c\}$ *and any hybrid DSm model, where* $a, b \cup c \not\equiv \emptyset$. *For* $m(a) = 0.1, m(a \cup c) = 0.1$, $m(b \cup c) = 0.8$ *and conditioning set* $A = a \cup b$. *We obtain* $m_{BCR1}(a|A) = 1$ *regardless to large value* $m(b \cup c)$. *Set* $a$ *may be disjoint from* $b \cup c$, *e.g. in Shafer's DSm model.*

For comparison with the other belief conditioning rules we can rewrite BCR1 as it follows:

$$m_{BCR1}(X|B) = m(X) + m(X)\,\frac{\sum_{W \in D_2} m(W)}{\sum_{Y \in D_1} m(Y)} + m(X)\,\frac{\sum_{W \in D_3} m(W)}{\sum_{Y \in D_1} m(Y)}.$$

Thus $m(X)$ is kept to be assigned to $X$ for all $X \subseteq A$, and $m(X)$ is proportionalized according to $m(Y)$ for all $Y \subseteq A$ (i.e., $Y \in D_1$) otherwise.

## 10.4 Belief conditioning rules BCR2–BCR11

Generalized basic belief masses of all focal elements $X$ from $D_1$ are kept to be assigned to $X$ again, and generalized basic belief masses of all focal elements $W$ from $D_2$ are in the same way proportionalized among all subsets of conditioning set $A$ by all belief conditioning rules BCR2–BCR11. These subsets are proportionalized according to belief masses $m(Y)$ of subsets $Y$ of the conditioning set $A$.

### 10.4.1 Belief conditioning rules BCR2–BCR6

Generalized basic belief masses of all focal elements $W$ from $D_3$ are in similar ways reallocated by belief conditioning rules BCR2–BCR6. Ways of this reallocation specify and mutually differ BCRs from this group. What does it mean $W \in D_3$? It means

[5] This feature depends from a hybrid DSm model which is used; it cannot occur in the case of the free DSm model, where there is no empty intersection.

that $W$ is neither from $D_1$, i.e. $W \not\subseteq A$, nor from $D_2$, thus some $\theta_i$ appear(s) in $W$ which appear(s) also in $A$, hence $W \cap A \not\equiv 0$.

Similarly to BCR1, none of BCR2 – BCR6 rules are defined for BFs, such that $Bel(A) = \sum_{Y \in D_1} m(Y) = 0$; these cases[6] are not even mentioned in [12].

We will repeat neither the original formulas from [12] for all BCRs, nor the new compact parametric ones from [6], and make only comments related to BCR2 and BCR6.

#### 10.4.1.1 BCR2 — intersection of $W \in D_3$ with conditioning set $A$

Rule BCR2 relocates $m(W)$ for $W \in D_3$ to $k$-largest ($k$-maximal) elements of $D_1$ which are subset of $W$, i.e. to $k$-largest ($k$-maximal) elements of $W \cap A$. The largest (maximal) subset of $W \cap A$ is $W \cap A$ itself and it is unique, thus it is 1-largest and we can write BCR2 as it follows:

$$m_{BCR2}(X|A) = m(X) + m(X)\,\frac{\sum_{W \in D_2} m(W)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X = W \cap A}} m(W),$$

for all bbas $m$ and conditioning sets $A$, such that $Bel(A) = \sum_{Y \in D_1} m(Y) \neq 0$. Hence all bbms $m(W)$ from $D_3$ are relocated to $W \cap A$ by this rule.

#### 10.4.1.2 BCR6 — all non-empty subsets of $W \cap A$

BCR6 splits bbms $m(W)$ from $D_3$ into same portions and redistributes them among all subsets of $W \cap A$, thus we can slightly simplify its formula as follows:

$$m_{BCR6}(X|A) = m(X) + m(X)\frac{\sum_{W \in D_2} m(W)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subseteq W}} \frac{m(W)}{Card\{V|\emptyset \not\equiv V \subseteq W \cap A\}}.$$

#### 10.4.1.3 Analysis of BCR2–BCR6

The problems which are presented in examples 1 and 2 do not occur using rules BCR2–BCR6 as bbms of sets from $D_3$ are not proportionalized according to $m(X)$ for $X \in D_1$. On the other hand these bbms $m(W)$ are blindly distributed among several or all subsets of $W \cap A$ by rules BCR3–BCR6, see continuation of Example 2. We also have to note, that as BCR1, rules BCR2–BCR6 proportionalize all bbms of sets from $D_2$ according to bbms $m(X)$ for $X \in D_1$; which can be often odd and not intuitive, see Example 3.

[6] The cases where $Bel(A) = 0$ are denoted to be degenerated in [13], and any BCR is defined as $m(A|A)$, $m(X|A) = 0$ for $X \neq A$ in the section on BCRs in [13] (the paper on new qualitative belief conditioning rules (QBCRs)). For more details see the appendix.

***Example 2*** (cont.). *Let us suppose the free DSm model $\mathcal{M}^f$ now. When redistributing bbm $m(b \cup c) = 0.8$ we have $W \cap A = (b \cup c) \cap (a \cup b) = b \cup (a \cap c)$ with DSm cardinality $Card_{DSm} = 5$, there are 9 proper subsets of $W \cap A$ in the free DSm model; $Card_{DSm} = 1 : a \cap b \cap c$, $Card_{DSm} = 2 : a \cap b, a \cap c, b \cap c$, $Card_{DSm} = 3 : a \cap (b \cup c), b \cap (a \cup c), c \cap (a \cup b)$, $Card_{DSm} = 4 : b, (a \cap b) \cup (a \cap c) \cup (b \cap c)$. $m(b \cup c)$ is relocated to whole $W \cap A$ by BCR2, $W \cap A = b \cup (a \cap c)$ in the free DSm model and it naturally can be different in various hybrid DSm models, e.g., just $b$ in Shafer's model. $m(b \cup c)$ is relocated to $a \cap b \cap c$ by BCR3 in the free DSm model; it is divided by 3 and redistributed among $a \cap (b \cup c), b \cap (a \cup c), c \cap (a \cup b)$ by BCR4 and BCR5 in the free DSm model; and it is divided by 10 and redistributed among all subsets of $b \cup (a \cap c)$ in the free DSm model. In this simple example, $m(b \cup c)$ is relocated to $b$ by all of BCR2–BCR6 rules in the case of Shafer's model.*

***Example 3.*** *Let us suppose $\Theta = \{a, b, c\}$ and the free DSm model $\mathcal{M}^f$ again. For $m(a) = 0.01, m(b) = 0.04, m(a \cup c) = 0.50$, $m(b \cup c) = 0.05, m(c) = 0.40$ and conditioning set $A = a \cup b$. $c$ is in $D_2$, thus $m(c)$ is proportionalized between $a$ and $b$ in the ration $m(a) : m(b)$, i.e. $1 : 4$, by all of BCR2–BCR6 rules, regardless the fact that $a$ is significantly more plausible through $m(a \cup c)$ than $b$ is through $m(b \cup c)$.*

*In the modified example $m'(a) = 0.001, m'(b \cup c) = 0.450, m'(c) = 0.549$ whole $m'(c)$ is relocated to $a$ by all BCR2–BCR6 rules; $m'(b \cup c)$ is relocated or redistributed among subsets of $m'(b \cup c) \cap (a \cup b)$ as it follows:*
$m'_{BCR2}(a|A) = 0.550, m'_{BCR2}((b \cup c) \cap (a \cup b)|A) = 0.450$,
$m'_{BCR3}(a|A) = 0.550, m'_{BCR3}(a \cap b \cap c|A) = 0.450$,
$m'_{BCR4}(a|A) = 0.550, m'_{BCR4}(a \cap (b \cup c)|A) = 0.150$,
$m'_{BCR4}(b \cap (a \cup c)|A) = 0.150, m'_{BCR4}(c \cap (a \cup b)|A) = 0.150$,
$m'_{BCR6}(a|A) = 0.550, m'_{BCR6}((b \cup c) \cap (a \cup b)|A) = 0.045, m'_{BCR6}(b|A) = 0.045$,
$m'_{BCR6}((a \cap b) \cup (a \cap) \cup (b \cap c)|A) = 0.045, m'_{BCR6}(a \cap (b \cup c)|A) = 0.045$,
$m'_{BCR6}(b \cap (a \cup c)|A) = 0.045, m'_{BCR6}(c \cap (a \cup b)|A) = 0.045$, $m'_{BCR6}(a \cap b|A) = 0.045, m'_{BCR6}(a \cap c|A) = 0.045$,
$m'_{BCR6}(b \cap c|A) = 0.045$, $m'_{BCR6}(a \cap b \cap c|A) = 0.045$
*(results by BCR5 are the same as those by BCR4 in this example).*
*For comparison we obtain $m'_{DRC}(a|A) = 0.0022, m'_{DRC}((b \cup c) \cap (a \cup b)|A) = 0.9978$, by the generalized Dempster's rule of conditioning [3].*

As it is mentioned in [12], bbms $m(W)$ for $W \in D_3$ are blindly divided by $k$ according the number of sets among those it should be redistributed, regardless of bbms of those sets.

The way of bbm relocation in BCR2 rule is referred as the most *pessimistic/prudent* one and that in BCR3 as the most *optimistic* one. Nevertheless the difference among the rules does not seem to be related to optimism/pessimism, but it is a question of addition or non-addition of an extra additional information when $m(W)$ is redistributed for $W \cap A \neq \emptyset, W \nsubseteq A$, or what additional information is added. Similarly relocation of all $m(W)$ to absorbing $\cap_{i=1,\ldots,n}\theta_i$ by BRC3 really does not express any optimism.

When relocating $m(W)$ to $W \cap A$ in BCR2 no additional information is added. All other redistributions of $m(W)$ for $W \in D_3$ in BRC3–BCR6 add to $m$ some additional information, which is mainly based on the specific rule and partly on bbms $m(Y)$ for $Y \in D_1$. Nevertheless there is no need to add any information within conditioning, and definitely there is no reasonable motivation for redistribution of $m(X)$ among sets given by $k$-median or by $k$-average as it is performed by BCR4 and BCR5.

Really, from conditioning set $A$ we only know that $m(W)$ should be located to $A$ and/or some of its subsets. From $m$ we only know that $m(W)$ should be located to $W$ (if it is acceptable by $A$). Hence we know that $m(W)$ conditioned by $A$ should be located to $W \cap A$. We do not know anything more using both the sources of information bba $m$ and conditioning set $A$. Any other precision of focal elements is addition of some kind of an extra information (out of $m$ and $A$).

### 10.4.2 Belief conditioning rules BCR7–BCR11

Analogically to the previous subsection, we can formulate also BCR7–BCR11 rules in the more compact parametric form, see [6]. Similarly to BCR1–BCR6, BCR7 – BCR11 rules are defined for BFs, such that $Bel(A) = \sum_{Y \in D_1} m(Y) \neq 0$ again. Rules BCR8–BCR11 really improve belief conditioning, as they remove blind redistribution of bbm $W$ (i.e., division by $k$) whenever $S(W) \neq 0$; $m(W)$ is redistributed respecting $W$ (only among subsets of $W$), nevertheless it is proportionalized according $m(Y)$ for $Y \subset A$, thus sensitivity to the values $m(Y)$ for $Y \subset A$ increases and the problem of relocation/redistribution of $m(W)$ for $W \in D_2$ continues.

Because a part of $\{m(W)|W \in D_2\}$ is redistributed among subsets of $W \cap A$ even by BCR7, which uses the $k$-largest element, the rule BCR7 also add some more additional information within the combination in comparison with BCR2. Thus the change obtained using fractions $m(W)/S(W)$ is counter intuitive in the case of BCR7.

## 10.5 Belief conditioning rules BCR12–BCR31

The rules from this large group start to distinguish whether $W \cap A$ is empty or non-empty for $W \in D_2$. $m(W)$ are relocated or redistributed in the same or analogical way as those form $D_3$ in the case of non-empty intersection with $A$. Thus we can use $D_S, D_I, D_D$ instead of $D_1, D_2, D_3$ for simplification and higher understandability of formulas, see [6]. Similarly to all the previous rules, BCR12 – BCR31 rules are defined[7] for BFs, such that $Bel(A) = \sum_{Y \in D_1} m(Y) \neq 0$.

### 10.5.1 Belief conditioning rules BCR12–BCR16

Bbms of focal elements from both intersective sets of $D_{2I}$ and $D_3$ are processed in the same way of particular redistribution; this corresponds to class $(D_2^p, D_3^p)$ in the

[7] None of BCRs is defined for $Bel(A) = 0$ in [12].
All BCRs are defined $m(A|A) = 1, m(X|A) = 0$ for $X \neq 0$ if $Bel(A) = 0$ in [13].

classification of the BCRs, see Section 5 in [12].

The rules really improve conditioning again, as $m(W)$ is redistributed only inside $W \cap A$ whenever it is non-empty. The difference among individual rules BCR12–BRC17 is again related to an extra additional information which is added to the original belief within conditioning. Why some information is added within conditioning? There is really no need for it. No information is added when $m(W)$ is relocated to $W \cap A$ by BCR12, thus the rule is the best from these rules. It also corresponds to the fact that it is one of two rules which are recommended by Dezert and Smarandache in [12]. Nevertheless, in the case of hybrid DSm models, the sensitivity with respect to $m(W)$ for $W \in D_1$ within conflicting bbm (when $W \cap A = \emptyset$) redistribution remains similarly to all other DSm BRC rules from [12]. This sensitivity was removed only in the case of the free DSm model, where $D_{2I} = D_2$ and $D_{2D} = \emptyset$, thus all $m(W)$ are redistributed inside $W \cap A$ for all $W \in D_2 \cup D_3$. This of course does not hold for hybrid DSm model in general.

***Example* 3** (cont.)*. Let us suppose the free DSm model $M^f$, again. $c \in D_2$, $c \cap (a \cup b) \not\equiv \emptyset$ in $M^f$, thus $c \in D_{2I}$ more precisely. Hence $m(c)$ is relocated to $c \cap (a \cup b)$ using BCR12, or redistributed among subsets of $c \cap (a \cup b)$ using BCR13–BCR16.*
*Let us suppose a hybrid DSm model with a constraint $c \cap (a \cup b) \equiv \emptyset$ now; it trivially holds e.g. in Shafer's model $M^0$. In this case $c \in D_{2D}$ and $m(C)$ is redistributed between $a$ and $b$ in the ratio $1 : 4$ by all BCR11–BCR16, in the same way as by BCR2–BCR6.*
*Similarly in the modified example under the constraint $c \cap (a \cup b) \equiv \emptyset$, the entire $m'(c)$ is relocated to the element $a$ by all BCR11–BCR16 in the same way as it is done by BCR2–BCR6.*

#### 10.5.1.1 A simplification of BCR12

Using of intersection $\cap$ instead of the superflous notion $k$-largest and the equation $\sum_{Y \in D_1} m(Y) = \sum_{Y \subseteq A} m(Y) = Bel(A)$ we can simplify formula for BCR12 as it follows (for detail see [6]):

$$m_{BCR12}(X|A) = \sum_{W \cap A \equiv X} m(W) \; + \; \frac{m(X)}{Bel(A)} \cdot \sum_{Z \cap A \equiv \emptyset} m(Z).$$

In the special case of the DSm free model we have[8]

$$m_{BCR12}(X|A) = \sum_{W \cap A \; \equiv_{M^f} \; X} m(W).$$

In Shafer's DSm model we have

$$m_{BCR12}(X|A) = \sum_{W \cap A = X} m(W) \; + \; \frac{m(X)}{Bel(A)} \cdot \sum_{Z \cap A = \emptyset} m(Z).$$

[8]Note, that this simplification for the free DSm model is already extended in the sense of Section 10.7.

### 10.5.2 Belief conditioning rules BCR17–BCR21

Focal elements of both $D_{2I}$ and $D_3$ are processed in the same way again, this time in the way of 'splitted redistribution'; this corresponds to class $(D_2^s, D_3^s)$, see Section 5 in [12].

Rules BCR17–BCR21 are improvements of BCR7–BCR11, as $m(W)$ is not relocated out of $W \cap A$ whenever non-empty. Moreover rules BCR18–BCR21 decrease a 'blind' redistribution (i.e. division by $k$) of $m(W)$ with respect to rules BCR13–BCR16. BCR17 does not add any additional information when $m(W)$ is relocated to $W \cap A$ for $W \in D_{2I}$ or $W \in D_3$ where $Bel(W \cap A) = 0$. On the other hand, similarly to BCR7, also BCR17 brings some additional information, which is not added by BCR12, this arises whenever $W \in D_{2I} \cup D_3 \,\&\, Bel(W \cap A) \neq 0$.

#### 10.5.2.1 BCR17

Analogically to BCR12, we can simplify BCR17 as it follows

$m_{BCR17}(X|A) =$

$$\sum_{\substack{W \cap A \equiv X \\ W \subseteq A \vee Bel(W \cap A)=0}} m(W) \;+\; \frac{m(X)}{Bel(A)} \cdot \sum_{Z \cap A \equiv \emptyset} m(Z) \;+\; m(X) \cdot \sum_{\substack{X \subseteq W,\, W \nsubseteq A \\ Bel(W \cap A) \neq 0}} \frac{m(W)}{Bel(W \cap A)}.$$

For special cases in the free DSm model and in Shafer's DSm model see [6].

### 10.5.3 The remaining belief conditioning rules

These rules are just variations of BCR17-BCR21, where the idea of proportionalization of $m(X)/Bel(W \cap A)$ (cf. $m(X)/S(W)$ in [12]) is applied only to $D_{2I}$ or to $D_3$. It is applied to $m(W)$ for $W$ from $D_3$ in BCR22–BCR26, whereas for $W$ from $D_{2I}$, in BCR27–BCR31.

We can notice that $Bel(W \cap A)$ is always 0 for $W \in D_{2D}$, thus difference between the above two groups of rules can arise only for $W \in D_{2I}$ and $W \in D_3$.
We can further notice that a part of $m(W)$ where $W \cap A \not\equiv \emptyset$ (i.e. $W \in D_{2I} \cup D_3$) is proportionalized, whereas the rest is blindly divided by $k$ (it correspods to classes $(D_2^p, D_3^s)$ and $(D_2^s, D_3^p)$ in [12]), unfortunately there is no reasonable explanation or motivation for it. This seems to be non-intuitive or even counter-intuitive; in correspondence with this, both of the groups of rules are counter-intuitive. It also corresponds to the fact, that there is no formula for any of BCR22–BCR31 presented in [12]. Hence there is no need for further analysis of these rules.

## 10.6 Comparison of BCRs with the classic rules

### 10.6.1 BCR1

As it was already mentioned in Section 10.3, BCR1 rule is just the generalization of the belief focusing rule [3]. From the generalized Dempster's rule of conditioning[9] (DRC) it differs by processing of $m(W)$ both for $W$ from $D_2$ and from $D_3$. Thus BCR1 coincides with DRC just for belief functions with focal elements from $D_1$, i.e., whenever all focal elements are subsets of the conditioning set $A$; $m(X|A) = m(X)$ in such a case.

In the same way all BCRs coincide with the generalized belief focusing rule and (hence also with DRC) whenever $Bel(A) = 1$, and differ from it otherwise.

### 10.6.2 BCR2–BCR6

All BCRs from this group differ from DRC by processing of $m(W)$ for $W$ from $D_2$. BCR2 coincides with DRC whenever BCR2 is defined and all focal elements are from $D_1 \cup D_3$. Rules BCR3–BCR6 add more additive information, than BCR2 does, and they differ from DRC also for BFs with focal elements from $D_3$.

### 10.6.3 BCR7–BCR11

All of these rules add more information than BCR2 does, thus all of these rules differ from DRC whenever their focal elements are out of $D_1$ (i.e., if there exists a focal element which is not subset of $A$, i.e. if $Bel(A) < 1$).

### 10.6.4 BCR12–BCR16

All BCRs from this group differ from DRC by processing of $m(W)$ for $W$ from $D_{2D}$. BCR12 coincides with DRC whenever it is defined and all the focal elements are from $D_1 \cup D_3 \cup D_{2I}$. Rules BCR13–BCR16 add more additive information, than BCR12 does, and they differ from DRC also for BFs with focal elements from $D_{2I} \cup D_3$.

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[9] The original rule was defined by Shafer in [9] and called Dempster's rule of conditioning there. This name is generally used in belief function literature, see e.g. [10]. Nevertheless the editors of this volume started to call the rule Shafer's conditioning rule in [12].

The generalization to hyper-power sets was defined by the author of this chapter in [3] as:

$$m(X|A) = K \sum_{A \cap Y \equiv X} m(Y) = \frac{\sum_{A \cap Y \equiv X} m(Y)}{\sum_{A \cap Y \not\equiv \emptyset} m(Y)}$$

for $\emptyset \not\equiv X \subseteq A$, $X, A \in D^{\Theta}_M$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{Y \in D^{\Theta},\, A \cap Y \equiv \emptyset} m(Y)$, and $m(X|A) = 0$ otherwise, i.e., for $X \equiv \emptyset$, $X \not\subseteq A$ and for $X \notin D^{\Theta}_M$. The rule is defined (applicable) whenever $\kappa < 1$, i.e., whenever there exists $Y \in D^{\Theta}_M$, $Y \cap A \not\equiv \emptyset$, such that $m(Y) > 0$.

To avoid any confusions with the name of the rule, we will not use any personal name in this chapter a denote the rule simply as (the generalized) DRC.

We have seen that the mechanism of BCR12 does not coincide with DRC in general as the rules handle focal elements from $D_{2D}$ in different ways. From the same reason, the rules do not coincide either in Shafer's model, see also the special case of the formula for BCR12 in Subsection 10.5.1.1. Hence we cannot consider BCR12 as a generalization[10] of DRC, which conservatively extends the original DRC.

### 10.6.5 BCR17–BCR21

All of these rules add more information than BCR12 does, thus all of these rules differ from DRC whenever their focal elements are out of $D_1$ (i.e., if there exists a focal element which is not subset of $A$).

### 10.6.6 BCR22–BCR31

Formulas for the rules BCR22–BCR26 (resp. BCR27-BCR31) are similar to those for BCR12–BCR16, but the rules add more information when processing $m(W)$ for $W \in D_3$ ($W \in D_{2I}$ resp.). Thus all BCRs from this group differ from DRC by processing of $m(W)$ for $W$ from $D_{2D} \cup D_3$ (from $D_{2D} \cup D_{2I} = D_2$ resp.). BCR22 coincides with DRC whenever all focal elements are from $D_1 \cup D_{2I}$. Rules BCR23–BCR26 add more additive information, than BCR22 does, and they differ from DRC also for BFs with focal elements from $D_{2I}$.

BCR27 coincides with DRC, similarly to BCR2 whenever all focal elements are from $D_1 \cup D_3$. Rules BCR28–BCR31 add more additive information, than BCR27 does, and they differ from DRC also for BFs with focal elements from $D_3$.

We have to mention, that the information added in the case of $W \in D_{2I}$ is different from that which is added by BCR2–BCR6. It is based on proportionalization according to $m(Y)$ for $Y \subseteq A$ in BCR2–BCR6, whereas on 'splitted proportionalization' according to $m(Y)$, $Y \cap A \not\equiv 0$ in BCR27-BCR31. Thus even if the coincidence with DRC is the same for two groups of BCRs BCR2–BCR6 and BCR27–BCR31, this does not mean that these two groups of rules coincide themselves in general. The coincidence of the whole groups holds only when these rules coincide with DRC and for other special situations.

### 10.6.7 Comparison of definition domains

All BCR1-BCR31, as they are defined in [12], have the same definition domain. These rules are not defined[11] whenever $Bel(A) = 0$, i.e., if $m(W) = 0$ for all $W \subseteq A$. DRC

[10] Let us note, that the extension of BRC12 from [13] does not coincide with the generalized DRC either in the DSm free model.

[11] We follow [12] here. In [13], there is $Dom(BCRs) = \{BFs\}$, i.e., the set of all belief functions. But the extension $m(A|A) = 1$ [13] has a nature of BCR1 and does not correspond with nature and mutual differences of other BCRs.

is not defined only when $m(W) = 0$ for all $W \cap A \neq 0$, i.e., when $Pl(A) = 0$. Thus the definition domain fo BCRs is the proper subset of that of DRC:

$$Dom(BCRs) = \{m|Bel(A) \neq 0\} \subset \{m|Pl(A) \neq 0\} = Dom(DRC).$$

From it, we can easily see again, that BCR12 (as it is published in [12]) is not a generalization of DRC either in the free DSm model.

Further we can see that both BCR2 and BCR12 do coincide with DRC neither for all belief functions, which should be processed in the same way, simply because BCR2 and BCR12 are not defined for some of them, thus they are not applicable in such cases.

## 10.7 Extension of applicability of BCRs

We can see from the above comparison in the previous section, that some of the rules can coincide with DRC even out of their definition domain. We will extend definition domains of BCRs as much as possible in this section. In the same time we will extend the applicability of the rules[12].

### 10.7.1 Extension of applicability in general DSm models

Limitation for definition domains of all BCRs is division by

$$\sum_{Y \in D_1} m(Y) = \sum_{Y \subseteq A} m(Y) = Bel(A)$$

which should be non-zero. E.g. we cannot conditionalize the vacuous belief function $VBF$ (where $m_{VBF}(\Theta) = 1$, $m_{VBF}(X) = 0$ otherwise) by any of BCRs.

We cannot do anything more with BCR1, thus its definition domain is really $\{m|Bel(A) \neq 0\}$. It also corresponds with definition domain of belief focusing.

Division by $\sum_{Y \in D_1} m(Y)$ appears when processing bbms from $D_2$ in BCR2–BCR11: in summand $m(X) \frac{\sum_{W \in D_2} m(W)}{\sum_{Y \in D_1} m(Y)}$. We can extend the definition with a formula without this summand for conditioning in the case of $\sum_{W \in D_2} m(W) = 0$. In this case there is no necessity to redistribute zero bbms of elements of $D_2$ and the rules produce correct bbas even without the problematic summand. In this way we extend the definition domain of BCR2–BCR11 also for all BFs such that $\sum_{W \in D_1 \cup D_2} m(W) = 0$. Thus $Dom(BCR2\text{–}BCR11) = \{m| \sum_{Y \in D_1} m(Y) \neq 0\} \cup \{m| \sum_{W \in D_1 \cup D_2} m(W) = 0\} = \{m| \sum_{Y \in D_1} m(Y) \neq 0\} \cup \{m| \sum_{W \in D_2} m(W) = 0\} = \{m|Bel(A) \neq 0\} \cup \{m| \sum_{W \in D_2} m(W) = 0\}$.

[12] We keep the original Smarandache & Dezert's ideas of BCRs [12] in this chapter. We only try to extend their definition domains and applicability as much as possible, thus we continue to speak about BCR1–BCR31. Just a reformulation and a completion of their definitions is suggested here, not any new rules.

***Example 4.*** *Making a new friend:*
*I've met an interesting person in a conference in Paris. His affiliation is in U.S., but I have a strong feeling that he has European origin. He speaks French very well, he has a French friend, he understands my weak Italian, he has spoken about Romania several times. What is my subjective belief about his origin?*

*Let us suppose a 4-element frame of discernment $\Theta = \{U, I, F, R\}$, where $U$ stands for U.S., $I$ for Italy, $F$ for France, and $R$ for Romania. An origin of my new friend may be mixed, thus application of hyper-power set is adequate. For simplicity, we do not suppose any constraints, and use the free DSm model on $\Theta$ in this example.*

*Let my belief be given by the following bba $m$: $m(F \cup R) = 0.6$, $m(I \cup R) = 0.1$, $m(\Theta) = 0.3$. Let us learn a sure evidence that my new friend's origin is American or Romanian, thus my belief represented by $m$ should be conditionalized by $U \cup R$: $F \cup R$, $I \cup R$, $\Theta \in D_3$, $\sum_{Y \in D_1} m(Y) = 0$, thus a conditioning by $U \cup R$ is not possible using the original definition of BCRs. $\sum_{Y \in D_2} m(Y) = 0$ as well, thus we can use the extending simplified formula for this case and perform conditionalization as it follows:*
$m_{BCR2}(R\,|\,U \cup R) = 0.7$, $m_{BCR2}(U \cup R\,|\,U \cup R) = 0.3$;
$m_{BCR3}(R\,|\,U \cup R) = 0.7$, $m_{BCR3}(U \cap R\,|\,U \cup R) = 0.3$;
$m_{BCR4}(R\,|\,U \cup R) = 0.85$, $m_{BCR4}(U\,|\,U \cup R) = 0.15$;
$m_{BCR5}(R\,|\,U \cup R) = 0.85$, $m_{BCR5}(U\,|\,U \cup R) = 0.15$;
$m_{BCR6}(R\,|\,U \cup R) = 0.775$, $m_{BCR6}(U\,|\,U \cup R) = 0.075$, $m_{BCR6}(U \cap R\,|\,U \cup R) = 0.075$, $m_{BCR6}(U \cup R\,|\,U \cup R) = 0.075$.
*Because of $\sum_{Y \in D_1} m(Y) = Bel(U \cup R) = 0$ it holds true also $S(W) = Bel((U \cup R) \cap W) = 0$, hence rules BCR7–BCR11 produce the same results as BCR2–BCR6 do in our example.*

We have seen in Example 4 that the definition domains of BCR2–BCR6 and BCR7 – BCR11 were really extended for a class of generalized belief functions given by bbas such that $\sum_{Y \in D_2} m(Y) = 0$. Of course our extension is not sufficient for conditioning of all bbas, see the modified version of Example 4.

***Example 4*** (modif.). *Let my belief be given by a modified bba $m'$: $m'(F \cup R) = 0.6$, $m'(I \cup R) = 0.1$, $m'(I \cup F) = 0.1$, $m'(\Theta) = 0.2$. It holds true that $\sum_{Y \in D_1} m'(Y) = 0$ again, $I \cup F \in D_2$ thus $\sum_{Y \in D_2} m'(Y) = m'(I \cup F) = 0.1 > 0$, hence we can use neither the original formulas for BCRs (because of division by zero "$\frac{0.1}{0}$") nor the simplified formulas (because their assumptions are not satisfied). Thus we cannot apply BCR2–BCR11 in this modified example.*

Division by $\sum_{Y \in D_1} m(Y)$ appears when processing bbms from $D_{2D}$ in BCR12–BCR21: in summand $m(X) \frac{\sum_{W \in D_{2D}} m(W)}{\sum_{Y \in D_1} m(Y)}$. Analogically to the previous group of BCRs, we can extend the definition with a formula without this summand for conditioning in the case of $\sum_{W \in D_{2D}} m(W) = 0$. In this case there is no necessity to redistribute zero bbms of elements of $D_{2D}$ and the rules produce correct bbas even without the problematic summand. In this way we extend the definition domain of BCR12–BCR21 also for all BFs such that $\sum_{W \in D_1 \cup D_{2D}} m(W) = 0$. Thus

$Dom(BCR12\text{–}BCR21) = \{m|\sum_{Y\in D_1} m(Y) \neq 0\} \cup \{m|\sum_{W\in D_{2D}} m(W) = 0\} = \{m|Bel(A) \neq 0\} \cup \{m|Pl(A) = 1\}$.

The same holds also for BCR22–BCR31, hence we obtain $Dom(BCR22\text{–}BCR31) = Dom(BCR12\text{–}BCR21) = \{m|Bel(A) \neq 0\} \cup \{m|Pl(A) = 1\}$.

***Example* 4** (cont.). *It holds true that $\sum_{Y\in D_2} m(Y) = 0$ in the example, thus it holds true also $\sum_{Y\in D_{2D}} m(Y) = 0$ and $\sum_{Y\in D_{2I}} m(Y) = 0$. From the first equality we can see that we can apply also BCR12-BCR31, and from the second one, that rules BCR12–BCR16 and BCR27–BCR31 produce the same results as rules BCR2–BCR6. From $Bel(U\cap R) = Bel(A) = 0$ it follows also $Bel(W\cap A) = 0$ and $S(W) = 0$, and that also rules BCR17–BCR21 and BCR22–BCR26 similarly to rules BCR7–BCR11 produce the same results as BCR2–BCR6 in this example. Thus we have:*
$m(R\,|\,U\cup R) = 0.7$, $m(U\cup R\,|\,U\cup R) = 0.3$ *also for BCR12,BCR17,BCR22,BCR27;*
$m(R\,|\,U\cup R) = 0.7$, $m(U\cap R\,|\,U\cup R) = 0.3$ *also for BCR13,BCR18,BCR23,BCR28;*
$m(R\,|\,U\cup R) = 0.85$, $m(U\,|\,U\cup R) = 0.15$ *also for BCR14,BCR19,BCR24,BCR29;*
$m(R\,|\,U\cup R) = 0.85$, $m(U\,|\,U\cup R) = 0.15$ *also for BCR15,BCR20,BCR25,BCR30;*
$m(R\,|\,U\cup R) = 0.775$, $m(U\,|\,U\cup R) = 0.075$, $m(U\cap R\,|\,U\cup R) = 0.075$, $m(U\cup R\,|\,U\cup R) = 0.075$ *also for BCR16,BCR21,BCR26,BCR31.*

For more distinguishing of BCRs we present the following example:

***Example* 5.** *Let us take a 3 colour R-G-B example from DSm web page now. Hence we have 3-element $\Theta = \{R, G, B\}$. Let us further suppose the free DSm model $\mathcal{M}^f$ and a simple bba m such that $m(G) = 0.5, m(R\cup G\cup B) = 0.5, m(X) = 0$ otherwise. Let us make a conditioning by $A = R\cup B$.*

*$\sum_{Y\in D_1} m(Y) = 0$ and $\sum_{Y\in D_2} m(Y) = m(G) = 0.5 > 0$ thus we cannot apply rules BCR2–BCR11 either in their extended versions. $G\cap(R\cup B) \not\equiv \emptyset$ in $\mathcal{M}^f$, thus $G \in D_{2I}$ and $\sum_{Y\in D_{2D}} m(Y) = 0$ in this example. Hence we can apply BCR12–BCR31as it follows:*
$m_{BCR12}(G\cap(R\cup B)\,|\,R\cup B) = 0.5$, $m_{BCR12}(R\cup B\,|\,R\cup B) = 0.5$;
$m_{BCR13}(R\cap G\cap B\,|\,R\cup B) = 1.0$;
$m_{BCR14}(R\cap G\,|\,R\cup B) = m_{BCR14}(B\cap G\,|\,R\cup B) = 0.25$,
$m_{BCR14}((R\cap G)\cup(R\cap B)\,|\,R\cup B) = m_{BCR14}((R\cap G)\cup(G\cap B)\,|\,R\cup B) =$
$m_{BCR14}((R\cap B)\cup(G\cap B)\,|\,R\cup B) = 0.1\overline{66}$;
$m_{BCR15}(R\cap G\,|\,R\cup B) = m_{BCR15}(B\cap G\,|\,R\cup B) = 0.25$,
$m_{BCR15}((R\cap G)\cup(R\cap B)\,|\,R\cup B) = m_{BCR15}((R\cap G)\cup(G\cap B)\,|\,R\cup B) =$
$m_{BCR15}((R\cap B)\cup(G\cap B)\,|\,R\cup B) = 0.1\overline{66}$;
$m_{BCR16}(R\cap G\cap B\,|\,R\cup B) = m_{BCR16}(R\cap G\,|\,R\cup B) = m_{BCR16}(B\cap G\,|\,R\cup B) =$
$m_{BCR16}((R\cap G)\cup(B\cap G)\,|\,R\cup B) = 0.163\overline{461538}$,
$m_{BCR16}(R\cap B\,|\,R\cup B) = m_{BCR16}((R\cap G)\cup(R\cap B)\,|\,R\cup B) =$
$m_{BCR16}((R\cap B)\cup(B\cap G)\,|\,R\cup B) = m_{BCR16}((R\cap G)\cup(B\cap G)\cup(R\cap B)\,|\,R\cup B) =$
$m_{BCR16}(R\,|\,R\cup B) = m_{BCR16}(B\,|\,R\cup B) = m_{BCR16}(R\cup(B\cap G)\,|\,R\cup B) = m_{BCR16}(B\cup (R\cap G)\,|\,R\cup B) = m_{BCR16}(R\cup B\,|\,R\cup B) = 0.038\overline{461538}$.

*From $Bel(A) = 0$ it follows that $Bel(A\cap W) = S(W) = 0$, hence rules BCR17–BCR21 produce the same results as rules BCR12–BCR16 do. The same holds true also for rules BCR22–BCR26 and BCR27–BCR31.*

***Example* 4** (modif. cont.). *Analogically, we can continue also modified example of new friend as* $(U \cup R) \cap (I \cup F) \not\equiv \emptyset$ *in the free DSm model and* $(I \cup F) \in D_{2I}$*, thus* $\sum_{Y \in D_{2D}} m(Y) = 0$ *again. In this example we obtain:*
$m'_{BCR12}(U \cup R \,|\, U \cup R) = 0.2$, $m'_{BCR12}((F \cup R) \cap (U \cup R) \,|\, U \cup R) = 0.6$,
$m'_{BCR12}((I \cup R) \cap (U \cup R) \,|\, U \cup R) = m'_{BCR12}((I \cup F) \cap (U \cup R) \,|\, U \cup R) = 0.1$;
$m'_{BCR13}(U \cap R \cap F \cap I \,|\, U \cup R) = 1.0$;
*etc.*

We can summarize our analysis and extension of the definition domains of BCRs now. In the extended case for general hybrid DSm models the following holds:

$$Dom(BCR1) \subset Dom(BCR2 - BCR11) \subset Dom(BCR12 - BCR31) \subset Dom(DRC),$$

$Dom(BCR1) = \{m|Bel(A) \neq 0\} \subset Dom(BCR2 - BCR11) \subset \{m|Bel(A) \neq 0\} \cup \{m|Pl(A) = 1\} = Dom(BCR12 - BCR31) \subset \{m|Pl(A) \neq 0\} = Dom(DRC)$.

Hence we can see that applicability of the extended BCRs is still less then that one of DRC in general.

### 10.7.2 Extension of applicability in the free DSm model

In the special case of the free DSm model $\mathcal{M}^f$, $X \cap Y \not\equiv \emptyset$ and $Pl(X) = Pl(Y) = 1$ always holds true for any $X, Y \in D^\Theta$. Therefore, we can remove the summand $m(X) \frac{\sum_{W \in D_{2D}} m(W)}{\sum_{Y \in D_1} m(Y)}$ from the definitions of the rules regardless of the condition $\sum_{W \in D_{2D}} m(W) = 0$ which always holds true in $\mathcal{M}^f$.

$$\begin{aligned} Dom(BCR1) \subset\ & Dom(BCR2 - BCR11) \\ & \subset Dom(BCR12 - BCR31) = Dom(DRC). \end{aligned}$$

Where $Dom(BCR1) = \{m|Bel(A) \neq 0\}$ as in a general case, and $Dom(BCR12 - BCR31)$ is a set of all bbas defined on $D^\Theta$ now.

Under this extension, we finally obtain BCR12 as a full generalization of DRC in $\mathcal{M}^f$, and BCR12 is completely equivalent to the generalized DRC in the DSm free model $\mathcal{M}^f$.

### 10.7.3 Extended definition of BCR12

As an example of full formal definition of BRC, we present here the extended version of BCR12, for extended versions of other BCRs see [6].

*The extended version of Belief Conditioning Rule no. 12* (*BCR12*) is defined for $X \subseteq A$ by the formula

$$m_{BCR12}(X|A) = \sum_{W \cap A \equiv X} m(W) \; + \; \frac{m(X)}{Bel(A)} \cdot \sum_{Z \cap A \equiv \emptyset} m(Z),$$

when $Bel(A) \neq 0$, and by the formula

$$m_{BCR12}(X|A) = \sum_{W \cap A \equiv X} m(W),$$

when $\sum_{W \in D_{2D}} m(W) = 0$.
$m_{BCR12}(X|A) = 0$ for $X \not\subseteq A$ as in the original definition. Our extended BCR12 is not defined for BFs such that $Bel(A) = 0$ & $\sum_{W \in D_{2D}} m(W) > 0$.

In the special case of the DSm free model $M^f$ we have

$$m_{BCR12}(X|A) = \sum_{W \cap A \ \equiv_{M^f} \ X} m(W)$$

in full generality for any BF.

Thus it is the real and complete generalization[13] of DRC in $M^f$.

## 10.8 Summary of comparison

### 10.8.1 Summary of coincidence of BCRs with DRC

As it was already mentioned, BCR12 is the best of BCRs as it does not add any additional information when processing $m(W)$ for $W$ from $D_3 \cup D_{2I}$. This rule has also the greatest coincidence with (the generalized) DRC. BCR12 coincides[14] with DRC for BFs where all focal element are from $D_1 \cup D_3 \cup D_{2I}$, i.e. if $Pl(A) = 1$; this trivially holds for any BF which is defined in the free DSm model $\mathcal{M}^f$.

This coincidence is based on the fact, that there are no conflicts in $\mathcal{M}^f$ and subsequently several combination rules, which are based on intersection of focal elements, mutually coincide in $\mathcal{M}^f$, see [2] and also Chapter 3 in [11]. Similarly, also DRC and BCR12 coincide in the free DSm model $\mathcal{M}^f$ with the conjunctive rule of combination (with the 2nd argument fixed to $m_A$, where $m_A(A) = 1$, $m_A(X) = 0$ for $X \neq A$), hence, also with the generalization of Dempster's rule of combination [2].

DRC performs a normalization, i.e., proportionalization of $m(X)$ according to $m(Y)$ for $Y$ such that $Y \cap A \not\equiv \emptyset$, whereas BCR12 performs a proportionalization according to $m(Y)$ for $Y \subseteq A$. Thus all bbms of $Y$ for $Y \cap A \not\equiv \emptyset$ & $Y \not\subseteq A$ are ignored within the proportionalization in BCR12, hence the rule is more sensitive with respect to bbms of $Y \subseteq A$.

BCR2 coincides with DRC for BFs where all focal element are from $D_1 \cup D_3$.
BCR27 coincides with DRC for BFs where all focal element are from $D_1 \cup D_3$.
BCR22 coincides with DRC for BFs where all focal element are from $D_1 \cup D_{2I}$.
All 27 other BCRs coincide with DRC only for BFs where all focal elements are from $D_1$ (i.e., if $Bel(A) = 1$), i.e., only in the case of trivial conditioning $m(X|A) = m(X)$ for $X \in A$, and $m(X|A) = 0$ otherwise.

[13] This evidently does not hold true for the extension from [13].

[14] We consider the generalized DRC and the new extended version of BCRs in this section.

### 10.8.2 Comparison of BCR1, BCR12 and BCR17 with classic rules of conditioning

Rule BCR12 is somewhere in between BRC1 (that is equivalent to the generalized belief focusing [3]) and the generalized DRC, as bbms of $X \subseteq A$ are kept located to $X$ by all 3 conditioning rules, bbms of $X$ for $X \cap A = \emptyset$ are proportionalized in the same way by BCR1 and BCR12 (according to $m(Y)$ for $Y \subseteq A$), whereas bbms of $X$ for $X \cap A \neq 0 \,\&\, X \nsubseteq A$ are in the same way relocated to $X \cap A$ by BCR12 and (the generalized) DRC.

The ways of relocation/redistribution of bbms $m(W)$ of focal elements in dependence on their relation to conditioning set $A$ are presented in Table 1.

There is a little bit more complicated situation for BCR17, which is somewhere between BCR1 and BCR12. Bbms $m(W)$ are either redistributed as by BCR1 or relocated as by BCR12 according to $S(W) = Bel(W \cap A)$ for $W \in D_{2I} \cup D_3$, bbms $m(W)$ are relocated as by BCR12 for other focal elements, i.e. for $W \in D_1 \cup D_{2D}$, see Table 1 again.

| | | | BCR1 | BCR12 | BCR17 | DRC |
|---|---|---|---|---|---|---|
| 1 | $D_1$ | $W \subseteq A$ | $W$ | $W$ | $W$ | $W$ |
| 2 | $D_{2D}$ | $W \cap A \equiv \emptyset$ | $Y : Y \subseteq A$ | $Y : Y \subseteq A$ | $Y : Y \subseteq A$ | $Y : Y \cap A \not\equiv \emptyset$ |
| 3 | $D_{2I}$ | $W \cap A \not\equiv \emptyset$ | $Y : Y \subseteq A$ | $W \cap A$ | $*$ | $W \cap A$ |
| | $D_3$ | $W \cap A \not\equiv \emptyset$ | $Y : Y \subseteq A$ | $W \cap A$ | $*$ | $W \cap A$ |

$*$ .... $m(W)$ should be redistributed among $Y : Y \subseteq W \cap A$ if $Bel(W \cap A) \neq 0$, or relocated to $W \cap A$ otherwise.

Table 10.1: Relocation/redistribution of bbm $m(W)$

The second column of the table contains the domain of focal element $W$, relation of focal element $W$ and of conditioning set $A$ is in the 3rd column, the 4th – 7th columns display the element of $D^{\Theta}_{\mathcal{M}}$ to which bbm $m(W)$ should be relocated or the set of elements of $D^{\Theta}_{\mathcal{M}}$ among them $m(W)$ should be distributed. Note: when $m(W)$ to be proportionalized among $Y$ such that $Y \subseteq A$ or $Y \cap A \neq 0$, $m(Y)$ must be positive; when $m(W)$ to be relocated to $W \cap A$ or redistributed among $Y$ such that $Y \subseteq W \cap A$, $m(W \cap A)$ and $m(Y)$ may be also equal to zero.

When computing BCRs according to Table 1, we have to keep the order[15] of steps (see the first column of the table), due to performing step 3 (redistribution of

[15] Notice, that in the changed order (step 3 before step 2) it would be necessary to normalize conflicts in DRC among all non-conflicting elements (i.e. among all elements of $D_{2I} \cup D_3$).

$D_{2I} \cup D_3$) before step 2 changes BCR12 to DRC as step 2 (redistribution of $D_{2D}$) differs DRC from BCR12.

In the case of other BCRs, it is not possible to say uniquely which rule adds more information within conditioning process in general, i.e. which rule is closer to DRC than another.

For a table of relocation/redistribution of bbm $m(W)$ for all BCRs see [6].

## 10.9 Conclusions

All of the 31 Belief Conditioning Rules (BCRs) are analysed in this chapter. The important role of the splitting of $D_2$ into $D_{2D}$ and $D_{2I}$ is underlined here. And a comparison of all 31 BCRs with Shafer's (i.e. Dempster's) rule of conditioning (DRC) is presented.

Based on the results of the presented analysis and the comparison, the definitions of BCRs are extended to be applicable to as wide definition domain as possible. A series of examples illuminating wider applicability of the new extended version of BCRs are displayed.

From the presented theoretical results it follows that BCR12 and BCR17 are really the best of all 31 BCRs, where BCR12 is better, as it adds less additive information within conditioning process. On the other hand, BCR12 cannot be considered a generalization of DRC. The real generalized DRC [3] is briefly recalled.

As the final recommendation for belief conditioning in DSmT, we recommend using BCR12 or the generalized DRC.

## 10.10 References


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## 10.11 Appendix: comments to implementation

The extension of BCRs defined in section on BCRs in [13] has already been mentioned in several footnotes in this chapter. It is simply defined for all BCRs as $m(A|A) = 1$, $m(X|A) = 0$ for $X \neq A$ when $Bel(A) = 0$. This definition extends definition domain of all BCRs to the entire set of all (generalized) belief functions. The extension enables implementation of BCRs which always produce a result. This extension fits with the nature of BCR1 which is based only on belief masses of focal elements from $D_1$, the other focal elements are simply ignored.

Nevertheless, the results of all other BCRs are related also to the focal elements from $D_3$ and results of BCR12–BCR31 also to the focal elements from $D_2$ which intersect conditioning set $A$. This is ignored by the extension from [13]. Our extension defined in this chapter extends BCRs respecting their nature as much as possible, see different results of particular BCRs in Examples 4 and 5. When applying the extension from [13], we obtain $m(U \cup R|U \cup R) = 1$, $m(X|U \cup R) = 0$ for $X \neq U \cup R$ for all BCRs in Example 4, and $m(R \cup B|R \cup B) = 1$, $m(X|R \cup B) = 0$ for $X \neq R \cup B$ for all BCRs in Example 5; all bbms of original focal elements from $D_2$ and $D_3$ are ignored by all BCRs.

Of course our extension has one disadvantage from the point of view of its implementation. There are still some possible non-trivial input belief functions, for which some (extended) BCRs are not defined, hence no implementation can provide any result for such an input. In such a case we can combine both the extensions: our from this chapter and that from [13] to extend definitions domains of BCRs as much as possible respecting the nature of particular BCRs. And whenever this extension is not defined we can apply the idea of the extension from [13] and transfer the conflicting belief masses to $m(A|A)$. Thus we obtain BCRs which are defined for all BFs and implementations which always produce some resulting bbms.

In the case of BCR12 we obtain the following formulas:

$$m_{BCR12}(X|A) = \sum_{W \cap A \equiv X} m(W) \; + \; \frac{m(X)}{Bel(A)} \cdot \sum_{Z \cap A \equiv \emptyset} m(Z),$$

for $X \subseteq A$ when $Bel(A) \neq 0$, and by the formulas

$$m_{BCR12}(A|A) = \sum_{A \subseteq W} m(W) + \sum_{W \in D_{2D}} m(W),$$

$$m_{BCR12}(X|A) = \sum_{W \cap A \equiv X} m(W),$$

for $X \subset A$ when $Bel(A) = 0$,

$m_{BCR12}(X|A) = 0$ for $X \not\subseteq A$ as in the original definition.

In a special case of BFs such that all focal elements are from $D_{2D}$ we obtain $m_{BCR12}(A|A) = 1$, $m_{BCR12}(X|A) = 0$ for $X \not\equiv A$. $D_{2D} = \emptyset$ in the free DSm model $\mathcal{M}^f$ thus the extension remains the same as it was in subsection 10.7.3 in the case of $\mathcal{M}^f$.

Considering this combined extension we can apply BCR2-BCR11 also in the case of modified example 4 obtaining the following results:
$m_{BCR2}(R\,|\,U\cup R) = 0.7$, $m_{BCR2}(U\cup R\,|\,U\cup R) = 0.3$;
$m_{BCR3}(R\,|\,U\cup R) = 0.7$, $m_{BCR3}(U\cap R\,|\,U\cup R) = 0.2$, $m_{BCR2}(U\cup R\,|\,U\cup R) = 0.1$;
$m_{BCR4}(R\,|\,U\cup R) = 0.8$, $m_{BCR4}(U\,|\,U\cup R) = 0.1$, $m_{BCR2}(U\cup R\,|\,U\cup R) = 0.1$;
$m_{BCR5}(R\,|\,U\cup R) = 0.8$, $m_{BCR5}(U\,|\,U\cup R) = 0.1$, $m_{BCR2}(U\cup R\,|\,U\cup R) = 0.1$;
$m_{BCR6}(R\,|\,U\cup R) = 0.75$, $m_{BCR6}(U\,|\,U\cup R) = 0.05$, $m_{BCR6}(U\cap R\,|\,U\cup R) = 0.05$, $m_{BCR6}(U\cup R\,|\,U\cup R) = 0.15$.
$Bel(A) = 0$ implies $S(W) = 0$ and that BCR7–BCR11 produce the same results as BCR2–BCR6 do. Analogically we can make conditioning by any other BCR in situations where our extension from Section 10.7 is not defined.

Applying the idea from this section, we obtain extensions of all BCRs to the set of all (generalized) belief functions such that the original ideas of BCRs [12] are conserved as much as possible.