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Application of intuitionistic neutrosophic graph structures in decision-making

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Abstract. In this research study, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

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1. Introduction

Graphical models are extensively useful tools for solving combinatorial problems of different fields including optimization, algebra, computer science, topology and operations research etc. Fuzzy graphical models are comparatively more close to nature, because in nature vagueness and ambiguity occurs. There are many complex phenomena and processes in science and technology having incomplete information. To deal such cases we needed a theory different from classical mathematics. Graph structures as generalized simple graphs are widely used for study of edge colored and edge signed graphs, also helpful and copiously used for studying large domains of computer science. Initially in 1965, Zadeh [29] proposed the notion of fuzzy sets to handle uncertainty in a lot of real applications. Fuzzy set theory is finding large number of applications in real time systems, where information inherent in systems has various levels of precision. Afterwards, Turksen [26] proposed the idea of interval-valued fuzzy set. But in various systems, there are membership and non-membership values, membership value is in favor of an event and non-membership value is against of that event. Atanassov [8] proposed the notion of intuitionistic
fuzzy set in 1986. The intuitionistic fuzzy sets are more practical and applicable in real-life situations. Intuitionistic fuzzy set deal with incomplete information, that is, degree of membership function, non-membership function but not indeterminate and inconsistent information that exists definitely in many systems, including belief system, decision-support systems etc. In 1998, Smarandache [24] proposed another notion of imprecise data named as neutrosophic sets. “Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra”. Neutrosophic set is recently proposed powerful formal framework. For convenient usage of neutrosophic sets in real-life situations, Wang et al. [27] proposed single-valued neutrosophic set as a generalization of intuitionistic fuzzy set[8]. A neutrosophic set has three independent components having values in unit interval [0, 1].

On the other hand, Bhowmik and Pal [10, 11] introduced the notions of intuitionistic neutrosophic sets and relations. Kauffman [16] defined fuzzy graph on the basis of Zadeh’s fuzzy relations [30]. Rosenfeld [21] investigated fuzzy analogue of various graph-theoretic ideas in 1975. Later on, Bhattacharya gave some remarks on fuzzy graph in 1987. Bhutani and Rosenfeld discussed M-strong fuzzy graphs with their properties in [12]. In 2011, Dinesh and Ramakrishnan [15] put forward fuzzy graph structures and investigated its properties. In 2016, Akram and Akmal [1] proposed the notion of bipolar fuzzy graph structures. Broumi et al. [13] portrayed single-valued neutrosophic graphs. Akram and Shahzadi [2] introduced the notion of neutrosophic soft graphs with applications. Akram and Shahzadi [1] highlighted some flaws in the definitions of Broumi et al. [13] and Shah-Hussain [22]. Akram et al. [5] also introduced the single-valued neutrosophic hypergraphs. Representation of graphs using intuitionistic neutrosophic soft sets was discussed in [3]. In this paper, we present concept of intuitionistic neutrosophic graph structures. We introduce the certain operations on intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [3, 6, 7, 9, 13, 14, 17, 18, 20, 22, 23, 25, 28, 30].

2. Intuitionistic Neutrosophic Graph Structures

**Definition 2.1.** ([23]). Let $\tilde{G}_1 = (P, P_1, P_2, \ldots, P_r)$ and $\tilde{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, Cartesian product of $\tilde{G}_1$ and $\tilde{G}_1$ is defined as:

$$\tilde{G}_1 \times \tilde{G}_2 = (P \times P', P_1 \times P'_1, P_2 \times P'_2, \ldots, P_r \times P'_r),$$

where $P_h \times P'_h = \{(k_1l_1)(k_2l_2) \mid l \in P', k_1k_2 \in P_h\} \cup \{(k_1l_1)(k_2l_2) \mid k \in P, l_1l_2 \in P'_h\},$ $h = (1, 2, \ldots, r)$.

**Definition 2.2.** ([23]). Let $\tilde{G}_1 = (P, P_1, P_2, \ldots, P_n)$ and $\tilde{G}_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, cross product of $\tilde{G}_1$ and $\tilde{G}_2$ is defined as:

$$\tilde{G}_1 \ast \tilde{G}_2 = (P \ast P', P_1 \ast P'_1, P_2 \ast P'_2, \ldots, P_r \ast P'_r),$$
where $P_h \ast P''_h = \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h, l_1l_2 \in P''_h\}, h = (1,2,\ldots,r)$.

**Definition 2.3.** ([23]). Let $G_1 = (P, P_1, P_2, \ldots, P_r)$ and $G_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, lexicographic product of $G_1$ and $G_2$ is defined as:

$$G_1 \circ G_2 = (P \circ P', P_1 \circ P'_1, P_2 \circ P'_2, \ldots, P_r \circ P'_r),$$

where $P_h \circ P''_h = \{(k_1l_1)(k_2l_2) \mid k_1, k_2 \in P_h \cup \{(k_1l_1)(k_2l_2) \mid k_2 \in P, l_1l_2 \in P''_h\}, h = (1,2,\ldots,r)$.

**Definition 2.4.** ([23]). Let $G_1 = (P, P_1, P_2, \ldots, P_r)$ and $G_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, strong product of $G_1$ and $G_2$ is defined as:

$$G_1 \boxtimes G_2 = (P \boxtimes P', P_1 \boxtimes P'_1, P_2 \boxtimes P'_2, \ldots, P_r \boxtimes P'_r),$$

where $P_h \boxtimes P''_h = \{(k_1l_1)(k_2l_2) \mid k_1, k_2 \in P_h \cup \{(k_1l_1)(k_2l_2) \mid k_2 \in P, l_1l_2 \in P''_h\}, h = (1,2,\ldots,r)$.

**Definition 2.5.** ([23]). Let $G_1 = (P, P_1, P_2, \ldots, P_r)$ and $G_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, composition of $G_1$ and $G_2$ is defined as:

$$G_1 \circ G_2 = (P \circ P', P_1 \circ P'_1, P_2 \circ P'_2, \ldots, P_r \circ P'_r),$$

where $P_h \circ P''_h = \{(k_1l_1)(k_2l_2) \mid k_1, k_2 \in P_h \cup \{(k_1l_1)(k_2l_2) \mid k_2 \in P, l_1l_2 \in P''_h\} \cup \{(k_1l_1)(k_2l_2) \mid k_1k_2 \in P_h, l_1l_2 \in P' \text{ such that } l_1 \neq l_2\}, h = (1,2,\ldots,r)$.

**Definition 2.6.** ([23]). Let $G_1 = (P, P_1, P_2, \ldots, P_r)$ and $G_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, union of $G_1$ and $G_2$ is defined as:

$$G_1 \cup G_2 = (P \cup P', P_1 \cup P'_1, P_2 \cup P'_2, \ldots, P_r \cup P'_r).$$

**Definition 2.7.** ([23]). Let $G_1 = (P, P_1, P_2, \ldots, P_r)$ and $G_2 = (P', P'_1, P'_2, \ldots, P'_r)$ be two GSs, join of $G_1$ and $G_2$ is defined as:

$$G_1 + G_2 = (P + P', P_1 + P'_1, P_2 + P'_2, \ldots, P_r + P'_r),$$

where $P + P' = P \cup P', P_h + P''_h = P_h \cup P'_h \cup P''_h$ for $h = (1,2,\ldots,r)$. $P''_h$ consists of all those edges which join the vertices of $P$ and $P'$.

**Definition 2.8.** ([19]). Let $V$ be a fixed set. A generalized intuitionistic fuzzy set $I$ of $V$ is an object having the form $I = \{(v, \mu_I(v), \nu_I(v)) \mid v \in V\}$, where the functions $\mu_I : V \rightarrow [0,1]$ and $\nu_I : V \rightarrow [0,1]$ define the degree of membership and degree of nonmembership of an element $v \in V$, respectively, such that

$$\min\{\mu_I(v), \nu_I(v)\} \leq 0.5, \text{ for all } v \in V.$$

This condition is called the generalized intuitionistic condition.

**Definition 2.9.** ([10, 11]). A set $I = \{I_I(v), I_I(v), F_I(v) : v \in V\}$ is said to be an intuitionistic neutrosophic (IN) set, if

(i) $\{T_I(v) \wedge I_I(v)\} \leq 0.5, \quad \{I_I(v) \wedge F_I(v)\} \leq 0.5, \quad \{F_I(v) \wedge T_I(v)\} \leq 0.5,$

(ii) $0 \leq T_I(v) + I_I(v) + F_I(v) \leq 2.$

**Definition 2.10.** An intuitionistic neutrosophic graph is a pair $G = (A, B)$ with underlying set $V$, where $T_A$, $F_A$, $I_A : V \rightarrow [0,1]$ denote the truth, falsity and indeterminacy membership values of the vertices in $V$ and $T_B$, $F_B$, $I_B : E \subseteq V \times V \rightarrow [0,1]$ denote the truth, falsity and indeterminacy membership values of the edges $kl \in E$ such that
(i) \( T_B(kl) \leq T_A(k) \land T_A(l), \quad F_B(kl) \leq F_A(k) \lor F_A(l), \quad I_B(kl) \leq I_A(k) \land I_A(l), \)
(ii) \( T_B(kl) \land I_B(kl) \leq 0.5, \quad T_B(kl) \lor F_B(kl) \leq 0.5, \quad I_B(kl) \land F_B(kl) \leq 0.5, \)
(iii) \( 0 \leq T_B(kl) + F_B(kl) + I_B(kl) \leq 2, \forall k, l \in V. \)

**Definition 2.11.** \( \hat{G}_i = (O, O_1, O_2, \ldots, O_r) \) is said to be an intuitionistic neutrosophic graph structure (INGS) of graph structure \( \hat{G} = (P, P_1, P_2, \ldots, P_r) \), if \( O = < k, T(k), I(k), F(k) > \) and \( O_h = < kl, T_h(kl), I_h(kl), F_h(kl) > \) are the intuitionistic neutrosophic (IN) sets on the sets \( P \) and \( P_h \), respectively such that
(i) \( T_h(kl) \leq T(k) \land T(l), \quad I_h(kl) \leq I(k) \land I(l), \quad F_h(kl) \leq F(k) \lor F(l), \)
(ii) \( T_h(kl) \land I_h(kl) \leq 0.5, \quad T_h(kl) \lor F_h(kl) \leq 0.5, \quad I_h(kl) \land F_h(kl) \leq 0.5, \)
(iii) \( 0 \leq T_h(kl) + I_h(kl) + F_h(kl) \leq 2, \quad \forall k, l \in O_h, \quad h \in \{1, 2, \ldots, r\}, \)
where, \( O \) and \( O_h \) are underlying vertex and \( h \)-edge sets of INGS \( \hat{G}_i, \quad h \in \{1, 2, \ldots, r\} \).

**Example 2.12.** An intuitionistic neutrosophic graph structure is represented in Fig. 1.

![Figure 1. An intuitionistic neutrosophic graph structure](image-url)

Now we define the operations on INGSs.

**Definition 2.13.** Let \( \hat{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r}) \) and \( \hat{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, O_{2r}) \) be INGSs of GSS \( \hat{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r}) \) and \( \hat{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r}) \), respectively.

Cartesian product of \( \hat{G}_{i1} \) and \( \hat{G}_{i2} \), denoted by
\( \hat{G}_{i1} \times \hat{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r}), \)
is defined as:
(i) \[
\left\{
\begin{array}{l}
T_{(O_1 \times O_2)}(kl) = (T_{O_1} \times T_{O_2})(kl) = T_{O_1}(k) \land T_{O_2}(l)
\end{array}\right.
\]
for all \( k \in P_1 \times P_2 \),
(ii) \[
\left\{
\begin{array}{l}
T_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \times T_{O_{2h}})(kl_1)(kl_2) = T_{O_{1h}}(k) \land T_{O_{2h}}(l_1l_2)
\end{array}\right.
\]
for all \( k \in P_1, l_1l_2 \in P_{2h} \),
Example 2.14. Consider $\tilde{G}_{11} = (O_1, O_{11}, O_{12})$ and $\tilde{G}_{12} = (O_2, O_{21}, O_{22})$ are two INGSs of GSs $\tilde{G}_1 = (P_1, P_{11}, P_{12})$ and $\tilde{G}_2 = (P_2, P_{21}, P_{22})$ respectively, as represented in Fig. 2, where $P_{11} = \{k_1, k_2\}$, $P_{12} = \{k_3, k_4\}$, $P_{21} = \{l_1, l_2\}$, $P_{22} = \{l_2, l_3\}$.

![Figure 2. Two INGSs $\tilde{G}_{11}$ and $\tilde{G}_{12}$](image)

Cartesian product of $\tilde{G}_{11}$ and $\tilde{G}_{12}$ defined as $\tilde{G}_{11} \times \tilde{G}_{12} = \{O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}\}$ is represented in Fig. 3.
Theorem 2.15. Cartesian product $G_1 \times G_2 = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r})$ of two INGSs of GSs $G_1$ and $G_2$ is an INGS of $G_1 \times G_2$.

Proof. We consider two cases:

**Case 1:** For $k \in P_1$, $l_1 l_2 \in P_{2h}$

$$T_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) = T_{O_1}(k) \land T_{O_{2h}}(l_1 l_2)$$

$$\leq T_{O_1}(k) \land [T_{O_2}(l_1) \land T_{O_2}(l_2)]$$

$$= [T_{O_1}(k) \land T_{O_2}(l_1)] \land [T_{O_1}(k) \land T_{O_2}(l_2)]$$

$$= T_{(O_1 \times O_2)}(kl_1) \land T_{(O_1 \times O_2)}(kl_2),$$

$$I_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) = I_{O_1}(k) \land I_{O_{2h}}(l_1 l_2)$$

$$\leq I_{O_1}(k) \land [I_{O_2}(l_1) \land I_{O_2}(l_2)]$$

$$= [I_{O_1}(k) \land I_{O_2}(l_1)] \land [I_{O_1}(k) \land I_{O_2}(l_2)]$$

$$= I_{(O_1 \times O_2)}(kl_1) \land I_{(O_1 \times O_2)}(kl_2),$$

$$F_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) = F_{O_1}(k) \lor F_{O_{2h}}(l_1 l_2)$$

$$\leq F_{O_1}(k) \lor [F_{O_2}(l_1) \lor F_{O_2}(l_2)]$$

$$= [F_{O_1}(k) \lor F_{O_2}(l_1)] \lor [F_{O_1}(k) \lor F_{O_2}(l_2)]$$

$$= F_{(O_1 \times O_2)}(kl_1) \lor F_{(O_1 \times O_2)}(kl_2),$$

for $kl_1, kl_2 \in P_1 \times P_2$.

**Case 2:** For $k \in P_2$, $l_1 l_2 \in P_{1h}$

$$T_{(O_{1h} \times O_{2h})}((l_1 k)(l_2 k)) = T_{O_2}(k) \land T_{O_{1h}}(l_1 l_2)$$

$$\leq T_{O_2}(k) \land [T_{O_1}(l_1) \land T_{O_1}(l_2)]$$

$$= [T_{O_1}(k) \land T_{O_1}(l_1)] \land [T_{O_2}(k) \land T_{O_1}(l_2)]$$

$$= T_{(O_1 \times O_2)}(l_1 k) \land T_{(O_1 \times O_2)}(l_2 k),$$
\[
I(O_{1k} \times O_{2h})(l_1k)(l_2k)) = I_{O_{1k}}(k) \wedge I_{O_{2h}}(l_1l_2) \\
\leq I_{O_{2k}}(k) \wedge [I_{O_{1k}}(l_1) \wedge I_{O_{2k}}(l_2)] \\
= [I_{O_{2k}}(k) \wedge I_{O_{1k}}(l_1)] \wedge [I_{O_{2k}}(k) \wedge I_{O_{2k}}(l_2)] \\
= I_{(O_{1k} \times O_{2h})}(l_1k) \wedge I_{(O_{1k} \times O_{2h})}(l_2k),
\]

\[
F(O_{1k} \times O_{2h})(l_1k)(l_2k)) = F_{O_{2k}}(k) \wedge F_{O_{1k}}(l_1l_2) \\
\leq F_{O_{2k}}(k) \wedge [F_{O_{1k}}(l_1) \wedge F_{O_{2k}}(l_2)] \\
= [F_{O_{2k}}(k) \wedge F_{O_{1k}}(l_1)] \wedge [F_{O_{2k}}(k) \wedge F_{O_{2k}}(l_2)] \\
= F_{(O_{1k} \times O_{2h})}(l_1k) \wedge F_{(O_{1k} \times O_{2h})}(l_2k),
\]

for \(l_1k, l_2k \in P_1 \times P_2\).

Both cases exists \(\forall h \in \{1, 2, \ldots, r\}\). This completes the proof. \(\square\)

**Definition 2.16.** Let \(\hat{G}_{1i} = (O_{1i}, O_{12}, \ldots, Q_{1r})\) and \(\hat{G}_{1r} = (O_{21}, O_{22}, \ldots, Q_{2r})\) be INGSs of GS \(\hat{G}_{11} = (P_{11}, P_{12}, \ldots, P_{1r})\) and \(\hat{G}_{2} = (P_{21}, P_{22}, \ldots, P_{2r})\), respectively. Cross product of \(\hat{G}_{1i}\) and \(\hat{G}_{1r}\), denoted by

\[
\hat{G}_{1i} \ast \hat{G}_{1r} = (O_{1i} \ast O_{21}, O_{12} \ast O_{22}, \ldots, O_{1r} \ast O_{2r}),
\]

is defined as:

1. \(T_{O_{1i} \ast O_{21}}(kl) = (T_{O_{1i}} \ast T_{O_{21}})(kl) = T_{O_{1i}}(k) \wedge T_{O_{21}}(l)\)
2. \(I_{O_{1i} \ast O_{21}}(kl) = (I_{O_{1i}} \ast I_{O_{21}})(kl) = I_{O_{1i}}(k) \wedge I_{O_{21}}(l)\)
3. \(F_{O_{1i} \ast O_{21}}(kl) = (F_{O_{1i}} \ast F_{O_{21}})(kl) = F_{O_{1i}}(k) \wedge F_{O_{21}}(l)\)

for all \(kl \in P_1 \times P_2\).

**Example 2.17.** Cross product of INGSs \(\hat{G}_{1i}\) and \(\hat{G}_{1r}\) shown in Fig. 2 is defined as \(\hat{G}_{1i} \ast \hat{G}_{1r} = \{O_{1i} \ast O_{21}, O_{12} \ast O_{22}\}\) and is represented in Fig. 4.

![Figure 4. \(\hat{G}_{1i} \ast \hat{G}_{1r}\)](image)

**Theorem 2.18.** Cross product \(\hat{G}_{1i} \ast \hat{G}_{1r} = (O_{1i} \ast O_{21}, O_{12} \ast O_{22}, \ldots, O_{1r} \ast O_{2r})\) of two INGSs of GS \(\hat{G}_{1i}\) and \(\hat{G}_{2}\) is an INGS of \(\hat{G}_{1i} \ast \hat{G}_{2}\).
Proof. For all $k_1, l_1, k_2, l_2 \in P_1 \ast P_2$

$$T_{(O_{1h} \ast O_{2h})}((k_1 l_1) (k_2 l_2)) = T_{O_{1h}}(k_1 k_2) \land T_{O_{2h}}(l_1 l_2)$$

$$\leq [T_{O_{1h}}(k_1) \land T_{O_{1h}}(k_2)] \land [T_{O_{2h}}(l_1) \land T_{O_{2h}}(l_2)]$$

$$= [T_{O_{1h}}(k_1) \land T_{O_{2h}}(l_1)] \land [T_{O_{1h}}(k_2) \land T_{O_{2h}}(l_2)]$$

$$= T_{(O_{1h} \ast O_{2h})}(k_1 l_1) \land T_{(O_{1h} \ast O_{2h})}(k_2 l_2),$$

$$I_{(O_{1h} \ast O_{2h})}((k_1 l_1) (k_2 l_2)) = I_{O_{1h}}(k_1 k_2) \land I_{O_{2h}}(l_1 l_2)$$

$$\leq [I_{O_{1h}}(k_1) \land I_{O_{1h}}(k_2)] \land [I_{O_{2h}}(l_1) \land I_{O_{2h}}(l_2)]$$

$$= [I_{O_{1h}}(k_1) \land I_{O_{2h}}(l_1)] \land [I_{O_{1h}}(k_2) \land I_{O_{2h}}(l_2)]$$

$$= I_{(O_{1h} \ast O_{2h})}(k_1 l_1) \land I_{(O_{1h} \ast O_{2h})}(k_2 l_2),$$

$$F_{(O_{1h} \ast O_{2h})}((k_1 l_1) (k_2 l_2)) = F_{O_{1h}}(k_1 k_2) \lor F_{O_{2h}}(l_1 l_2)$$

$$\leq [F_{O_{1h}}(k_1) \lor F_{O_{1h}}(k_2)] \lor [F_{O_{2h}}(l_1) \lor F_{O_{2h}}(l_2)]$$

$$= [F_{O_{1h}}(k_1) \lor F_{O_{2h}}(l_1)] \lor [F_{O_{1h}}(k_2) \lor F_{O_{2h}}(l_2)]$$

$$= F_{(O_{1h} \ast O_{2h})}(k_1 l_1) \lor F_{(O_{1h} \ast O_{2h})}(k_2 l_2),$$

for $h \in \{1, 2, \ldots, r\}$. This completes the proof. \qed

Definition 2.19. Let $G_{11} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $G_{12} = (O_2, O_{21}, O_{22}, \ldots, O_{2r})$ be INGSs of GSs $G_{11} = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $G_{12} = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Lexicographic product of $G_{11}$ and $G_{12}$, denoted by

$$G_{11} \circ G_{12} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r}),$$

is defined as:

(i) $T_{(O_{1h} \ast O_{2h})}(kl) = (T_{O_{1h}} \bullet T_{O_{2h}})(kl) = T_{O_{1h}}(k) \land T_{O_{2h}}(l)$

(ii) $I_{(O_{1h} \ast O_{2h})}(kl) = (I_{O_{1h}} \bullet I_{O_{2h}})(kl) = I_{O_{1h}}(k) \land I_{O_{2h}}(l)$

(iii) $F_{(O_{1h} \ast O_{2h})}(kl) = (F_{O_{1h}} \bullet F_{O_{2h}})(kl) = F_{O_{1h}}(k) \lor F_{O_{2h}}(l)$

for all $k, l \in P_1 \times P_2$.

Example 2.20. Lexicographic product of INGSs $G_{11}$ and $G_{12}$ shown in Fig. 2 is defined as $G_{1i} \circ G_{1j} = \{O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}\}$ and is represented in Fig. 5.
We consider two cases:

**Theorem 2.21.** Lexicographic product $\tilde{G}_{11} \cdot \tilde{G}_{12} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r})$ of two INGSs of the GSs $\tilde{G}_1$ and $\tilde{G}_2$ is an INGS of $\tilde{G}_1 \cdot \tilde{G}_2$.

**Proof.** We consider two cases:

**Case 1:** For $k \in P_1$, $l_1 l_2 \in P_{2h}$

$$T_{(O_{1h} \bullet O_{2h})}((kl_1)(kl_2)) = T_{O_1}(k) \land T_{O_{2h}}(l_1 l_2)$$

$$\leq T_{O_1}(k) \land [T_{O_2}(l_1) \land T_{O_2}(l_2)]$$

$$= [T_{O_1}(k) \land T_{O_2}(l_1)] \land [T_{O_1}(k) \land T_{O_2}(l_2)]$$

$$= T_{(O_1 \bullet O_2)}(kl_1) \land T_{(O_1 \bullet O_2)}(kl_2),$$

$$I_{(O_{1h} \bullet O_{2h})}((kl_1)(kl_2)) = I_{O_1}(k) \land I_{O_{2h}}(l_1 l_2)$$

$$\leq I_{O_1}(k) \land [I_{O_2}(l_1) \land I_{O_2}(l_2)]$$

$$= [I_{O_1}(k) \land I_{O_2}(l_1)] \land [I_{O_1}(k) \land I_{O_2}(l_2)]$$

$$= I_{(O_1 \bullet O_2)}(kl_1) \land I_{(O_1 \bullet O_2)}(kl_2),$$

$$F_{(O_{1h} \bullet O_{2h})}((kl_1)(kl_2)) = F_{O_1}(k) \lor F_{O_{2h}}(l_1 l_2)$$

$$\leq F_{O_1}(k) \lor [F_{O_2}(l_1) \lor F_{O_2}(l_2)]$$

$$= [F_{O_1}(k) \lor F_{O_2}(l_1)] \lor [F_{O_1}(k) \lor F_{O_2}(l_2)]$$

$$= F_{(O_1 \bullet O_2)}(kl_1) \lor F_{(O_1 \bullet O_2)}(kl_2),$$

for $kl_1, kl_2 \in P_1 \cdot P_2$. 

\[ \text{Figure 5. } \tilde{G}_{11} \cdot \tilde{G}_{12} \]
Case 2: For $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$
\[
T_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = T_{O_{1h}}(k_1k_2) \land T_{O_{2h}}(l_1l_2)
\leq [T_{O_{1h}}(k_1) \land T_{O_{2h}}(k_2)] \land [T_{O_{1h}}(l_1) \land T_{O_{2h}}(l_2)]
= [T_{O_{1h}}(k_1) \land T_{O_{2h}}(l_1)] \land [T_{O_{1h}}(k_2) \land T_{O_{2h}}(l_2)]
= T_{(O_{1h} \bullet O_{2h})}(k_1l_1) \land T_{(O_{1h} \bullet O_{2h})}(k_2l_2),
\]
\[
I_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = I_{O_{1h}}(k_1k_2) \land I_{O_{2h}}(l_1l_2)
\leq [I_{O_{1h}}(k_1) \land I_{O_{2h}}(k_2)] \land [I_{O_{1h}}(l_1) \land I_{O_{2h}}(l_2)]
= [I_{O_{1h}}(k_1) \land I_{O_{2h}}(l_1)] \land [I_{O_{1h}}(k_2) \land I_{O_{2h}}(l_2)]
= I_{(O_{1h} \bullet O_{2h})}(k_1l_1) \land I_{(O_{1h} \bullet O_{2h})}(k_2l_2),
\]
\[
F_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = F_{O_{1h}}(k_1k_2) \lor F_{O_{2h}}(l_1l_2)
\leq [F_{O_{1h}}(k_1) \lor F_{O_{2h}}(k_2)] \lor [F_{O_{1h}}(l_1) \lor F_{O_{2h}}(l_2)]
= [F_{O_{1h}}(k_1) \lor F_{O_{2h}}(l_1)] \lor [F_{O_{1h}}(k_2) \lor F_{O_{2h}}(l_2)]
= F_{(O_{1h} \bullet O_{2h})}(k_1l_1) \lor F_{(O_{1h} \bullet O_{2h})}(k_2l_2),
\]
for $k_1l_1, k_2l_2 \in P_1 \bullet P_2$.
Both cases hold for $h \in \{1, 2, \ldots, r\}$. This completes the proof.

\begin{definition}
Let $\hat{G}_{1i} = (O_{1}, O_{11}, O_{12}, \ldots, O_{1r})$ and $\hat{G}_{12} = (O_{2}, O_{21}, O_{22}, \ldots, O_{2r})$ be INGSs of GSs $\hat{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\hat{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. Strong product of $\hat{G}_{1i}$ and $\hat{G}_{12}$, denoted by
\[
\hat{G}_{1i} \boxtimes \hat{G}_{12} = (O_{1} \boxtimes O_{2}, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \ldots, O_{1r} \boxtimes O_{2r}),
\]
is defined as:

\begin{enumerate}
\item For all $k \in P_{1}, l \in P_{2}$,
\[
\begin{align*}
T_{(O_{1h} \boxtimes O_{2h})}(kl) &= (T_{O_{1h}} \boxtimes T_{O_{2h}})(kl) = T_{O_{1h}}(k) \land T_{O_{2h}}(l),
I_{(O_{1h} \boxtimes O_{2h})}(kl) &= (I_{O_{1h}} \boxtimes I_{O_{2h}})(kl) = I_{O_{1h}}(k) \land I_{O_{2h}}(l),
F_{(O_{1h} \boxtimes O_{2h})}(kl) &= (F_{O_{1h}} \boxtimes F_{O_{2h}})(kl) = F_{O_{1h}}(k) \lor F_{O_{2h}}(l),
\end{align*}
\]
\item For all $k \in P_{1h}, l \in P_{2h}$,
\[
\begin{align*}
T_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1) \lor T_{O_{2h}}(l_1) \land T_{O_{1h}}(k_2) \land T_{O_{2h}}(l_2),
I_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1) \land I_{O_{2h}}(l_1) \lor I_{O_{1h}}(k_2) \land I_{O_{2h}}(l_2),
F_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1) \lor F_{O_{2h}}(l_1) \lor F_{O_{1h}}(k_2) \land F_{O_{2h}}(l_2),
\end{align*}
\]
\item For all $l \in P_{1h}, k \in P_{2h}$,
\[
\begin{align*}
T_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) &= (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l)(k_2l) = T_{O_{1h}}(k_1) \land T_{O_{2h}}(l),
I_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) &= (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l)(k_2l) = I_{O_{1h}}(k_1) \lor I_{O_{2h}}(l),
F_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) &= (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l)(k_2l) = F_{O_{1h}}(k_1) \lor F_{O_{2h}}(l),
\end{align*}
\]
\item For all $k \in P_{1h}, l \in P_{2h}$,
\[
\begin{align*}
T_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1) \land T_{O_{2h}}(l_1) \lor T_{O_{1h}}(k_2) \land T_{O_{2h}}(l_2),
I_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1) \lor I_{O_{2h}}(l_1) \land I_{O_{1h}}(k_2) \land I_{O_{2h}}(l_2),
F_{(O_{1h} \boxtimes O_{2h})}(k_1l_1)(k_2l_2) &= (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1) \lor F_{O_{2h}}(l_1) \lor F_{O_{1h}}(k_2) \land F_{O_{2h}}(l_2),
\end{align*}
\]
\end{enumerate}

Example 2.23. Strong product of INGSs $\hat{G}_{11}$ and $\hat{G}_{12}$ shown in Fig. 2 is defined as $\hat{G}_{11} \boxtimes \hat{G}_{12} = \{O_{1} \boxtimes O_{2}, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}\}$ and is represented in Fig. 6.
Theorem 2.24. Strong product $\mathcal{G}_{i1} \boxtimes \mathcal{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \ldots, O_{1r} \boxtimes O_{2r})$ of two INGSs of the GSs $\mathcal{G}_1$ and $\mathcal{G}_2$ is an INGS of $\mathcal{G}_1 \boxtimes \mathcal{G}_2$.

Proof. There are three cases:

Case 1: For $k \in P_1$, $l_1 l_2 \in P_{2h}$

\[
T_{(O_{1h} \boxtimes O_{2h})}((kl_1)(kl_2)) = T_{O_1}(k) \land T_{O_{2h}}(l_1 l_2) \\
\leq T_{O_1}(k) \land [T_{O_2}(l_1) \land T_{O_2}(l_2)] \\
= [T_{O_1}(k) \land T_{O_2}(l_1)] \land [T_{O_1}(k) \land T_{O_2}(l_2)] \\
= T_{(O_1 \boxtimes O_2)(kl_1)} \land T_{(O_1 \boxtimes O_2)(kl_2)},
\]

\[
I_{(O_{1h} \boxtimes O_{2h})}((kl_1)(kl_2)) = I_{O_1}(k) \land I_{O_{2h}}(l_1 l_2) \\
\leq I_{O_1}(k) \land [I_{O_2}(l_1) \land I_{O_2}(l_2)] \\
= [I_{O_1}(k) \land I_{O_2}(l_1)] \land [I_{O_1}(k) \land I_{O_2}(l_2)] \\
= I_{(O_1 \boxtimes O_2)(kl_1)} \land I_{(O_1 \boxtimes O_2)(kl_2)},
\]
This completes the proof.

Case 2: For \( k \in P_2, l_1l_2 \in P_{1h} \)

\[
T_{(O_{1h} \circ O_{2h})}((l_1k)(l_2)) = T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2)
\]

\[
\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)]
\]

\[
= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)]
\]

\[
= T_{(O_1 \circ O_{2h})}(l_1k) \wedge T_{(O_1 \circ O_{2h})}(l_2k),
\]

\[
I_{(O_{1h} \circ O_{2h})}((l_1k)(l_2)) = I_{O_2}(k) \wedge I_{O_{1h}}(l_1l_2)
\]

\[
\leq I_{O_2}(k) \wedge [I_{O_1}(l_1) \wedge I_{O_1}(l_2)]
\]

\[
= [I_{O_2}(k) \wedge I_{O_1}(l_1)] \wedge [I_{O_2}(k) \wedge I_{O_1}(l_2)]
\]

\[
= I_{(O_1 \circ O_{2h})}(l_1k) \wedge I_{(O_1 \circ O_{2h})}(l_2k),
\]

\[
F_{(O_{1h} \circ O_{2h})}((l_1k)(l_2)) = F_{O_2}(k) \vee F_{O_{1h}}(l_1l_2)
\]

\[
\leq F_{O_2}(k) \vee [F_{O_1}(l_1) \vee F_{O_1}(l_2)]
\]

\[
= [F_{O_2}(k) \vee F_{O_1}(l_1)] \vee [F_{O_2}(k) \vee F_{O_1}(l_2)]
\]

\[
= F_{(O_1 \circ O_{2h})}(l_1k) \vee F_{(O_1 \circ O_{2h})}(l_2k),
\]

for \( l_1k, l_2k \in P_1 \otimes P_2 \).

Case 3: For every \( k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h} \)

\[
T_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = T_{O_1}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2)
\]

\[
\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)]
\]

\[
= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_2}(k_2) \wedge T_{O_2}(l_2)]
\]

\[
= T_{(O_1 \circ O_{2h})}(k_1l_1) \wedge T_{(O_1 \circ O_{2h})}(k_2l_2),
\]

\[
I_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = I_{O_1}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2)
\]

\[
\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]
\]

\[
= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_2}(k_2) \wedge I_{O_2}(l_2)]
\]

\[
= I_{(O_1 \circ O_{2h})}(k_1l_1) \wedge I_{(O_1 \circ O_{2h})}(k_2l_2),
\]

\[
F_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = F_{O_1}(k_1k_2) \vee F_{O_{2h}}(l_1l_2)
\]

\[
\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]
\]

\[
= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_2}(k_2) \vee F_{O_2}(l_2)]
\]

\[
= F_{(O_1 \circ O_{2h})}(k_1l_1) \vee F_{(O_1 \circ O_{2h})}(k_2l_2),
\]

for \( k_1l_1, k_2l_2 \in P_1 \otimes P_2 \), and \( h = 1, 2, \ldots, r \).

This completes the proof. □
Definition 2.25. Let $\hat{G}_1 = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\hat{G}_2 = (O_2, O_{21}, O_{22}, \ldots, O_{2r})$ be INGSs of GSs $\hat{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\hat{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$, respectively. The composition of $\hat{G}_1$ and $\hat{G}_2$, denoted by

$$\hat{G}_1 \circ \hat{G}_2 = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r}),$$

is defined as:

\[
\begin{align*}
& (i) \quad \left\{ \begin{array}{l}
T_{(O_1 \circ O_2)}(kl) = (T_{O_1} \circ T_{O_2})(kl) = T_{O_1}(k) \land T_{O_2}(l) \\
I_{(O_1 \circ O_2)}(kl) = (I_{O_1} \circ I_{O_2})(kl) = I_{O_1}(k) \land I_{O_2}(l) \\
F_{(O_1 \circ O_2)}(kl) = (F_{O_1} \circ F_{O_2})(kl) = F_{O_1}(k) \lor F_{O_2}(l)
\end{array} \right.
\text{for all } kl \in P_1 \times P_2, \\
& (ii) \quad \left\{ \begin{array}{l}
T_{(O_{1h} \circ O_{2h})}(kl) = (T_{O_{1h}} \circ T_{O_{2h}})(kl) = T_{O_{1h}}(k) \land T_{O_{2h}}(l) \\
I_{(O_{1h} \circ O_{2h})}(kl) = (I_{O_{1h}} \circ I_{O_{2h}})(kl) = I_{O_{1h}}(k) \land I_{O_{2h}}(l) \\
F_{(O_{1h} \circ O_{2h})}(kl) = (F_{O_{1h}} \circ F_{O_{2h}})(kl) = F_{O_{1h}}(k) \lor F_{O_{2h}}(l)
\end{array} \right.
\text{for all } k \in P_{1h}, l \in P_{2h}, \\
& (iii) \quad \left\{ \begin{array}{l}
T_{(O_{1h} \circ O_{2h})}(kl) = (T_{O_{1h}} \circ T_{O_{2h}})(kl) = T_{O_{1h}}(k) \land T_{O_{2h}}(l) \\
I_{(O_{1h} \circ O_{2h})}(kl) = (I_{O_{1h}} \circ I_{O_{2h}})(kl) = I_{O_{1h}}(k) \land I_{O_{2h}}(l) \\
F_{(O_{1h} \circ O_{2h})}(kl) = (F_{O_{1h}} \circ F_{O_{2h}})(kl) = F_{O_{1h}}(k) \lor F_{O_{2h}}(l)
\end{array} \right.
\text{for all } k \in P_{1h}, l \in P_{2h}, \\
& (iv) \quad \left\{ \begin{array}{l}
T_{(O_{1h} \circ O_{2h})}(kl) = (T_{O_{1h}} \circ T_{O_{2h}})(kl) = T_{O_{1h}}(k) \land T_{O_{2h}}(l) \\
I_{(O_{1h} \circ O_{2h})}(kl) = (I_{O_{1h}} \circ I_{O_{2h}})(kl) = I_{O_{1h}}(k) \land I_{O_{2h}}(l) \\
F_{(O_{1h} \circ O_{2h})}(kl) = (F_{O_{1h}} \circ F_{O_{2h}})(kl) = F_{O_{1h}}(k) \lor F_{O_{2h}}(l)
\end{array} \right.
\text{for all } k \in P_{1h}, l \in P_{2h} \text{ such that } k \neq l.
\]

Example 2.26. The composition of INGSs $\hat{G}_1$ and $\hat{G}_2$ shown in Fig. 2 is defined as:

$$\hat{G}_1 \circ \hat{G}_2 = \{O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}\}$$

and is represented in Fig. 7.
Theorem 2.27. The composition $\hat{G}_{i1} \circ \hat{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r})$ of two INGSs of GSs $G_1$ and $G_2$ is an INGS of $\hat{G}_1 \circ \hat{G}_2$.

Proof. We consider three cases:

**Case 1:** For $k \in P_1$, $l_1l_2 \in P_{2h}$

\[
T_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) = T_{O_1}(k) \land T_{O_{2h}}(l_1l_2) \\
\leq T_{O_1}(k) \land [T_{O_2}(l_1) \land T_{O_2}(l_2)] \\
= [T_{O_1}(k) \land T_{O_2}(l_1)] \land [T_{O_1}(k) \land T_{O_2}(l_2)] \\
= T_{(O_1 \circ O_2)}(kl_1) \land T_{(O_1 \circ O_2)}(kl_2),
\]

\[
I_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) = I_{O_1}(k) \land I_{O_{2h}}(l_1l_2) \\
\leq I_{O_1}(k) \land [I_{O_2}(l_1) \land I_{O_2}(l_2)] \\
= [I_{O_1}(k) \land I_{O_2}(l_1)] \land [I_{O_1}(k) \land I_{O_2}(l_2)] \\
= I_{(O_1 \circ O_2)}(kl_1) \land I_{(O_1 \circ O_2)}(kl_2),
\]

\[
F_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) = F_{O_1}(k) \lor F_{O_{2h}}(l_1l_2) \\
\leq F_{O_1}(k) \lor [F_{O_2}(l_1) \lor F_{O_2}(l_2)] \\
= [F_{O_1}(k) \lor F_{O_2}(l_1)] \lor [F_{O_1}(k) \lor F_{O_2}(l_2)] \\
= F_{(O_1 \circ O_2)}(kl_1) \lor F_{(O_1 \circ O_2)}(kl_2),
\]

for $kl_1, kl_2 \in P_1 \circ P_2$.

**Case 2:** For $k \in P_2$, $l_1l_2 \in P_{1h}$

\[
T_{(O_{1h} \circ O_{2h})}((l_1k)(l_2k)) = T_{O_2}(k) \land T_{O_{1h}}(l_1l_2) \\
\leq T_{O_2}(k) \land [T_{O_1}(l_1) \land T_{O_1}(l_2)] \\
= [T_{O_2}(k) \land T_{O_1}(l_1)] \land [T_{O_2}(k) \land T_{O_1}(l_2)] \\
= T_{(O_1 \circ O_2)}(l_1k) \land T_{(O_1 \circ O_2)}(l_2k),
\]
\[
I_{(O_{1h} \circ O_{2h})}((l_1k)(l_2k)) = I_{O_2}(k) \land I_{O_{1h}}(l_1l_2) \\
\leq I_{O_2}(k) \land [I_{O_1}(l_1) \land I_{O_1}(l_2)] \\
= [I_{O_2}(k) \land I_{O_1}(l_1)] \land [I_{O_2}(k) \land I_{O_1}(l_2)] \\
= I_{(O_1 \circ O_2)}(l_1k) \land I_{(O_1 \circ O_2)}(l_2k),
\]
\[
F_{(O_{1h} \circ O_{2h})}((l_1k)(l_2k)) = F_{O_2}(k) \lor F_{O_{1h}}(l_1l_2) \\
\leq F_{O_2}(k) \lor [F_{O_1}(l_1) \lor F_{O_1}(l_2)] \\
= [F_{O_2}(k) \lor F_{O_1}(l_1)] \lor [F_{O_2}(k) \lor F_{O_1}(l_2)] \\
= F_{(O_1 \circ O_2)}(l_1k) \lor F_{(O_1 \circ O_2)}(l_2k),
\]
for \(l_1k, l_2k \in P_1 \circ P_2.

**Case 3:** For \(k_1k_2 \in P_{1h}, l_1, l_2 \in P_2\) such that \(l_1 \neq l_2\)
\[
T_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = T_{O_{1h}}(k_1k_2) \land T_{O_2}(l_1l_2) \\
\leq [T_{O_1}(k_1) \land T_{O_1}(k_2)] \land T_{O_2}(l_1l_2) \\
= [T_{O_1}(k_1) \land T_{O_2}(l_1l_2)] \land [T_{O_1}(k_2) \land T_{O_2}(l_1l_2)] \\
= T_{(O_1 \circ O_2)}(k_1l_1) \land T_{(O_1 \circ O_2)}(k_2l_2),
\]
\[
I_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = I_{O_{1h}}(k_1k_2) \land I_{O_2}(l_1l_2) \\
\leq [I_{O_1}(k_1) \land I_{O_1}(k_2)] \land [I_{O_2}(l_1) \land I_{O_2}(l_2)] \\
= [I_{O_1}(k_1) \land I_{O_2}(l_1l_2)] \land [I_{O_1}(k_2) \land I_{O_2}(l_1l_2)] \\
= I_{(O_1 \circ O_2)}(k_1l_1) \land I_{(O_1 \circ O_2)}(k_2l_2),
\]
\[
F_{(O_{1h} \circ O_{2h})}((k_1l_1)(k_2l_2)) = F_{O_{1h}}(k_1k_2) \lor F_{O_2}(l_1l_2) \\
\leq [F_{O_1}(k_1) \lor F_{O_1}(k_2)] \lor [F_{O_2}(l_1) \lor F_{O_2}(l_2)] \\
= [F_{O_1}(k_1) \lor F_{O_2}(l_1l_2)] \lor [F_{O_1}(k_2) \lor F_{O_2}(l_1l_2)] \\
= F_{(O_1 \circ O_2)}(k_1l_1) \lor F_{(O_1 \circ O_2)}(k_2l_2),
\]
for \(k_1l_1, k_2l_2 \in P_1 \circ P_2.

All cases holds for \(h = 1, 2, \ldots, r\). This completes the proof. \(\square\)

**Definition 2.28.** Let \(\hat{G}_{11} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})\) and \(\hat{G}_{12} = (O_2, O_{21}, O_{22}, \ldots, O_{2r})\) be INGSs of GSs \(\hat{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})\) and \(\hat{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})\), respectively. The union of \(\hat{G}_{11}\) and \(\hat{G}_{12}\), denoted by \(\hat{G}_{11} \cup \hat{G}_{12} = (O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r})\), is defined as:

\[
\begin{align*}
(i) \quad & \begin{cases}
T_{(O_1 \cup O_2)}(k) = (T_{O_1} \cup T_{O_2})(k) = T_{O_1}(k) \lor T_{O_2}(k) \\
I_{(O_1 \cup O_2)}(k) = (I_{O_1} \cup I_{O_2})(k) = I_{O_1}(k) \land I_{O_2}(k) \\
F_{(O_1 \cup O_2)}(k) = (F_{O_1} \cup F_{O_2})(k) = F_{O_1}(k) \lor F_{O_2}(k)
\end{cases} \\
& \text{for all } k \in P_1 \cup P_2, \\
(ii) \quad & \begin{cases}
T_{(O_{1h} \cup O_{2h})}(kl) = (T_{O_{1h}} \cup T_{O_{2h}})(kl) = T_{O_{1h}}(kl) \lor T_{O_{2h}}(kl) \\
I_{(O_{1h} \cup O_{2h})}(kl) = (I_{O_{1h}} \cup I_{O_{2h}})(kl) = I_{O_{1h}}(kl) \land I_{O_{2h}}(kl) \\
F_{(O_{1h} \cup O_{2h})}(kl) = (F_{O_{1h}} \cup F_{O_{2h}})(kl) = F_{O_{1h}}(kl) \lor F_{O_{2h}}(kl)
\end{cases} \\
& \text{for all } kl \in P_{1h} \cup P_{2h}.
\end{align*}
\]
Example 2.29. The union of two INGSs $\tilde{G}_{i1}$ and $\tilde{G}_{i2}$ shown in Fig. 2 is defined as

$$\tilde{G}_{i1} \cup \tilde{G}_{i2} = \{ O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22} \}$$

and is represented in Fig. 8.

![Figure 8. $\tilde{G}_{i1} \cup \tilde{G}_{i2}$](image)

Theorem 2.30. The union $\tilde{G}_{i1} \cup \tilde{G}_{i2} = \{ O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r} \}$

of two INGSs of the GSs $\tilde{G}_1$ and $\tilde{G}_2$ is an INGS of $\tilde{G}_{i1} \cup \tilde{G}_{i2}$.

Proof. Let $k_1, k_2 \in P_{1h} \cup P_{2h}$. There are two cases:

**Case 1:** For $k_1, k_2 \in P_1$, by definition 2.28, $T_{O_1}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) = 1$. Thus,

$$T_{(O_{1h} \cup O_{2h})}(k_1k_2) = T_{O_{1h}}(k_1k_2) \lor T_{O_{2h}}(k_1k_2)$$
$$= T_{O_{1h}}(k_1k_2) \lor 0$$
$$\leq [T_{O_1}(k_1) \land T_{O_1}(k_2)] \lor 0$$
$$= [T_{O_1}(k_1) \lor 0] \land [T_{O_1}(k_2) \lor 0]$$
$$= [T_{O_1}(k_1) \lor T_{O_2}(k_1)] \land [T_{O_1}(k_2) \lor T_{O_2}(k_2)]$$
$$= T_{(O_1 \lor O_2)}(k_1) \land T_{(O_1 \lor O_2)}(k_2),$$

$$I_{(O_{1h} \cup O_{2h})}(k_1k_2) = I_{O_{1h}}(k_1k_2) \lor I_{O_{2h}}(k_1k_2)$$
$$= I_{O_{1h}}(k_1k_2) \lor 0$$
$$\leq [I_{O_1}(k_1) \land I_{O_1}(k_2)] \lor 0$$
$$= [I_{O_1}(k_1) \lor 0] \land [I_{O_1}(k_2) \lor 0]$$
$$= [I_{O_1}(k_1) \lor I_{O_2}(k_1)] \land [I_{O_1}(k_2) \lor I_{O_2}(k_2)]$$
$$= I_{(O_1 \lor O_2)}(k_1) \land I_{(O_1 \lor O_2)}(k_2),$$
\[ F_{(O_{1h} \cup O_{2h})}(k_1k_2) = F_{O_{1h}}(k_1k_2) \land F_{O_{2h}}(k_1k_2) \\
= F_{O_{1h}}(k_1k_2) \land 1 \\
\leq [F_{O_{1h}}(k_1) \lor F_{O_{1h}}(k_2)] \land 1 \\
= [F_{O_{1h}}(k_1) \land 1] \lor [F_{O_{1h}}(k_2) \land 1] \\
= [F_{O_{1h}}(k_1) \land F_{O_{2h}}(k_1)] \lor [F_{O_{1h}}(k_2) \land F_{O_{2h}}(k_2)] \\
= F_{(O_{1h} \cup O_{2h})}(k_1) \lor F_{(O_{1h} \cup O_{2h})}(k_2), \]

for \( k_1, k_2 \in P_1 \cup P_2 \).

**Case 2:** For \( k_1, k_2 \in P_2 \), by definition 2.28, \( T_{O_{1h}}(k_1) = T_{O_{1h}}(k_2) = T_{O_{2h}}(k_1k_2) = 0 \), \( I_{O_{1h}}(k_1) = I_{O_{1h}}(k_2) = I_{O_{2h}}(k_1k_2) = 0 \), \( F_{O_{1h}}(k_1) = F_{O_{2h}}(q_2) = F_{O_{2h}}(k_1k_2) = 1 \), so

\[ T_{(O_{1h} \cup O_{2h})}(k_1k_2) = T_{O_{1h}}(k_1k_2) \lor T_{O_{2h}}(k_1k_2) \\
= T_{O_{2h}}(k_1k_2) \lor 0 \\
\leq [T_{O_{2h}}(k_1) \land T_{O_{2h}}(k_2)] \lor 0 \\
= [T_{O_{2h}}(k_1) \lor 0] \land [T_{O_{2h}}(k_2) \lor 0] \\
= [T_{O_{1h}}(k_1) \lor T_{O_{2h}}(k_2)] \land [T_{O_{1h}}(k_2) \lor T_{O_{2h}}(k_2)] \\
= T_{(O_{1h} \cup O_{2h})}(k_1) \land T_{(O_{1h} \cup O_{2h})}(k_2), \]

\[ I_{(O_{1h} \cup O_{2h})}(q_1k_2) = I_{O_{1h}}(k_1k_2) \lor I_{O_{2h}}(k_1k_2) \\
= I_{O_{2h}}(k_1k_2) \lor 0 \\
\leq [I_{O_{2h}}(k_1) \land I_{O_{2h}}(k_2)] \lor 0 \\
= [I_{O_{2h}}(k_1) \lor 0] \land [I_{O_{2h}}(k_2) \lor 0] \\
= [I_{O_{1h}}(k_1) \lor I_{O_{2h}}(k_1)] \land [I_{O_{1h}}(k_2) \lor I_{O_{2h}}(k_2)] \\
= I_{(O_{1h} \cup O_{2h})}(k_1) \land I_{(O_{1h} \cup O_{2h})}(k_2), \]

\[ F_{(O_{1h} \cup O_{2h})}(k_1k_2) = F_{O_{1h}}(k_1k_2) \land F_{O_{2h}}(k_1k_2) \\
= F_{O_{2h}}(k_1k_2) \land 1 \\
\leq [F_{O_{2h}}(k_1) \lor F_{O_{2h}}(k_2)] \land 1 \\
= [F_{O_{2h}}(k_1) \land 1] \lor [F_{O_{2h}}(k_2) \land 1] \\
= [F_{O_{1h}}(k_1) \land F_{O_{2h}}(k_1)] \lor [F_{O_{1h}}(k_2) \land F_{O_{2h}}(k_2)] \\
= F_{(O_{1h} \cup O_{2h})}(k_1) \lor F_{(O_{1h} \cup O_{2h})}(k_2), \]

for \( k_1, k_2 \in P_1 \cup P_2 \).

Both cases hold \( \forall h \in \{1, 2, \ldots, r\} \). This completes the proof. \( \Box \)

**Theorem 2.31.** Let \( \tilde{G} = (P_1 \cup P_2, P_{11} \cup P_{21}, P_{12} \cup P_{22}, \ldots, P_{1r} \cup P_{2r}) \) be the union of two GSs \( \tilde{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r}) \) and \( \tilde{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r}) \). Then every INGS \( \tilde{G}_1 = (O, O_1, O_2, \ldots, O_r) \) of \( \tilde{G} \) is union of the two INGSs \( \tilde{G}_{11} \) and \( \tilde{G}_{12} \) of GSs \( \tilde{G}_1 \) and \( \tilde{G}_2 \), respectively.
Proof. Firstly, we define $O_1, O_2, O_{1h}$ and $O_{2h}$ for $h \in \{1, 2, \ldots, r\}$ as:

- $T_{O_1}(k) = T_O(k), I_{O_1}(k) = I_O(k), F_{O_1}(k) = F_O(k)$, if $k \in P_1$,
- $T_{O_2}(k) = T_O(k), I_{O_2}(k) = I_O(k), F_{O_2}(k) = F_O(k)$, if $k \in P_2$,
- $T_{O_{1h}}(k_1k_2) = T_O(k_1k_2), I_{O_{1h}}(k_1k_2) = I_O(k_1k_2), F_{O_{1h}}(k_1k_2) = F_O(k_1k_2)$, if $k_1k_2 \in P_{1h}$,
- $T_{O_{2h}}(k_1k_2) = T_O(k_1k_2), I_{O_{2h}}(k_1k_2) = I_O(k_1k_2), F_{O_{2h}}(k_1k_2) = F_O(k_1k_2)$, if $k_1k_2 \in P_{2h}$.

Then $O = O_1 \cup O_2$ and $O_h = O_{1h} \cup O_{2h}$, $h \in \{1, 2, \ldots, r\}$.

Now for $k_1k_2 \in P_{1h}$, $l = 1, 2, h = 1, 2, \ldots, r$,

- $T_{O_{1h}}(k_1k_2) = T_{O_h}(k_1k_2) \leq T_O(k_1) \cap T_O(k_2) = T_O(k_1) \cap T_{O_h}(k_2)$,
- $I_{O_{1h}}(k_1k_2) = I_{O_h}(k_1k_2) \leq I_O(k_1) \cap I_O(k_2) = I_O(k_1) \cap I_{O_h}(k_2)$,
- $F_{O_{1h}}(k_1k_2) = F_{O_h}(k_1k_2) \leq F_O(k_1) \lor F_O(k_2) = F_O(k_1) \lor F_{O_h}(k_2)$, i.e.,

Thus $\tilde{G}_l = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ is an INGS of $\tilde{G}_l$, $l = 1, 2$.

Thus $\tilde{G}_l = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$, an INGS of $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$, is the union of the two INGSs $\tilde{G}_{i1}$ and $\tilde{G}_{i2}$.

\[\tilde{G}_{i1} + \tilde{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),\]

is defined as:

(i) \[
\begin{align*}
T_{(O_1+O_2)}(k) &= T_{(O_1\cup O_2)}(k) \\
I_{(O_1+O_2)}(k) &= I_{(O_1\cup O_2)}(k) \\
F_{(O_1+O_2)}(k) &= F_{(O_1\cup O_2)}(k)
\end{align*}
\]
for all $k \in P_1 \cup P_2$,

(ii) \[
\begin{align*}
T_{(O_{1h}+O_{2h})}(kl) &= T_{(O_{1h}\cup O_{2h})}(kl) \\
I_{(O_{1h}+O_{2h})}(kl) &= I_{(O_{1h}\cup O_{2h})}(kl) \\
F_{(O_{1h}+O_{2h})}(kl) &= F_{(O_{1h}\cup O_{2h})}(kl)
\end{align*}
\]
for all $kl \in P_{1h} \cup P_{2h}$,

(iii) \[
\begin{align*}
T_{(O_{1h}+O_{2h})}(kl) &= \left(T_{O_{1h}} + T_{O_{2h}}\right)(kl) = T_{O_h}(kl) \land T_{O_l}(l) \\
I_{(O_{1h}+O_{2h})}(kl) &= \left(I_{O_{1h}} + I_{O_{2h}}\right)(kl) = I_{O_h}(kl) \land I_{O_l}(l) \\
F_{(O_{1h}+O_{2h})}(kl) &= \left(F_{O_{1h}} + F_{O_{2h}}\right)(kl) = F_{O_h}(kl) \lor F_{O_l}(l)
\end{align*}
\]
for all $k \in P_{1h}, l \in P_2$.

Example 2.33. The join of two INGSs $\tilde{G}_{i1}$ and $\tilde{G}_{i2}$ shown in Fig. 2 is defined as $\tilde{G}_{i1} + \tilde{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 9.

Theorem 2.34. The join $\tilde{G}_{i1} + \tilde{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r})$ of two INGSs of GSs $\tilde{G}_1$ and $\tilde{G}_2$ is INGS of $\tilde{G}_1 + \tilde{G}_2$.

Proof. Let $k_1k_2 \in P_{1h} + P_{2h}$. There are three cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.32, $T_{O_1}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) = 0$. 

\[\tilde{G}_{i1} + \tilde{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),\]

for all $k \in P_1 \cup P_2$.

Example 2.33. The join of two INGSs $\tilde{G}_{i1}$ and $\tilde{G}_{i2}$ shown in Fig. 2 is defined as $\tilde{G}_{i1} + \tilde{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 9.

Theorem 2.34. The join $\tilde{G}_{i1} + \tilde{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r})$ of two INGSs of GSs $\tilde{G}_1$ and $\tilde{G}_2$ is INGS of $\tilde{G}_1 + \tilde{G}_2$.

Proof. Let $k_1k_2 \in P_{1h} + P_{2h}$. There are three cases:

Case 1: For $k_1, k_2 \in P_1$, by definition 2.32, $T_{O_1}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1k_2) = 0$, $I_{O_1}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1k_2) = 0$, $F_{O_1}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1k_2) = 0$. 

\[\tilde{G}_{i1} + \tilde{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),\]

for all $k \in P_1 \cup P_2$. 

Example 2.33. The join of two INGSs $\tilde{G}_{i1}$ and $\tilde{G}_{i2}$ shown in Fig. 2 is defined as $\tilde{G}_{i1} + \tilde{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 9.
1, so,

\[ T_{O_{1h} + O_{2h}}(k_1k_2) = T_{O_{1h}}(k_1k_2) \lor T_{O_{2h}}(k_1k_2) \]
\[ = T_{O_{1h}}(k_1k_2) \lor \emptyset \]
\[ \leq [T_{O_1}(k_1) \land T_{O_2}(k_2)] \lor \emptyset \]
\[ = [T_{O_1}(k_1) \lor \emptyset] \land [T_{O_2}(k_2) \lor \emptyset] \]
\[ = [T_{O_1}(k_1) \lor T_{O_2}(k_1)] \land [T_{O_1}(k_2) \lor T_{O_2}(k_2)] \]
\[ = T_{(O_1 + O_2)(k_1)} \land T_{(O_1 + O_2)(k_2)}, \]

\[ I_{O_{1h} + O_{2h}}(k_1k_2) = I_{O_{1h}}(k_1k_2) \lor I_{O_{2h}}(k_1k_2) \]
\[ = I_{O_{1h}}(k_1k_2) \lor \emptyset \]
\[ \leq [I_{O_1}(k_1) \land I_{O_2}(k_2)] \lor \emptyset \]
\[ = [I_{O_1}(k_1) \lor \emptyset] \land [I_{O_2}(k_2) \lor \emptyset] \]
\[ = [I_{O_1}(k_1) \lor I_{O_2}(k_1)] \land [I_{O_1}(k_2) \lor I_{O_2}(k_2)] \]
\[ = I_{(O_1 + O_2)(k_1)} \land I_{(O_1 + O_2)(k_2)}, \]

\[ F_{O_{1h} + O_{2h}}(k_1k_2) = F_{O_{1h}}(k_1k_2) \land F_{O_{2h}}(k_1k_2) \]
\[ = F_{O_{1h}}(k_1k_2) \land 1 \]
\[ \leq [F_{O_1}(k_1) \lor F_{O_2}(k_2)] \land 1 \]
\[ = [F_{O_1}(k_1) \land 1] \lor [F_{O_2}(k_2) \land 1] \]
\[ = [F_{O_1}(k_1) \land F_{O_2}(k_1)] \lor [F_{O_1}(k_2) \land F_{O_2}(k_2)] \]
\[ = F_{(O_1 + O_2)(k_1)} \lor F_{(O_1 + O_2)(k_2)}, \]

for \( k_1, k_2 \in P_1 + P_2 \).
Case 2: For $k_1, k_2 \in P_2$, by definition 2.32, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_1}(k_1 k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_1}(k_1 k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(k_2) = F_{O_1}(k_1 k_2) = 1$, so

\[
T_{(O_{1h} + O_{2h})}(k_1 k_2) = T_{O_{1h}}(k_1 k_2) \lor T_{O_{2h}}(k_1 k_2) \\
= T_{O_{2h}}(k_1 k_2) \lor 0 \\
\leq [T_{O_{2h}}(k_1) \land T_{O_{2h}}(k_2)] \lor 0 \\
= [T_{O_{2h}}(k_1) \lor 0] \land [T_{O_{2h}}(k_2) \lor 0] \\
= [T_{O_1}(k_1) \lor T_{O_2}(k_1)] \land [T_{O_1}(k_2) \lor T_{O_2}(k_2)] \\
= T_{(O_1 + O_2)}(k_1) \land T_{(O_1 + O_2)}(k_2),
\]

\[
I_{(O_{1h} + O_{2h})}(k_1 k_2) = I_{O_{1h}}(k_1 k_2) \lor I_{O_{2h}}(k_1 k_2) \\
= I_{O_{2h}}(k_1 k_2) \lor 0 \\
\leq [I_{O_{2h}}(k_1) \land I_{O_{2h}}(k_2)] \lor 0 \\
= [I_{O_{2h}}(k_1) \lor 0] \land [I_{O_{2h}}(k_2) \lor 0] \\
= [I_{O_1}(k_1) \lor I_{O_2}(k_1)] \land [I_{O_1}(k_2) \lor I_{O_2}(k_2)] \\
= I_{(O_1 + O_2)}(k_1) \land I_{(O_1 + O_2)}(k_2),
\]

\[
F_{(O_{1h} + O_{2h})}(k_1 k_2) = F_{O_{1h}}(k_1 k_2) \land F_{O_{2h}}(k_1 k_2) \\
= F_{O_{2h}}(k_1 k_2) \land 1 \\
\leq [F_{O_{2h}}(k_1) \lor F_{O_{2h}}(k_2)] \land 1 \\
= [F_{O_{2h}}(k_1) \land 1] \lor [F_{O_{2h}}(k_2) \land 1] \\
= [F_{O_1}(k_1) \lor F_{O_2}(k_1)] \lor [F_{O_1}(k_2) \lor F_{O_2}(k_2)] \\
= F_{(O_1 + O_2)}(k_1) \lor F_{(O_1 + O_2)}(k_2),
\]

for $q_1, q_2 \in P_1 + P_2$.

Case 3: For $k_1 \in P_1$, $k_2 \in P_2$, by definition 2.32, $T_{O_2}(k_1) = T_{O_2}(k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = 0$, $F_{O_1}(k_1) = F_{O_2}(k_1) = 1$, so

\[
T_{(O_{1h} + O_{2h})}(k_1 k_2) = T_{O_1}(q_1) \land T_{O_2}(k_2) \\
= [T_{O_1}(k_1) \lor 0] \land [T_{O_2}(k_2) \lor 0] \\
= [T_{O_1}(k_1) \lor T_{O_2}(k_1)] \land [T_{O_2}(k_2) \lor T_{O_1}(k_2)] \\
= T_{(O_1 + O_2)}(k_1) \land T_{(O_1 + O_2)}(k_2),
\]

\[
I_{(O_{1h} + O_{2h})}(k_1 k_2) = I_{O_1}(k_1) \land I_{O_2}(k_2) \\
= [I_{O_1}(k_1) \lor 0] \land [I_{O_2}(k_2) \lor 0] \\
= [I_{O_1}(k_1) \lor I_{O_2}(k_1)] \land [I_{O_2}(k_2) \lor I_{O_1}(k_2)] \\
= I_{(O_1 + O_2)}(k_1) \land I_{(O_1 + O_2)}(k_2),
\]

20
According to IMF data, 1.75 billion people are living in poverty, their living is estimated to be less than two dollars a day. Poverty changes by region, for example in Europe it is 3%, and in the Sub-Saharan Africa it is up to 65%. We rank the countries of the World as poor or rich, using their GDP per capita as scale. Poor countries are trying to catch up with rich or developed countries. But this ratio is very small, that’s why trade of poor countries among themselves is very important. There are different types of trade among poor countries, for example: agricultural or food items, raw minerals, medicines, textile materials, industrials goods etc. Using INGS, we can estimate between any two poor countries which trade is comparatively stronger than others. Moreover, we can decide (judge) which country has large number of resources trying to catch up with rich or developed countries. But this ratio is very small, that’s why trade of poor countries among themselves is very important. According to the definitions 2.28 and 2.32, $O = O_1 \cup O_2 = O_1 + O_2$ and $O_h = O_{1h} \cup O_{2h} = O_{1h} + O_{2h}$, $\forall k_1 k_2 \in P_{1h} \cup P_{2h}$.

When $k_1 k_2 \in P_{1h} + P_{2h}$, $P_{1h} \cup P_{2h}$, i.e., $k_1 \in P_1$ and $k_2 \in P_2$, $T_{O_1}(k_1 k_2) = T_O(k_1) \land T_O(k_2) = T_O(k_1) \land T_O(k_2)$, $I_{O_1}(k_1 k_2) = I_O(k_1) \land I_O(k_2) = I_O(k_1) \land I_O(k_2)$, $F_{O_1}(k_1 k_2) = F_O(k_1) \lor F_O(k_2) = F_O(k_1) \lor F_O(k_2)$, when $k_1 \in P_2$, $k_2 \in P_1$, we get similar calculations. It’s true for $h = 1, 2, \ldots, r$. This completes the proof. □

3. Application

According to IMF data, 1.75 billion people are living in poverty, their living is estimated to be less than two dollars a day. Poverty changes by region, for example in Europe it is 3%, and in the Sub-Saharan Africa it is up to 65%. We rank the countries of the World as poor or rich, using their GDP per capita as scale. Poor countries are trying to catch up with rich or developed countries. But this ratio is very small, that’s why trade of poor countries among themselves is very important. There are different types of trade among poor countries, for example: agricultural or food items, raw minerals, medicines, textile materials, industrials goods etc. Using INGS, we can estimate between any two poor countries which trade is comparatively stronger than others. Moreover, we can decide (judge) which country has large number of resources for particular type of goods and better circumstances for its trade. We can figure out, for which trade, an external investor can invest his money in these poor countries. Further, it will be easy to judge that in which field these poor countries are trying to
Table 1. IN set $O$ of nine poor countries on globe

<table>
<thead>
<tr>
<th>Poor Country</th>
<th>$T$</th>
<th>$I$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Congo</td>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>Liberia</td>
<td>0.4</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>Burundi</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Tanzania</td>
<td>0.5</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>Uganda</td>
<td>0.4</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>Sierra Leone</td>
<td>0.5</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Zimbabwe</td>
<td>0.3</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Kenya</td>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Zambia</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
</tbody>
</table>

be better, and can be helped. It will also help in deciding that in which trade they are weak, and should be facilitated, so that they can be independent and improve their living standards.

We consider a set of nine poor countries in the World:

$$P = \{\text{Congo, Liberia, Burundi, Tanzania, Uganda, Sierra Leone, Zimbabwe, Kenya, Zambia}\}.$$ 

Let $O$ be the IN set on $P$, as defined in Table 1. In Table 1, symbol $T$ demonstrates the positive aspects of that poor country, symbol $I$ indicates its negative aspects, whereas $F$ denotes the percentage of ambiguity of its problems for the World. Let we use following alphabets for country names:

CO = Congo, L = Liberia, B = Burundi, T = Tanzania, U = Uganda, SL = Sierra Leone, ZI = Zimbabwe, K = Kenya, ZA = Zambia. For every pair of poor countries in set $P$, different trades with their $T$, $I$ and $F$ values are demonstrated in Tables 2–8, where $T$, $F$ and $I$ indicates the percentage of occurrence, non-occurrence and uncertainty, respectively of a particular trade between those two poor countries.
### Table 3. IN set of different types of trade between Liberia and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>(L, B)</th>
<th>(L, T)</th>
<th>(L, U)</th>
<th>(L, SL)</th>
<th>(L, ZI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.4, 0.2, 0.2)</td>
<td>(0.4, 0.3, 0.2)</td>
<td>(0.3, 0.3, 0.4)</td>
<td>(0.3, 0.3, 0.3)</td>
<td>(0.2, 0.3, 0.3)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.2, 0.2, 0.4)</td>
<td>(0.1, 0.4, 0.3)</td>
<td>(0.3, 0.3, 0.3)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.1, 0.3, 0.3)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.1, 0.1, 0.4)</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.1, 0.1, 0.4)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.2, 0.2, 0.3)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.3, 0.1, 0.3)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.2, 0.1, 0.4)</td>
<td>(0.3, 0.2, 0.3)</td>
<td>(0.2, 0.1, 0.3)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.1, 0.3, 0.4)</td>
<td>(0.1, 0.3, 0.3)</td>
<td>(0.2, 0.1, 0.3)</td>
<td>(0.1, 0.2, 0.3)</td>
<td>(0.2, 0.2, 0.3)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.2, 0.1, 0.4)</td>
<td>(0.2, 0.1, 0.3)</td>
<td>(0.3, 0.1, 0.3)</td>
<td>(0.4, 0.1, 0.1)</td>
<td>(0.3, 0.1, 0.1)</td>
</tr>
</tbody>
</table>

### Table 4. IN set of different types of trade between Burundi and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>(B, T)</th>
<th>(B, U)</th>
<th>(B, SL)</th>
<th>(B, ZI)</th>
<th>(B, K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.4, 0.1, 0.2)</td>
<td>(0.3, 0.3, 0.1)</td>
<td>(0.3, 0.3, 0.2)</td>
<td>(0.3, 0.2, 0.2)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.1, 0.2, 0.3)</td>
<td>(0.2, 0.1, 0.3)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.3, 0.4, 0.3)</td>
<td>(0.3, 0.3, 0.1)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.1, 0.1, 0.4)</td>
<td>(0.2, 0.3, 0.4)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.2, 0.2, 0.5)</td>
<td>(0.1, 0.3, 0.4)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.2, 0.1, 0.3)</td>
<td>(0.4, 0.2, 0.3)</td>
<td>(0.4, 0.2, 0.4)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.4, 0.2, 0.2)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.4, 0.1, 0.2)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.3, 0.2, 0.3)</td>
<td>(0.3, 0.4, 0.3)</td>
<td>(0.1, 0.4, 0.2)</td>
<td>(0.2, 0.4, 0.2)</td>
<td>(0.2, 0.3, 0.4)</td>
</tr>
</tbody>
</table>

### Table 5. IN set of different types of trade between Tanzania and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>(T, U)</th>
<th>(T, SL)</th>
<th>(T, ZI)</th>
<th>(T, K)</th>
<th>(T, ZA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.4, 0.2, 0.1)</td>
<td>(0.5, 0.1, 0.1)</td>
<td>(0.3, 0.1, 0.2)</td>
<td>(0.4, 0.3, 0.2)</td>
<td>(0.3, 0.2, 0.2)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.2, 0.3, 0.4)</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.4, 0.1, 0.4)</td>
<td>(0.3, 0.4, 0.4)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.1, 0.3, 0.3)</td>
<td>(0.4, 0.1, 0.3)</td>
<td>(0.3, 0.4, 0.2)</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.1, 0.3, 0.3)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.3, 0.3, 0.4)</td>
<td>(0.4, 0.3, 0.3)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.4, 0.2, 0.3)</td>
<td>(0.3, 0.2, 0.3)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.2, 0.4, 0.4)</td>
<td>(0.1, 0.3, 0.4)</td>
<td>(0.2, 0.3, 0.2)</td>
<td>(0.4, 0.1, 0.2)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.3, 0.4, 0.3)</td>
<td>(0.4, 0.3, 0.4)</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.2, 0.2, 0.2)</td>
<td>(0.4, 0.3, 0.3)</td>
</tr>
</tbody>
</table>

### Table 6. IN set of different types of trade between Sierra Leone and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>(SL, ZI)</th>
<th>(SL, K)</th>
<th>(SL, ZA)</th>
<th>(SL, CO)</th>
<th>(L, K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.3, 0.3, 0.2)</td>
<td>(0.4, 0.2, 0.1)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.5, 0.1, 0.1)</td>
<td>(0.4, 0.1, 0.2)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.2, 0.3, 0.4)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.2, 0.4, 0.4)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.2, 0.3, 0.3)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.1, 0.3, 0.4)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.3, 0.4, 0.2)</td>
<td>(0.5, 0.2, 0.1)</td>
<td>(0.3, 0.3, 0.3)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.5, 0.2, 0.1)</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.3, 0.3, 0.3)</td>
<td>(0.4, 0.1, 0.2)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.2, 0.4, 0.2)</td>
<td>(0.3, 0.2, 0.3)</td>
<td>(0.2, 0.2, 0.4)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.3, 0.3, 0.2)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.1, 0.2, 0.4)</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.4, 0.1, 0.2)</td>
<td>(0.3, 0.2, 0.3)</td>
</tr>
</tbody>
</table>
Many relations can be defined on the set \( P \), we define following relations on set \( P \) as:

\[ P_1 = \text{Food items}, \quad P_2 = \text{Chemicals}, \quad P_3 = \text{Oil}, \quad P_4 = \text{Raw minerals}, \quad P_5 = \text{Textile products}, \quad P_6 = \text{Gold and diamonds} \]

such that \((P, P_1, P_2, P_3, P_4, P_5, P_6)\) is a GS. Any element of a relation demonstrates a particular trade between those two poor countries. As \((P, P_1, P_2, P_3, P_4, P_5, P_6)\) is GS, that’s why any element can appear in only one relation. Therefore, any element will be considered in that relation, whose value of \( T \) is high, and values of \( I, F \) are comparatively low, using data of above tables.

Write down \( T, I \) and \( F \) values of the elements in relations according to above data, such that \( O_1, O_2, O_3, O_4, O_5, O_6 \) are IN sets on relations \( P_1, P_2, P_3, P_4, P_5, P_6 \), respectively.

Let \( P_1 = \{(\text{Burundi, Congo}), (\text{Sierra Leone, Congo}), (\text{Burundi, Zambia})\} \), \( P_2 = \{(\text{Kenya, Congo})\} \), \( P_3 = \{(\text{Congo, Zambia}), (\text{Congo, Tanzania}), (\text{Zimbabwe, Congo})\} \), \( P_4 = \{(\text{Congo, Uganda}), (\text{Sierra Leone, Kenya}), (\text{Zambia, Kenya})\} \), \( P_5 = \{(\text{Burundi, Zimbabwe}), (\text{Tanzania, Burundi})\} \), \( P_6 = \{(\text{Sierra Leone, Liberia}), (\text{Uganda, Sierra Leone}), (\text{Zimbabwe, Sierra Leone})\} \).

Let \( O_1 = \{(B, CO), 0.4, 0.2, 0.1\}, (SL, CO), 0.5, 0.1, 0.1\}, (B, ZA), 0.4, 0.2, 0.1\} \), \( O_2 = \{(K, CO), 0.5, 0.1, 0.1\}, (CO, ZA), 0.4, 0.1, 0.1\}, (CO, T), 0.5, 0.1, 0.2\}, \( (ZI, CO), 0.3, 0.1, 0.1\} \), \( O_4 = \{(CO, U), 0.4, 0.1, 0.2\}, (SL, K), 0.5, 0.2, 0.1\}, (ZA, K), 0.4, 0.1, 0.1\} \), \( O_5 = \{(B, ZT), 0.3, 0.2, 0.1\}, (T, B), 0.3, 0.1, 0.1\} \), \( O_6 = \{(SL, L), 0.4, 0.1, 0.1\}, (U, SL), 0.4, 0.2, 0.1\}, (ZI, SL), 0.3, 0.1, 0.1\} \). Obviously, \((O, O_1, O_2, O_3, O_4, O_5, O_6)\) is an INGS as shown in Fig. 10.

### Table 7. IN set of different types of trade between Zimbabwe and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>((ZI, K))</th>
<th>((ZI, ZA))</th>
<th>((ZI, U))</th>
<th>((ZI, CO))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.2, 0.1, 0.1)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.3, 0.3, 0.2)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.2, 0.1, 0.2)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.1, 0.3, 0.3)</td>
<td>(0.1, 0.4, 0.4)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.3, 0.1, 0.1)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.3, 0.1, 0.2)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.3, 0.2, 0.3)</td>
<td>(0.2, 0.3, 0.1)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.2, 0.2, 0.2)</td>
<td>(0.2, 0.4, 0.3)</td>
<td>(0.2, 0.3, 0.3)</td>
<td>(0.2, 0.3, 0.1)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.3, 0.3, 0.1)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.3, 0.2, 0.1)</td>
</tr>
</tbody>
</table>

### Table 8. IN set of different types of trade between Zambia and other poor countries in \( P \)

<table>
<thead>
<tr>
<th>Type of trade</th>
<th>((ZA, CO))</th>
<th>((ZA, L))</th>
<th>((ZA, B))</th>
<th>((ZA, K))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food items</td>
<td>(0.3, 0.1, 0.2)</td>
<td>(0.3, 0.1, 0.2)</td>
<td>(0.4, 0.2, 0.1)</td>
<td>(0.3, 0.1, 0.3)</td>
</tr>
<tr>
<td>Chemicals</td>
<td>(0.2, 0.2, 0.2)</td>
<td>(0.2, 0.2, 0.1)</td>
<td>(0.3, 0.2, 0.2)</td>
<td>(0.3, 0.1, 0.1)</td>
</tr>
<tr>
<td>Oil</td>
<td>(0.4, 0.1, 0.1)</td>
<td>(0.2, 0.1, 0.1)</td>
<td>(0.3, 0.2, 0.1)</td>
<td>(0.3, 0.2, 0.2)</td>
</tr>
<tr>
<td>Raw minerals</td>
<td>(0.3, 0.1, 0.1)</td>
<td>(0.4, 0.1, 0.1)</td>
<td>(0.4, 0.2, 0.2)</td>
<td>(0.4, 0.1, 0.1)</td>
</tr>
<tr>
<td>Textile products</td>
<td>(0.2, 0.2, 0.2)</td>
<td>(0.2, 0.2, 0.3)</td>
<td>(0.2, 0.3, 0.2)</td>
<td>(0.3, 0.1, 0.2)</td>
</tr>
<tr>
<td>Gold and diamonds</td>
<td>(0.1, 0.2, 0.4)</td>
<td>(0.4, 0.3, 0.2)</td>
<td>(0.2, 0.3, 0.2)</td>
<td>(0.3, 0.2, 0.1)</td>
</tr>
</tbody>
</table>
Every edge of this INGS demonstrates the prominent trade between two poor countries, for example prominent trade between Congo and Zambia is Oil, its T, F and I values are 0.4, 0.1 and 0.1, respectively. According to these values, despite of poverty, circumstances of Congo and Zambia are 40% favorable for oil trade, 10% are unfavorable, and 10% are uncertain, that is, sometimes they may be favorable and sometimes unfavorable. We can observe that Congo is vertex with highest vertex degree for relation oil and Sierra Leone is vertex with highest vertex degree for relation gold and diamonds. That is, among these nine poor countries, Congo is most favorable for oil trade, and Sierra Leone is most favorable for trade of gold and diamonds. This INGS will be useful for those investors, who are interested to invest in these nine poor countries. For example an investor can invest in oil in Congo. And if someone wants to invest in gold and diamonds, this INGS will help him that Sierra Leone is most favorable.

A big advantage of this INGS is that United Nations, IMF, World Bank, and rich countries can be aware of the fact that in which fields of trade, these poor countries are trying to be better and can be helped to make their economic conditions better. Moreover, INGS of poor countries can be very beneficial for them, it may increase trade as well as foreign aid and economic help from the World, and can present their
better aspects before the World.

We now explain general procedure of this application by following algorithm.

Algorithm:
1. Input a vertex set \( P = \{C_1, C_2, \ldots, C_n\} \) and a IN set \( O \) defined on set \( P \).
2. Input IN set of trade of any vertex with all other vertices and calculate \( T(C_i,C_j) \leq \min(T(C_i), T(C_j)), \)
   \( F(C_i,C_j) \leq \max(F(C_i), F(C_j)), \) and \( I(C_i,C_j) \leq \min(I(C_i), I(C_j)). \)
3. Repeat Step 2 for each vertex in set \( P \).
4. Define relations \( P_1, P_2, \ldots, P_n \) on the set \( P \) such that \((P, P_1, P_2, \ldots, P_n)\) is a GS.
5. Consider an element of that relation, for which its value of \( T \) is comparatively high, and its values of \( F \) and \( I \) are low than other relations.
6. Write down all elements in relations with \( T, F \) and \( I \) values, corresponding relations \( O_1, O_2, \ldots, O_n \) are IN sets on \( P_1, P_2, P_3, \ldots, P_n \), respectively and \((O, O_1, O_2, \ldots, O_n)\) is an INGS.

4. Conclusions

Fuzzy graphical models are highly utilized in applications of computer science. Especially in database theory, cluster analysis, image capturing, data mining, control theory, neural networks, expert systems and artificial intelligence. In this research paper, we have introduced certain operations on intuitionistic neutrosophic graph structures. We have discussed a novel and worthwhile real-life application of intuitionistic neutrosophic graph structure in decision-making. We have intensions to generalize our concepts to (1) Applications of IN soft GSs in decision-making (2) Applications of IN rough fuzzy GSs in decision-making, (3) Applications of IN fuzzy soft GSs in decision-making, and (4) Applications of IN rough fuzzy soft GSs in decision-making.

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Neutrosophic quadruple algebraic hyperstructures

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Abstract. The objective of this paper is to develop neutrosophic quadruple algebraic hyperstructures. Specifically, we develop neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings and we present elementary properties which characterize them.

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1. Introduction

The concept of neutrosophic quadruple numbers was introduced by Florentin Smarandache [18]. It was shown in [18] how arithmetic operations of addition, subtraction, multiplication and scalar multiplication could be performed on the set of neutrosophic quadruple numbers. In [1], Akinleye et.al. introduced the notion of neutrosophic quadruple algebraic structures. Neutrosophic quadruple rings were studied and their basic properties were presented. In the present paper, two hyperoperations \( \hat{+} \) and \( \hat{\times} \) are defined on the neutrosophic set \( NQ \) of quadruple numbers to develop new algebraic hyperstructures which we call neutrosophic quadruple algebraic hyperstructures. Specifically, it is shown that \( (NQ, \hat{\times}) \) is a neutrosophic quadruple semihypergroup, \( (NQ, \hat{+}) \) is a neutrosophic quadruple canonical hypergroup and \( (NQ, \hat{+}, \hat{\times}) \) is a neutrosophic quadruple hyperring and their basic properties are presented.

Definition 1.1 ([18]). A neutrosophic quadruple number is a number of the form \((a, bT, cI, dF)\) where \(T, I, F\) have their usual neutrosophic logic meanings and \(a, b, c, d \in \mathbb{R} \) or \(\mathbb{C} \). The set \( NQ \) defined by

\[
NQ = \{ (a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C} \}
\]
is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number \((a, bT, cI, dF)\) representing any entity which may be a number, an idea, an object, etc, \(a\) is called the known part and \((bT, cI, dF)\) is called the unknown part.

**Definition 1.2.** Let \(a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in NQ\). We define the following:

\[
(1.2) \quad a + b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F).
\]

\[
(1.3) \quad a - b = (a_1 - b_1, (a_2 - b_2)T, (a_3 - b_3)I, (a_4 - b_4)F).
\]

**Definition 1.3.** Let \(a = (a_1, a_2, a_3, a_4) \in NQ\) and let \(\alpha\) be any scalar which may be real or complex, the scalar product \(\alpha.a\) is defined by

\[
(1.4) \quad \alpha.a = \alpha.(a_1, a_2, a_3, a_4) = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4).
\]

If \(\alpha = 0\), then we have \(0.a = (0, 0, 0, 0)\) and for any non-zero scalars \(m\) and \(n\) and \(b = (b_1, b_2, b_3, b_4)\), we have:

\[
\begin{align*}
(m + n)a &= ma + na, \\
ma + mb &= m.(a + b), \\
mn(a) &= m.(na), \\
-a &= (-a_1, -a_2, -a_3, -a_4).
\end{align*}
\]

**Definition 1.4 ([18]).** [Absorbance Law] Let \(X\) be a set endowed with a total order \(x < y\), named "\(x\) prevails by \(y\)" or "\(x\) less preferred than \(y\)". \(x \leq y\) is considered as "\(x\) prevailed by or equal to \(y\)" or "\(x\) less preferred or equal to \(y\)".

For any elements \(x, y \in X\), with \(x \leq y\), absorbance law is defined as

\[
(1.5) \quad x.y = y.x = \text{absorb}(x, y) = \max\{x, y\} = y
\]

which means that the bigger element absorbs the smaller element (the big fish eats the small fish). It is clear from (1.5) that

\[
(1.6) \quad x.x = x^2 = \text{absorb}(x, x) = \max\{x, x\} = x \quad \text{and}
\]

\[
(1.7) \quad x_1.x_2 \cdots x_n = \max\{x_1, x_2, \cdots, x_n\}.
\]

Analogously, if \(x > y\), we say that "\(x\) prevails to \(y\)" or "\(x\) is stronger than \(y\)" or "\(x\) is preferred to \(y\)". Also, if \(x \geq y\), we say that "\(x\) prevails or is equal to \(y\)" or "\(x\) is stronger than or equal to \(y\)" or "\(x\) is preferred or equal to \(y\).

**Definition 1.5.** Consider the set \(\{T, I, F\}\). Suppose in an optimistic way we consider the prevalence order \(T > I > F\). Then we have:

\[
\begin{align*}
(1.8) \quad TI &= IT = \max\{T, I\} = T, \\
(1.9) \quad TF &= FT = \max\{T, F\} = T, \\
(1.10) \quad IF &= FI = \max\{I, F\} = I, \\
(1.11) \quad TT &= T^2 = T, \\
(1.12) \quad II &= I^2 = I, \\
(1.13) \quad FF &= F^2 = F.
\end{align*}
\]
Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

\begin{align*}
TI & = IT = \max\{T, I\} = I, \\
TF & = FT = \max\{T, F\} = F, \\
IF & = FI = \max\{I, F\} = F, \\
TT & = T^2 = T, \\
II & = I^2 = I, \\
FF & = F^2 = F.
\end{align*}

(1.14) (1.15) (1.16) (1.17) (1.18) (1.19)

Except otherwise stated, we will consider only the prevalence order $T < I < F$ in this paper.

**Definition 1.6.** Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. Then

\[ a \cdot b = (a_1, a_2T, a_3I, a_4F)(b_1, b_2T, b_3I, b_4F) \]

\[ = (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \]

\[ (a_1b_4 + a_2b_4 + a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \]

(1.20)

**Theorem 1.7** ([1]). $(NQ, +)$ is an abelian group.

**Theorem 1.8** ([1]). $(NQ, \cdot)$ is a commutative monoid.

**Theorem 1.9** ([1]). $(NQ, \cdot)$ is not a group.

**Theorem 1.10** ([1]). $(NQ, +, \cdot)$ is a commutative ring.

**Definition 1.11.** Let $NQR$ be a neutrosophic quadruple ring and let $NQS$ be a nonempty subset of $NQR$. Then $NQS$ is called a neutrosophic quadruple subring of $NQR$, if $(NQS, +, \cdot)$ is itself a neutrosophic quadruple ring. For example, $NQR(n\mathbb{Z})$ is a neutrosophic quadruple subring of $NQR(\mathbb{Z})$ for $n = 1, 2, 3, \ldots$.

**Definition 1.12.** Let $NQJ$ be a nonempty subset of a neutrosophic quadruple ring $NQR$. $NQJ$ is called a neutrosophic quadruple ideal of $NQR$, if for all $x, y \in NQJ, r \in NQR$, the following conditions hold:

(i) $x - y \in NQJ$, 
(ii) $xr \in NQJ$ and $rx \in NQJ$.

**Definition 1.13** ([1]). Let $NQR$ and $NQS$ be two neutrosophic quadruple rings and let $\phi : NQR \to NQS$ be a mapping defined for all $x, y \in NQR$ as follows:

(i) $\phi(x + y) = \phi(x) + \phi(y)$, 
(ii) $\phi(xy) = \phi(x)\phi(y)$, 
(iii) $\phi(T) = T, \phi(I) = I$ and $\phi(F) = F$, 
(iv) $\phi((1, 0, 0, 0)) = (1, 0, 0, 0)$.

Then $\phi$ is called a neutrosophic quadruple homomorphism. Neutrosophic quadruple monomorphism, endomorphism, isomorphism, and other morphisms can be defined in the usual way.

**Definition 1.14.** Let $\phi : NQR \to NQS$ be a neutrosophic quadruple ring homomorphism.
Theorem 1.15 \([1]\). Let \(H\) be a non-empty set and let \(+\) be a hyperoperation on \(H\). The couple \((H, +)\) is called a canonical hypergroup if the following conditions hold:

(i) \(x + y = y + x\), for all \(x, y \in H\),
(ii) \(x + (y + z) = (x + y) + z\), for all \(x, y, z \in H\),
(iii) there exists a neutral element \(0 \in H\) such that \(x + 0 = \{x\} = 0 + x\), for all \(x \in H\),
(iv) for every \(x \in H\), there exists a unique element \(-x \in H\) such that \(0 \in x + (-x) \cap (-x) + x\),
(v) \(z \in x + y \) implies \(y \in -x + z\) and \(x \in z - y\), for all \(x, y, z \in H\).

A nonempty subset \(A\) of \(H\) is called a subcanonical hypergroup, if \(A\) is a canonical hypergroup under the same hyperaddition as that of \(H\) that is, for every \(a, b \in A\), \(a - b \in A\). If in addition \(a + A - a \subseteq A\) for all \(a \in H\), \(A\) is said to be normal.

Definition 1.17. A hyperring is a triple \((R, +, .)\) satisfying the following axioms:

(i) \((R, +)\) is a canonical hypergroup,
(ii) \((R, .)\) is a semi-hyperring such that \(x.0 = 0.x = 0\) for all \(x \in R\), that is, 0 is a bilaterally absorbing element,
(iii) for all \(x, y, z \in R\),
\[x.(y + z) = x.y + x.z \text{ and } (x + y).z = x.z + y.z.\]

That is, the hyperoperation \(\cdot\) is distributive over the hyperoperation \(+\).

Definition 1.18. Let \((R, +, .)\) be a hyperring and let \(A\) be a nonempty subset of \(R\). \(A\) is said to be a subhyperring of \(R\) if \((A, +, .)\) is itself a hyperring.

Definition 1.19. Let \((R, +, .)\) be a hyperring and let \(A\) be a nonempty subset of \(R\). \(A\) is said to be a subhyperring of \(R\) if \((A, +, .)\) is itself a hyperring.

Definition 1.20. Let \(A\) be a subhyperring of a hyperring \(R\). Then

(i) \(A\) is called a left hyperideal of \(R\) if \(r.\alpha \subseteq A\) for all \(r \in R, \alpha \in A\),
(ii) \(A\) is called a right hyperideal of \(R\) if \(\alpha.\beta \subseteq A\) for all \(r \in R, \alpha \in A\),
(iii) \(A\) is called a hyperideal of \(R\) if \(A\) is both left and right hyperideal of \(R\).

Definition 1.21. Let \(A\) be a hyperideal of a hyperring \(R\). \(A\) is said to be normal in \(R\), if \(r + A - r \subseteq A\), for all \(r \in R\).

For full details about hypergroups, canonical hypergroups, hyperrings, neutrosophic canonical hypergroups and neutrosophic hyperrings, the reader should see [3, 14]
2. Development of Neutrosophic Quadruple Canonical Hypergroups and Neutrosophic Quadruple Hyperrings

In this section, we develop two neutrosophic hyperquadruple algebraic hyperstructures namely neutrosophic quadruple canonical hypergroup and neutrosophic quadruple hyperring. In what follows, all neutrosophic quadruple numbers will be real neutrosophic quadruple numbers i.e. \( a, b, c, d \in \mathbb{R} \) for any neutrosophic quadruple number \( (a, bT, cI, dF) \in NQ \).

**Definition 2.1.** Let \( +, \text{ and } \cdot \) be hyperoperations on \( \mathbb{R} \) that is \( x + y \subseteq \mathbb{R}, x, y \subseteq \mathbb{R} \) for all \( x, y \in \mathbb{R} \). Let \( \hat{+} \) and \( \hat{\cdot} \) be hyperoperations on \( NQ \). For \( x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F) \in NQ \) with \( x_i, y_i \in \mathbb{R}, i = 1, 2, 3, 4 \), define:

\[
\begin{align*}
x \hat{+} y &= \{(a, bT, cI, dF) : a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}, \\
x \hat{\cdot} y &= \{(a, bT, cI, dF) : a \in x_1, b \in (x_1, y_1) \cup (x_2, y_2), c \in (x_3, y_3), d \in (x_4, y_4) \}\end{align*}
\]

**Theorem 2.2.** \( (NQ, \hat{+}) \) is a canonical hypergroup.

**Proof.** Let \( x = (x_1, x_2T, x_3I, x_4F), y = (y_1, y_2T, y_3I, y_4F), z = (z_1, z_2T, z_3I, z_4F) \in NQ \) be arbitrary with \( x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4 \).

(i) To show that \( x \hat{+} y = y \hat{+} x \), let

\[
x \hat{+} y = \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1 + y_1, a_2 \in x_2 + y_2, a_3 \in x_3 + y_3, a_4 \in x_4 + y_4\},
\]

\[
y \hat{+} x = \{b = (b_1, b_2T, b_3I, b_4F) : b_1 \in y_1 + x_1, b_2 \in y_2 + x_2, b_3 \in y_3 + x_3, b_4 \in y_4 + x_4\}.
\]

Since \( a_i, b_i \in \mathbb{R}, i = 1, 2, 3, 4 \), it follows that \( x \hat{+} y = y \hat{+} x \).

(ii) To show that that \( x \hat{+} (y \hat{+} z) = (x \hat{+} y) \hat{+} z \), let

\[
y \hat{+} z = \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2, w_3 \in y_3 + z_3, w_4 \in y_4 + z_4\}.
\]

Now,

\[
x \hat{+} (y \hat{+} z) = x \hat{+} w = \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + w_1, p_2 \in x_2 + w_2, p_3 \in x_3 + w_3, p_4 \in x_4 + w_4\} = \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1 + (y_1 + z_1), p_2 \in x_2 + (y_2 + z_2), p_3 \in x_3 + (y_3 + z_3), p_4 \in x_4 + (y_4 + z_4)\}.
\]

Also, let \( x \hat{+} y = \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1 + y_1, u_2 \in x_2 + y_2, u_3 \in x_3 + y_3, u_4 \in x_4 + y_4\} \) so that

\[
(x \hat{+} y) \hat{+} z = u \hat{+} z = \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in u_1 + z_1, q_2 \in u_2 + z_2, q_3 \in u_3 + z_3, q_4 \in u_4 + z_4\} = \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in (x_1 + y_1) + z_1, q_2 \in (x_2 + y_2) + z_2, q_3 \in (x_3 + y_3) + z_3, q_4 \in (x_4 + y_4) + z_4\}.
\]

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Since \( u_i, p_i, q_i, w_i, x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4 \), it follows that \( x^+ (y^+ z) = (x^+ y)^+ z \).

(iii) To show that \( 0 = (0, 0, 0, 0) \in NQ \) is a neutral element, consider
\[
x^+ (0, 0, 0, 0) = \{ a = (a_1, a_2 T, a_3 I, a_4 F) : a_1 \in x_1 + 0, a_2 \in x_2 + 0, a_3 \in x_3 + 0, a_4 \in x_4 + 0 \}
= \{ a = (a_1, a_2 T, a_3 I, a_4 F) : a_1 \in \{ x_1 \}, a_2 \in \{ x_2 \}, a_3 \in \{ x_3 \}, a_4 \in \{ x_4 \} \}
= \{ x \}.
\]
Similarly, it can be shown that \( (0, 0, 0, 0) \textsuperscript{−} x = \{ x \} \). Hence \( 0 = (0, 0, 0, 0) \in NQ \) is a neutral element.

(iv) To show that for every \( x \in NQ \), there exists a unique element \( \textsuperscript{−} x \in NQ \) such that \( 0 \in x^+ (\textsuperscript{−} x) \cap (\textsuperscript{−} x)^+ x \), consider
\[
x^+ (\textsuperscript{−} x) \cap (\textsuperscript{−} x)^+ x = \{ a = (a_1, a_2 T, a_3 I, a_4 F) : a_1 \in x_1 - x_1, a_2 \in x_2 - x_2,
\quad a_3 \in x_3 - x_3, a_4 \in x_4 - x_4 \} \cap \{ b = (b_1, b_2 T, b_3 I, b_4 F) : b_1 \in -x_1 + x_1, b_2 \in -x_2 + x_2, b_3 \in -x_3 + x_3, b_4 \in -x_4 + x_4 \}
= \{(0, 0, 0, 0)\}.
\]
This shows that for every \( x \in NQ \), there exists a unique element \( \textsuperscript{−} x \in NQ \) such that \( 0 \in x^+ (\textsuperscript{−} x) \cap (\textsuperscript{−} x)^+ x \).

(v) Since for all \( x, y, z \in NQ \) with \( x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4 \), it follows that \( z \in x+y \) implies \( y \in -x+z \) and \( x \in z+(−y) \). Hence, \((NQ, +)\) is a canonical hypergroup.

\[\square\]

**Lemma 2.3.** Let \((NQ, +)\) be a neutrosophic quadruple canonical hypergroup. Then
\begin{enumerate}
  \item \( \textsuperscript{−} (\textsuperscript{−} x) = x \) for all \( x \in NQ \),
  \item \( 0 = (0, 0, 0, 0) \) is the unique element such that for every \( x \in NQ \), there is an element \( \textsuperscript{−} x \in NQ \) such that \( 0 \in x^+ (\textsuperscript{−} x) \),
  \item \( \textsuperscript{−} 0 = 0 \),
  \item \( \textsuperscript{−} (x+y) = \textsuperscript{−} x \textsuperscript{−} y \) for all \( x, y \in NQ \).
\end{enumerate}

**Example 2.4.** Let \( NQ = \{ 0, x, y \} \) be a neutrosophic quadruple set and let \( + \) be a hyperoperation on \( NQ \) defined in the table below.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>x</td>
<td>x</td>
<td>{0, x, y}</td>
<td>y</td>
</tr>
<tr>
<td>y</td>
<td>y</td>
<td>{0, y}</td>
<td></td>
</tr>
</tbody>
</table>

Then \((NQ, +)\) is a neutrosophic quadruple canonical hypergroup.

**Theorem 2.5.** \((NQ, \times)\) is a semi-hypergroup.

**Proof.** Let \( x = (x_1, x_2 T, x_3 I, x_4 F), y = (y_1, y_2 T, y_3 I, y_4 F), z = (z_1, z_2 T, z_3 I, z_4 F) \in NQ \) be arbitrary with \( x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4 \).
To show that $NH$ is a neutrosophic quadruple hypergroup, if the following conditions hold:

Remark 2.6.

(i) $x \hat{\times} y = \{a = (a_1, a_2T, a_3I, a_4F) : a_1 \in x_1y_1, a_2 \in x_1y_2 \cup x_2y_1 \cup x_2y_2, a_3 \in x_1y_3 \cup x_2y_3 \cup x_3y_1 \cup x_3y_2 \cup x_3y_3, a_4 \in x_1y_4 \cup x_2y_4 \cup x_3y_4 \cup x_4y_1 \cup x_4y_2 \cup x_4y_3 \cup x_4y_4 \} \subseteq NQ$.

(ii) To show that $x \hat{\times} (y \hat{\times} z) = (x \hat{\times} y) \hat{\times} z$, let

$$y \hat{\times} z = \{w = (w_1, w_2T, w_3I, w_4F) : w_1 \in y_1z_1, w_2 \in y_1z_2 \cup y_2z_1 \cup y_2z_2, w_3 \in y_1z_3 \cup y_2z_3 \cup y_3z_1 \cup y_3z_2 \cup y_3z_3, w_4 \in y_1z_4 \cup y_2z_4 \}$$

so that

$$x \hat{\times} (y \hat{\times} z) = x \hat{\times} w = \{p = (p_1, p_2T, p_3I, p_4F) : p_1 \in x_1w_1, p_2 \in x_1w_2 \cup x_2w_1 \cup x_2w_2, p_3 \in x_1w_3 \cup x_2w_3 \cup x_3w_1 \cup x_3w_2 \cup x_3w_3, p_4 \in x_1w_4 \cup x_2w_4 \cup x_3w_4 \cup x_4w_3 \cup x_4w_4 \}$$

Also, let

$$x \hat{\times} y = \{u = (u_1, u_2T, u_3I, u_4F) : u_1 \in x_1y_1, u_2 \in x_1y_2 \cup x_2y_1 \cup x_2y_2, u_3 \in x_1y_3 \cup x_2y_3 \cup x_3y_1 \cup x_3y_2 \cup x_3y_3, u_4 \in x_1y_4 \cup x_2y_4 \}$$

so that

$$(x \hat{\times} y) \hat{\times} z = u \hat{\times} z = \{q = (q_1, q_2T, q_3I, q_4F) : q_1 \in u_1z_1, q_2 \in u_1z_2 \cup u_2z_1 \cup u_2z_2, q_3 \in u_1z_3 \cup u_2z_3 \cup u_3z_1 \cup u_3z_2 \cup u_3z_3, q_4 \in u_1z_4 \cup u_2z_4 \cup u_3z_4 \cup u_4z_3 \cup u_4z_4 \}$$

Substituting $w_i$ of (2.3) in (2.4) and also substituting $u_i$ of (2.5) in (2.6), where $i = 1, 2, 3, 4$ and since $p_i, q_i, u_i, w_i, x_i, z_i \in R$, it follows that $x \hat{\times} (y \hat{\times} z) = (x \hat{\times} y) \hat{\times} z$. Consequently, $(NQ, \hat{\times})$ is a semi-hypergroup which we call neutrosophic quadruple semi-hypergroup.

Remark 2.6. $(NQ, \hat{\times})$ is not a hypergroup.

Definition 2.7. Let $(NQ, \hat{\hat{+}})$ be a neutrosophic quadruple canonical hypergroup. For any subset $NH$ of $NQ$, we define

$$\hat{-}NH = \{\hat{-}x : x \in NH \}.$$ 

A nonempty subset $NH$ of $NQ$ is called a neutrosophic quadruple subcanonical hypergroup, if the following conditions hold:

(i) $0 = (0, 0, 0, 0) \in NH$,

(ii) $x \hat{-} y \subseteq NH$ for all $x, y \in NH$.

A neutrosophic quadruple subcanonical hypergroup $NH$ of a neutrosophic quadruple canonical hypergroup $NQ$ is said to be normal, if $x \hat{+} NH \hat{-} x \subseteq NH$ for all $x \in NQ$.
Definition 2.8. Let \((NQ, \dot{+})\) be a neutrosophic quadruple canonical hypergroup. For \(x_i \in NQ\) with \(i = 1, 2, 3, \ldots, n \in \mathbb{N}\), the heart of \(NQ\) denoted by \(NQ_\omega\) is defined by

\[
NQ_\omega = \bigcup_{i=1}^{n} (x_i \dot{-} x_i) .
\]

In Example 2.4, \(NQ_\omega = NQ\).

Definition 2.9. Let \((NQ_1, \dot{+})\) and \((NQ_2, \dot{+}')\) be two neutrosophic quadruple canonical hypergroups. A mapping \(\phi : NQ_1 \rightarrow NQ_2\) is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) \(\phi(x \dot{+} y) = \phi(x) \dot{+}' \phi(y)\) for all \(x, y \in NQ_1\),
(ii) \(\phi(T) = T\),
(iii) \(\phi(I) = I\),
(iv) \(\phi(F) = F\),
(v) \(\phi(0) = 0\).

If in addition \(\phi\) is a bijection, then \(\phi\) is called a neutrosophic quadruple strong isomorphism and we write \(NQ_1 \cong NQ_2\).

Definition 2.10. Let \(\phi : NQ_1 \rightarrow NQ_2\) be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then the set \(\{x \in NQ_1 : \phi(x) = 0\}\) is called the kernel of \(\phi\) and it is denoted by \(\text{Ker}\phi\). Also, the set \(\{\phi(x) : x \in NQ_1\}\) is called the image of \(\phi\) and it is denoted by \(\text{Im}\phi\).

Theorem 2.11. \((NQ, \dot{+}, \dot{*})\) is a hyperring.

Proof. That \((NQ, \dot{+})\) is a canonical hypergroup follows from Theorem 2.2. Also, that \((NQ, \dot{*})\) is a semi hypergroup follows from Theorem 2.4.

Next, let \(x = (x_1, x_2 T, x_3 I, x_4 F) \in NQ\) be arbitrary with \(x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4\). Then

\[
x \dot{*} 0 = \big\{ u = (u_1, u_2 T, u_3 I, u_4 F) : u_1 \in x_1, 0, u_2 \in x_1.0 \cup x_2.0 \cup x_2.0, u_3 \in x_1.0
\]

\[
\cup x_2.0 \cup x_3.0 \cup x_3.0 \cup x_3.0, u_4 \in x_1.0 \cup x_2.0 \cup x_3.0 \cup x_4.0 \cup x_4.0
\]

\[
\cup x_4.0 \cup x_4.0\big\} = \big\{ u = (u_1, u_2 T, u_3 I, u_4 F) : u_1 \in \{0\}, u_2 \in \{0\}, u_3 \in \{0\}, u_4 \in \{0\}\big\}
\]

\[
= \{0\}.
\]

Similarly, it can be shown that \(0 \dot{*} x = \{0\}\). Since \(x\) is arbitrary, it follows that \(x \dot{*} 0 = 0 \dot{*} x = \{0\}\), for all \(x \in NQ\). Hence, \(0 = (0, 0, 0, 0)\) is a bilaterally absorbing element.

To complete the proof, we have to show that \(x \dot{*} (y \dot{+} z) = (x \dot{*} y) \dot{+} (x \dot{*} z)\), for all \(x, y, z \in NQ\). To this end, let \(x = (x_1, x_2 T, x_3 I, x_4 F), y = (y_1, y_2 T, y_3 I, y_4 F), z = (z_1, z_2 T, z_3 I, z_4 F) \in NQ\) be arbitrary with \(x_i, y_i, z_i \in \mathbb{R}, i = 1, 2, 3, 4\). Let

\[
\begin{align*}
y \dot{+} z & = \{ w = (w_1, w_2 T, w_3 I, w_4 F) : w_1 \in y_1 + z_1, w_2 \in y_2 + z_2, w_3 \in y_3 + z_3, \\
w_4 & \in y_4 + z_4 \}\end{align*}
\]
so that
\[ x \hat{x}(y+z) = x \hat{x}w \]
\[ = \{ p = (p_1, p_2 T, p_3 I, p_4 F) : p_1 \in x_1 w_1, p_2 \in x_1 w_2 \cup x_2 w_1 \cup x_2 w_2, \]
\[ p_3 \in x_1 w_3 \cup x_2 w_3 \cup x_3 w_1 \cup x_3 w_2 \cup x_3 y_3, p_4 \in x_1 w_4 \cup x_2 w_4 \]
\[ \cup x_3 w_4 \cup x_4 w_1 \cup x_4 w_2 \cup x_4 w_3 \cup x_4 w_4 \}. \]

Substituting \( w_i, i = 1, 2, 3, 4 \) of (2.7) in (2.8), we obtain the following:
\[ p_1 \in x_1(y_1 + z_1), \]
\[ p_2 \in x_1(y_2 + z_2) \cup x_2(y_1 + z_1) \cup x_2(y_2 + z_2), \]
\[ p_3 \in x_1(y_3 + z_3) \cup x_2(y_3 + z_3) \cup x_3(y_1 + z_1) \cup x_3(y_2 + z_2) \cup x_3(y_3 + z_3), \]
\[ p_4 \in x_1(y_4 + z_4) \cup x_2(y_4 + z_4) \cup x_3(y_4 + z_4) \cup x_4(y_1 + z_1) \cup x_4(y_2 + z_2), \]
\[ \cup x_4(y_3 + z_3) \cup x_4(y_4 + z_4). \]

Also, let
\[ x \hat{x} y = \{ u = (u_1, u_2 T, u_3 I, u_4 F) : u_1 \in x_1 y_1, u_2 \in x_1 y_2 \cup x_2 y_1 \cup x_2 y_2, \]
\[ u_3 \in x_1 y_3 \cup x_2 y_3 \cup x_3 y_1 \cup x_3 y_2 \cup x_3 y_3, u_4 \in x_1 y_4 \cup x_2 y_4 \]
\[ \cup x_3 y_4 \cup x_4 y_1 \cup x_4 y_2 \cup x_4 y_3 \cup x_4 y_4 \}, \]
\[ x \hat{x} z = \{ v = (v_1, v_2 T, v_3 I, v_4 F) : v_1 \in x_1 z_1, v_2 \in x_1 z_2 \cup x_2 z_1 \cup x_2 z_2, \]
\[ v_3 \in x_1 z_3 \cup x_2 z_3 \cup x_3 z_1 \cup x_3 z_2 \cup x_3 z_3, v_4 \in x_1 z_4 \cup x_2 z_4 \]
\[ \cup x_3 z_4 \cup x_4 z_1 \cup x_4 z_2 \cup x_4 z_3 \cup x_4 z_4 \}, \]
so that
\[ (x \hat{x} y) + (x \hat{x} z) = u + v \]
\[ = \{ q = (q_1, q_2 T, q_3 I, q_4 F) : q_1 \in u_1 + v_1, q_2 \in u_2 + v_2, \]
\[ q_3 \in u_3 + v_3, q_4 \in u_4 + v_4 \}. \]

Substituting \( u_i \) of (2.13) and \( v_i \) of (2.14) in (2.15), we obtain the following:
\[ q_1 \in u_1 + v_1 \subseteq x_1 y_1 + x_1 z_1 \subseteq x_1(y_1 + z_1), \]
\[ q_2 \in u_2 + v_2 \subseteq (x_1 y_2 \cup x_2 y_1 \cup x_2 y_2) \]
\[ + (x_1 z_2 \cup x_2 z_1 \cup x_2 z_2) \]
\[ \subseteq x_1(y_2 + z_2) \cup x_2(y_1 + z_1) \cup x_2(y_2 + z_2), \]
\[ q_3 \in u_3 + v_3 \subseteq (x_1 y_3 \cup x_2 y_3 \cup x_3 y_1 \cup x_3 y_2 \cup x_3 y_3) \]
\[ + (x_1 z_3 \cup x_2 z_3 \cup x_3 z_1 \cup x_3 z_2 \cup x_3 z_3) \]
\[ \subseteq x_1(y_3 + z_3) \cup x_2(y_3 + z_3) \cup x_3(y_1 + z_1) \cup x_3(y_2 + z_2) \cup x_3(y_3 + z_3), \]
\[ q_4 \in u_4 + v_4 \subseteq (x_1 y_4 \cup x_2 y_4 \cup x_3 y_4) \cup x_4 y_1 \cup x_4 y_2 \cup x_4 y_3 \cup x_4 y_4 \]
\[ + (x_1 z_4 \cup x_2 z_4 \cup x_3 z_4) \cup x_4 z_1 \cup x_4 z_2 \cup x_4 z_3 \cup x_4 z_4) \]
\[ \subseteq x_1(y_4 + z_4) \cup x_2(y_4 + z_4) \cup x_3(y_4 + z_4) \cup x_4(y_1 + z_1) \cup x_4(y_2 + z_2), \]
\[ \cup x_4(y_3 + z_3) \cup x_4(y_4 + z_4). \]

Comparing (2.9), (2.10), (2.11) and (2.12) respectively with (2.16), (2.17), (2.18) and (2.19), we obtain \( p_i = q_i, i = 1, 2, 3, 4 \). Hence, \( x \hat{x}(y+z) = (x \hat{x}y) + (x \hat{x}z) \), for all
x, y, z ∈ NQ. Thus, \((NQ, \hat{+}, \hat{\times})\) is a hyperring which we call neutrosophic quadruple hyperring.

\[\Box\]

**Theorem 2.12.** \((NQ, \hat{+}, \circ)\) is a Krasner hyperring where \(\circ\) is an ordinary multiplicative binary operation on \(NQ\).

**Definition 2.13.** Let \((NQ, \hat{+}, \hat{\times})\) be a neutrosophic quadruple hyperring. A nonempty subset \(NJ\) of \(NQ\) is called a neutrosophic quadruple subhyperring of \(NQ\), if \((NJ, \hat{+}, \hat{\times})\) is itself a neutrosophic quadruple hyperring.

\(NJ\) is called a neutrosophic quadruple hyperideal if the following conditions hold:

(i) \((NJ, \hat{+})\) is a neutrosophic quadruple subcanonical hypergroup.

(ii) For all \(x \in NJ\) and \(r \in NQ\), \(x \hat{\times} r, r \hat{\times} x \subseteq NJ\).

A neutrosophic quadruple hyperideal \(NJ\) of \(NQ\) is said to be normal in \(NQ\), if \(x \hat{+} NJ \hat{-} x \subseteq NJ\), for all \(x \in NQ\).

**Definition 2.14.** Let \((NQ_1, \hat{+}, \hat{\times})\) and \((NQ_2, \hat{+}', \hat{\times}')\) be two neutrosophic quadruple hyperrings. A mapping \(\phi: NQ_1 \to NQ_2\) is called a neutrosophic quadruple strong homomorphism, if the following conditions hold:

(i) \(\phi(x + y) = \phi(x) +' \phi(y)\), for all \(x, y \in NQ_1\),

(ii) \(\phi(x \hat{\times} y) = \phi(x) \hat{\times}' \phi(y)\), for all \(x, y \in NQ_1\),

(iii) \(\phi(T) = T\),

(iv) \(\phi(I) = I\),

(v) \(\phi(F) = F\),

(vi) \(\phi(0) = 0\).

If in addition \(\phi\) is a bijection, then \(\phi\) is called a neutrosophic quadruple strong isomorphism and we write \(NQ_1 \cong NQ_2\).

**Definition 2.15.** Let \(\phi: NQ_1 \to NQ_2\) be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings. Then the set \(\{x \in NQ_1 : \phi(x) = 0\}\) is called the kernel of \(\phi\) and it is denoted by \(\text{Ker} \phi\). Also, the set \(\{\phi(x) : x \in NQ_1\}\) is called the image of \(\phi\) and it is denoted by \(\text{Im} \phi\).

**Example 2.16.** Let \((NQ, \hat{+}, \hat{\times})\) be a neutrosophic quadruple hyperring and let \(NX\) be the set of all strong endomorphisms of \(NQ\). If \(\oplus\) and \(\odot\) are hyperoperations defined for all \(\phi, \psi \in NX\) and for all \(x \in NQ\) as

\[\phi \oplus \psi = \{\nu(x) : \nu(x) \in \phi(x) \hat{+} \psi(x)\}\],

\[\phi \odot \psi = \{\nu(x) : \nu(x) \in \phi(x) \hat{\times} \psi(x)\}\],

then \((NX, \oplus, \odot)\) is a neutrosophic quadruple hyperring.

### 3. Characterization of Neutrosophic Quadruple Canonical Hypergroups and Neutrosophic Hyperrings

In this section, we present elementary properties which characterize neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings.

**Theorem 3.1.** Let \(NG\) and \(NH\) be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup \((NQ, \hat{+})\). Then

(1) \(NG \cap NH\) is a neutrosophic quadruple subcanonical hyperring of \(NQ\),
(2) \( NG \times NH \) is a neutrosophic quadruple subcanonical hypergroup of \( NQ \).

**Theorem 3.2.** Let \( NH \) be a neutrosophic quadruple subcanonical hypergroup of a neutrosophic quadruple canonical hypergroup \((NQ, \hat{+})\). Then

1. \( NH \hat{+} NH = NH \),
2. \( x \hat{+} NH = NH \), for all \( x \in NH \).

**Theorem 3.3.** Let \((NQ, \hat{+})\) be a neutrosophic quadruple canonical hypergroup. \( NQ_\omega \), the heart of \( NQ \) is a normal neutrosophic quadruple subcanonical hypergroup of \( NQ \).

**Theorem 3.4.** Let \( NG \) and \( NH \) be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup \((NQ, \hat{+})\). Then

1. If \( NG \subseteq NH \) and \( NG \) is normal, then \( NG \) is normal.
2. If \( NG \) is normal, then \( NG \hat{+} NH \) is normal.

**Definition 3.5.** Let \( NG \) and \( NH \) be neutrosophic quadruple subcanonical hypergroups of a neutrosophic quadruple canonical hypergroup \((NQ, \hat{+})\). The set \( NG \hat{+} NH \) is defined by

\[
(3.1) \quad NG \hat{+} NH = \{x \hat{+} y : x \in NG, y \in NH\}.
\]

It is obvious that \( NG \hat{+} NH \) is a neutrosophic quadruple subcanonical hypergroup of \((NQ, \hat{+})\).

If \( x \in NH \), the set \( x \hat{+} NH \) is defined by

\[
(3.2) \quad x \hat{+} NH = \{x \hat{+} y : y \in NH\}.
\]

If \( x \) and \( y \) are any two elements of \( NH \) and \( \tau \) is a relation on \( NH \) defined by \( x \tau y \) if \( x \in y \hat{+} NH \), it can be shown that \( \tau \) is an equivalence relation on \( NH \) and the equivalence class of any element \( x \in NH \) determined by \( \tau \) is denoted by \([x]\).

**Lemma 3.6.** For any \( x \in NH \), we have

1. \([x] = x \hat{+} NH\),
2. \([\bar{x}] = \hat{-}[x]\).

**Proof.** (1)

\[
[x] = \{y \in NH : x \tau y\} = \{y \in NH : y \in x \hat{+} NH\} = x \hat{+} NH.
\]

(2) Obvious. \( \square \)

**Definition 3.7.** Let \( NQ/NH \) be the collection of all equivalence classes of \( x \in NH \) determined by \( \tau \). For \([x], [y] \in NQ/NH\), we define the set \([x] \boxplus[y]\) as

\[
(3.3) \quad [x] \boxplus[y] = \{[z] : z \in x \hat{+} y\}.
\]

**Theorem 3.8.** \((NQ/NH, \hat{\oplus})\) is a neutrosophic quadruple canonical hypergroup.

**Proof.** Same as the classical case and so omitted. \( \square \)
Theorem 3.9. Let \((NQ, \hat{+})\) be a neutrosophic quadruple canonical hypergroup and let \(NH\) be a normal neutrosophic quadruple subcanonical hypergroup of \(NQ\). Then, for any \(x, y \in NH\), the following are equivalent:

1. \(x \in y + NH\),
2. \(y - x \subseteq NH\),
3. \((y - x) \cap NH \neq \emptyset\)

**Proof.** Same as the classical case and so omitted. \(\Box\)

Theorem 3.10. Let \(\phi : NQ_1 \rightarrow NQ_2\) be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then

1. \(\ker \phi\) is not a neutrosophic quadruple subcanonical hypergroup of \(NQ_1\),
2. \(\text{Im} \phi\) is a neutrosophic quadruple subcanonical hypergroup of \(NQ_2\).

**Proof.** (1) Since it is not possible to have \(\phi((0, T, 0, 0)) = \phi((0, 0, 0, 0)), \phi((0, 0, I, 0)) = \phi((0, 0, 0, 0))\) and \(\phi((0, 0, 0, F)) = \phi((0, 0, 0, 0))\), it follows that \((0, T, 0, 0), (0, 0, I, 0)\) and \((0, 0, 0, F)\) cannot be in the kernel of \(\phi\). Consequently, \(\ker \phi\) cannot be a neutrosophic quadruple subcanonical hypergroup of \(NQ_1\).

(2) Obvious. \(\Box\)

Remark 3.11. If \(\phi : NQ_1 \rightarrow NQ_2\) is a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups, then \(\ker \phi\) is a subcanonical hypergroup of \(NQ_1\).

Theorem 3.12. Let \(\phi : NQ_1 \rightarrow NQ_2\) be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple canonical hypergroups. Then

1. \(NQ_1/\ker \phi\) is not a neutrosophic quadruple canonical hypergroup,
2. \(NQ_1/\ker \phi\) is a canonical hypergroup.

Theorem 3.13. Let \(NH\) be a neutrosophic quadruple subcanonical hypergroup of the neutrosophic quadruple canonical hypergroup \((NQ, \hat{+})\). Then the mapping \(\phi : NQ \rightarrow NQ/NH\) defined by \(\phi(x) = x + NH\) is not a neutrosophic quadruple strong homomorphism.


Lemma 3.15. Let \(NJ\) be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring \((NQ, \hat{+}, \hat{\times})\). Then

1. \(\sim NJ = NJ\),
2. \(x + NJ = NJ\), for all \(x \in NJ\),
3. \(x \hat{\times} NJ = NJ\), for all \(x \in NJ\).

Theorem 3.16. Let \(NJ\) and \(NK\) be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring \((NQ, \hat{+}, \hat{\times})\). Then

1. \(NJ \cap NK\) is a neutrosophic quadruple hyperideal of \(NQ\),
2. \(NJ \hat{\times} NK\) is a neutrosophic quadruple hyperideal of \(NQ\),
3. \(NJ \hat{+} NK\) is a neutrosophic quadruple hyperideal of \(NQ\).

Theorem 3.17. Let \(NJ\) be a normal neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring \((NQ, \hat{+}, \hat{\times})\). Then
(1) \((x+NJ)\hat{+} (y+NJ) = (x+y)\hat{+} NJ\), for all \(x, y \in NJ\),
(2) \((x+NJ)\hat{\times} (y+NJ) = (x\times y)\hat{+} NJ\), for all \(x, y \in NJ\),
(3) \(x+NJ = y+NJ\), for all \(y \in x+NJ\).

**Theorem 3.18.** Let \(NJ\) and \(NK\) be neutrosophic quadruple hyperideals of a neutrosophic quadruple hyperring \((NQ, \hat{+}, \hat{\times})\) such that \(NJ\) is normal in \(NQ\). Then

(1) \(NJ \cap NK\) is normal in \(NJ\),
(2) \(NJ\hat{+} NK\) is normal in \(NQ\),
(3) \(NJ\) is normal in \(NJ\hat{+} NK\).

Let \(NJ\) be a neutrosophic quadruple hyperideal of a neutrosophic quadruple hyperring \((NQ, \hat{+}, \hat{\times})\). For all \(x \in NQ\), the set \(NQ/NJ\) is defined as

\[
NQ/NJ = \{ x\hat{+} NJ : x \in NQ \}.
\]

For \([x], [y] \in NQ/NJ\), we define the hyperoperations \(\hat{\circ}\) and \(\hat{\otimes}\) on \(NQ/NJ\) as follows:

\[
[x] \hat{\circ} [y] = \{ [z] : z \in x\hat{+} y \},
\]
\[
[x] \hat{\otimes} [y] = \{ [z] : z \in x\hat{\times} y \}.
\]

It can easily be shown that \((NQ/NH, \hat{\circ}, \hat{\otimes})\) is a neutrosophic quadruple hyperring.

**Theorem 3.19.** Let \(\phi : NQ \rightarrow NR\) be a neutrosophic quadruple strong homomorphism of neutrosophic quadruple hyperrings and let \(NJ\) be a neutrosophic quadruple hyperideal of \(NQ\). Then

(1) \(\ker \phi\) is not a neutrosophic quadruple hyperideal of \(NQ\),
(2) \(\text{Im}\phi\) is a neutrosophic quadruple hyperideal of \(NR\),
(3) \(NQ/\ker \phi\) is not a neutrosophic quadruple hyperring,
(4) \(NQ/\text{Im}\phi\) is a neutrosophic quadruple hyperring,
(5) The mapping \(\psi : NQ \rightarrow NQ/NJ\) defined by \(\psi(x) = x\hat{+} NJ\), for all \(x \in NQ\) is not a neutrosophic quadruple strong homomorphism.

**Remark 3.20.** The classical isomorphism theorems of hyperrings do not hold in neutrosophic quadruple hyperrings.

4. Conclusion

We have developed neutrosophic quadruple algebraic hyperstructures in this paper. In particular, we have developed new neutrosophic algebraic hyperstructures namely neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. We have presented elementary properties which characterize the new neutrosophic algebraic hyperstructures.

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The category of neutrosophic crisp sets

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ABSTRACT. We introduce the category $\text{NCSet}$ consisting of neutrosophic crisp sets and morphisms between them. And we study $\text{NCSet}$ in the sense of a topological universe and prove that it is Cartesian closed over $\text{Set}$, where $\text{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them.

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1. Introduction

In 1965, Zadeh [20] had introduced a concept of a fuzzy set as the generalization of a crisp set. In 1986, Atanassove [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets considering the degree of membership and non-membership. In 1998 Smarandache [19] introduced the concept of a neutrosophic set considering the degree of membership, the degree of indeterminacy and the degree of non-membership. Moreover, Salama et al. [15, 16, 18] applied the concept of neutrosophic crisp sets to topology and relation.

After that time, many researchers [2, 3, 4, 5, 7, 8, 10, 12, 13, 14] have investigated fuzzy sets in the sense of category theory, for instance, $\text{Set}(H)$, $\text{Set}_f(H)$, $\text{Set}_g(H)$, $\text{Fuz}(H)$. Among them, the category $\text{Set}(H)$ is the most useful one as the "standard" category, because $\text{Set}(H)$ is very suitable for describing fuzzy sets and mappings between them. In particular, Carrega [2], Dubuc [3], Eytan [4], Goguen [5], Pittes [12], Ponasse [13, 14] had studied $\text{Set}(H)$ in topos view-point. However Hur et al. investigated $\text{Set}(H)$ in topological view-point. Moreover, Hur et al. [8] introduced the category $\text{ISet}(H)$ consisting of intuitionistic $H$-fuzzy sets and morphisms between them, and studied $\text{ISet}(H)$ in the sense of topological universe. Recently, Lim et al [10] introduced the new category $\text{VSet}(H)$ and investigated it in the sense of topological universe.
The concept of a topological universe was introduced by Nel [11], which implies a Cartesian closed category and a concrete quasitopos. Furthermore the concept has already been up to effective use for several areas of mathematics.

In this paper, first, we obtain some properties of neutrosophic crisp sets proposed by Salama and Smarandache [17] in 2015. Second, we introduce the category $\text{NCSet}$ consisting of neutrosophic crisp sets and morphisms between them. And we prove that the category $\text{NCSet}$ is topological and cotopological over $\text{Set}$ (See Theorem 4.6 and Corollary 4.8), where $\text{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we prove that final episinks in $\text{NCSet}$ are preserved by pullbacks (See Theorem 4.10) and $\text{NCSet}$ is Cartesian closed over $\text{Set}$ (See Theorem 4.15).

2. Preliminaries

In this section, we list some basic definitions and well-known results from [6, 9, 11] which are needed in the next sections.

**Definition 2.1** ([9]). Let $A$ be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in $A$ indexed by a class $J$. For any set $X$, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by $J$. Then an $A$-structure $\xi$ on $X$ is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), (Y_j, \xi_j))_J$, if it satisfies the following conditions:

(i) For each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an $A$-morphism,

(ii) If $(Z, \rho)$ is an $A$-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an $A$-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an $A$-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in $A$.

Dual notion: cotopological category.

**Result 2.2** ([9], Theorem 1.5). A concrete category $A$ is topological if and only if it is cotopological.

**Result 2.3** ([9], Theorem 1.6). Let $A$ be a topological category over $\text{Set}$, then it is complete and cocomplete.

**Definition 2.4** ([9]). Let $A$ be a concrete category.

(i) The $A$-fibre of a set $X$ is the class of all $A$-structures on $X$.

(ii) $A$ is said to be properly fibred over $\text{Set}$, it satisfies the followings:

(a) (Fibre-smallness) for each set $X$, the $A$-fibre of $X$ is a set,

(b) (Terminal separator property) for each singleton set $X$, the $A$-fibre of $X$ has precisely one element,

(c) if $\xi$ and $\eta$ are $A$-structures on a set $X$ such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are $A$-morphisms, then $\xi = \eta$.

**Definition 2.5** ([6]). A category $A$ is said to be Cartesian closed, if it satisfies the following conditions:

(i) For each $A$-object $A$ and $B$, there exists a product $A \times B$ in $A$,

(ii) Exponential objects exist in $A$, i.e., for each $A$-object $A$, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any $A$-object $B$, there exist an $A$-object $B^A$ and a $A$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any
A-smallness condition; (c) A-definition 2.6: s satisfies the following conditions:

Definition 2.6 ([6]). A category A is called a topological universe over Set, if it satisfies the following conditions:

(i) A is well-structured, i.e. (a) A is concrete category; (b) A satisfies the fibre-smallness condition; (c) A has the terminal separator property,

(ii) A is cotopological over Set,

(iii) final episinks in A are preserved by pullbacks, i.e., for any episink (g_j : X_j → Y)_j and any A-morphism f : W → Y, the family (e_j : U_j → W)_j, obtained by taking the pullback f and g_j, for each j ∈ J, is again a final episink.

3. Neutrosophic crisp sets

In [17], Salama and Smarandache introduced the concept of a neutrosophic crisp set in a set X and defined the inclusion between two neutrosophic crisp sets, the intersection [union] of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic crisp empty [resp., whole] set as more than two types. And they studied some properties related to neutrosophic crisp set operations. However, by selecting only one type, we define the inclusion, the intersection [union], and the complement of a neutrosophic crisp set, the neutrosophic crisp empty [resp., whole] set again and find some properties.

Definition 3.1. Let X be a non-empty set. Then A is called a neutrosophic crisp set in X if A has the form A = (A_1, A_2, A_3), where A_1, A_2, and A_3 are subsets of X.

The neutrosophic crisp empty [resp., whole] set, denoted by φ_N [resp., X_N] is an NCS in X defined by φ_N = (φ, φ, X) [resp., X_N = (X, X, φ)]. We will denote the set of all NCSs in X as NCS(X).

In particular, Salama and Smarandache [17] classified a neutrosophic crisp set as the follows:

(i) neutrosophic crisp set of Type 1 (in short, NCS-Type 1), if it satisfies

\[ A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi, \]

(ii) neutrosophic crisp set of Type 2 (in short, NCS-Type 2), if it satisfies

\[ A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X, \]

(iii) neutrosophic crisp set of Type 3 (in short, NCS-Type 3), if it satisfies

\[ A_1 \cap A_2 \cap A_3 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X. \]

We will denote the set of all NCSs-Type 1 [resp., Type 2 and Type 3] as NCS_1(X) [resp., NCS_2(X) and NCS_3(X)].

Definition 3.2. Let \( A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X). \) Then

(i) A is said to be contained in B, denoted by A ⊆ B, if

\[ A_1 \subseteq B_1, A_2 \subseteq B_2 \text{ and } A_3 \supseteq B_3, \]

(ii) A is said to equal to B, denoted by A = B, if

\[ A \subseteq B \text{ and } B \subseteq A, \]

(iii) the complement of A, denoted by \( A^c, \) is an NCS in X defined as:

\[ A^c = (A_3, A_2^c, A_1), \]
Proposition 3.3. Let \( \phi \) defined as:
\[
\phi = (1) \subset NCS(\mathbb{X}),
\]
then \( \phi_N \subset A \subset X_N \).

Proof. (1) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).
(2) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).
(3) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).
(4) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).
(5) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).
(6) \( \phi_N \subset A \subset X_N \), then \( \phi_N \subset A \subset X_N \).

Also the followings are the immediate results of Definition 3.2.

Proposition 3.4. Let \( A, B, C \in NCS(X) \). Then

(1) \( A \cap B = A \cap B \),
(2) \( A \cap B = A \cap B \),
(3) \( A \cap B = A \cap B \),
(4) \( A \cap B = A \cap B \),
(5) \( A \cap B = A \cap B \),
(6) \( A \cap B = A \cap B \),
(7) \( A^c = A^c \),
(8a) \( A \cup \phi = A \),
(8b) \( A \cap X_N = X_N \),
(8c) \( X_N^c = \phi_N \),
(8d) in general, \( A \cup A^c \neq X_N \), \( A \cap A^c \neq \phi_N \).

Proposition 3.5. Let \( A \in NCS(X) \) and let \( (A_j)_{j \in J} \subset NCS(X) \). Then

(1) \( (\bigcap A_j)^c = \bigcup A_j^c \), \( (\bigcup A_j)^c = \bigcap A_j^c \),
(2) \( A \cap (\bigcup A_j) = \bigcup (A \cap A_j) \), \( A \cap (\bigcap A_j) = \bigcap (A \cap A_j) \).

Proof. (1) \( A_j = (A_{j,1}, A_{j,2}, A_{j,3}) \). Then \( \bigcap A_j = (\bigcap A_{j,1} \cap \bigcap A_{j,2} \cap \bigcap A_{j,3} \bigcap) \). Thus \( (\bigcap A_j)^c = (\bigcup A_{j,3} \cup A_{j,1}^c, \bigcap A_{j,2}^c, \bigcap A_{j,3}) = (\bigcup A_{j,3}^c, \bigcup A_{j,1}^c, \bigcup A_{j,2}^c) \).
Similarly, the second part is proved.
(2) Let \( A = (A_1, A_2, A_3) \). Then \( A \cap (\bigcap A_j) = (A_1 \cap (\bigcap A_{j,1}), A_2 \cap (\bigcap A_{j,2}), A_3 \cap (\bigcap A_{j,3})) \).
Definition 3.6. Let \( f : X \to Y \) be a mapping, and let \( A = (A_1, A_2, A_3) \in \text{NCS}(X) \) and \( B = (B_1, B_2, B_3) \in \text{NCS}(Y) \). Then
(i) the image of \( A \) under \( f \), denoted by \( f(A) \), is an NCS in \( Y \) defined as:
\[
f(A) = (f(A_1), f(A_2), f(A_3)),
\]
(ii) the preimage of \( B \), denoted by \( f^{-1}(B) \), is an NCS in \( X \) defined as:
\[
f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3)).
\]

Proposition 3.7. Let \( f : X \to Y \) be a mapping and let \( A, B, C \in \text{NCS}(X) \), \((A_j)_{j \in J} \subset \text{NCS}(X)\) and \( D, E, F \in \text{NCS}(Y) \), \((D_k)_{k \in K} \subset \text{NCS}(Y) \). Then the followings hold:
1. if \( B \subset C \), then \( f(B) \subset f(C) \) and if \( E \subset F \), then \( f^{-1}(E) \subset f^{-1}(F) \).
2. \( A \subset f^{-1}(f(A)) \) and if \( f \) is injective, then \( A = f^{-1}(f(A)) \).
3. \( f(f^{-1}(D)) \subset D \) and if \( f \) is surjective, then \( f(f^{-1}(D)) = D \).
4. \( f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k), f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k) \).
5. \( f(A_j) = \bigcup f(A_j), f(\bigcap A_j) \subset \bigcap f(A_j) \).
6. \( f(A) = \phi_N \) if and only if \( A = \phi_N \) and hence \( f(\phi_N) = \phi_N \), in particular if \( f \)
is surjective, then \( f(X_N) = Y_N \).
7. \( f^{-1}(Y_N) = Y_N, f^{-1}(\phi_N) = \phi \).

Definition 3.8 ([17]). Let \( A = (A_1, A_2, A_3) \in \text{NCS}(X) \), where \( X \) is a set having at least distinct three points. Then \( A \) is called a neutrosophic crisp point (in short, NCP) in \( X \), if \( A_1, A_2 \) and \( A_3 \) are distinct singleton sets in \( X \).

Let \( A_1 = \{p_1\} \), \( A_2 = \{p_2\} \) and \( A_3 = \{p_3\} \), where \( p_1 \neq p_2 \neq p_3 \in X \). Then \( A = (A_1, A_2, A_3) \) is an NCP in \( X \). In this case, we will denote \( A \) as \( p = (p_1, p_2, p_3) \). Furthermore, we will denote the set of all NCPs in \( X \) as \( \text{NCP}(X) \).

Definition 3.9. Let \( A = (A_1, A_2, A_3) \in \text{NCS}(X) \) and let \( p = (p_1, p_2, p_3) \in \text{NCP}(X) \). Then \( p \) is said to belong to \( A \), denoted by \( p \in A \), if \( \{p_1\} \subset A_1, \{p_2\} \subset A_2 \) and \( \{p_3\} \subset A_3 \), i.e., \( p_1 \in A_1, p_2 \in A_2 \) and \( p_3 \in A_3 \).

Proposition 3.10. Let \( A = (A_1, A_2, A_3) \in \text{NCS}(X) \). Then
\[
A = \bigcup \{p \in \text{NCP}(X) : p \in A\}.
\]

Proof. Let \( p = (p_1, p_2, p_3) \in \text{NCP}(X) \). Then
\[
\bigcup \{p \in \text{NCP}(X) : p \in A\} = (\bigcup \{p_1 \in X : p_1 \in A_1\}, \bigcup \{p_2 \in X : p_2 \in A_2\}, \bigcap \{p_3 \in X : p_3 \in A_3\}) = A.
\]

Proposition 3.11. Let \( A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in \text{NCS}(X) \). Then \( A \subset B \) if and only if \( p \in B \), for each \( p \in A \).

Proof. Suppose \( A \subset B \) and let \( p = (p_1, p_2, p_3) \in A \). Then
\[
A_1 \subset B_1, A_2 \subset B_2, A_3 \subset B_3
\]
Proposition 3.12. Let \((A_j)_{j\in J} \subset NCS(X)\) and let \(p \in NCP(X)\).

1. \(p \in \bigcap A_j\) if and only if \(p \in A_j\) for each \(j \in J\).
2. \(p \in \bigcup A_j\) if and only if there exists \(j \in J\) such that \(p \in A_j\).

Proof. Let \(A_j = (A_{j,1}, A_{j,2}, A_{j,3})\) for each \(j \in J\) and let \(p = (p_1, p_2, p_3)\).

1. Suppose \(p \in \bigcap A_j\). Then \(p_1 \in \bigcap A_{j,1}, p_2 \in \bigcap A_{j,2}, p_3 \in \bigcup A_{j,3}\). Thus \(p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}\), for each \(j \in J\). So \(p \in A_j\) for each \(j \in J\).

We can easily see that the sufficient condition holds.

2. Suppose the necessary condition holds. Then there exists \(j \in J\) such that

\[
p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}.
\]

Thus \(p \in \bigcup A_{j,1}, p_2 \in \bigcup A_{j,2}, p_3 \in (\bigcap A_{j,3})^c\). So \(p \in \bigcup A_j\).

We can easily prove that the necessary condition holds. \(\square\)

Definition 3.13. Let \(f : X \to Y\) be an injective mapping, where \(X, Y\) are sets having at least distinct three points. Let \(p = (p_1, p_2, p_3) \in NCP(X)\). Then the image of \(p\) under \(f\), denoted by \(f(p)\), is an NCP in \(Y\) defined as:

\[
f(p) = (f(p_1), f(p_2), f(p_3)).
\]

Remark 3.14. In Definition 3.13, if either \(X\) or \(Y\) has two points, or \(f\) is not injective, then \(f(p)\) is not an NCP in \(Y\).

Definition 3.15 ([17]). Let \(A = (A_1, A_2, A_3) \in NCS(X)\) and \(B = (B_1, B_2, B_3) \in NCS(Y)\). Then the Cartesian product of \(A\) and \(B\), denoted by \(A \times B\), is an NCS in \(X \times Y\) defined as: \(A \times B = (A_1 \times B_1, A_2 \times B_2, A_3 \times B_3)\).

4. Properties of NCSet

Definition 4.1. A pair \((X, A)\) is called a neutrosophic crisp space (in short, NCSp), if \(A \in NCS(X)\).

Definition 4.2. A pair \((X, A)\) is called a neutrosophic crisp space-Type \(j\) (in short, NCSp-Type \(j\)), if \(A \in NCS_j(X)\), \(j = 1, 2, 3\).

Definition 4.3. Let \((X, A_X), (Y, A_Y)\) be two NCSp or NCSp-Type \(j\), \(j = 1, 2, 3\) and let \(f : X \to Y\) be a mapping. Then \(f : (X, A_X) \to (Y, A_Y)\) is called a morphism, if \(A_X \subset f^{-1}(A_Y)\), equivalently,

\[
A_{X,1} \subset f^{-1}(A_{Y,1}), A_{X,2} \subset f^{-1}(A_{Y,2}) \text{ and } A_{X,3} \supset f^{-1}(A_{Y,3}),
\]

where \(A_X = (A_X, A_{X,2}, A_{X,3})\) and \(A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3})\).

In particular, \(f : (X, A_X) \to (Y, A_Y)\) is called an epimorphism [resp., a monomorphism and an isomorphism], if it is surjective [resp., injective and bijective].

From Definitions 3.9, 4.3 and Proposition 3.11, it is obvious that

\[
f : (X, A_X) \to (Y, A_Y)\]

is a morphism

if and only if

\[
p = (p_1, p_2, p_3) \in f^{-1}(A_Y), \text{ for each } p = (p_1, p_2, p_3) \in A_X, \text{ i.e.,}
\]

\[
f(p_1) \in A_{Y,1}, f(p_2) \in A_{Y,2}, f(p_3) \notin A_{Y,3}, \text{ i.e.,}
\]

\[
f(p_1) \subseteq A_{Y,1}, f(p_2) \subseteq A_{Y,2}, f(p_3) \supseteq A_{Y,3}.
\]
\[ f(p) = (f(p_1), f(p_2), f(p_3)) \in A_Y. \]

The following is an immediate result of Definitions 4.3.

**Proposition 4.4.** For each NCSp or each NCSp-Type \( j (X, A_X) \), \( j = 1, 2, 3 \), the identity mapping \( \text{id} : (X, A_X) \to (X, A_X) \) is a morphism.

**Proposition 4.5.** Let \((X, A_X), (Y, A_Y), (Z, A_Z)\) be NCSp or NCSp-Type \( j, j = 1, 2, 3 \) and let \( f : X \to Y, g : Y \to Z \) be mappings. If \( f : (X, A_X) \to (Y, A_Y) \) and \( f : (Y, A_Y) \to (Z, A_Z) \) are morphisms, then \( g \circ f : (X, A_X) \to (Z, A_Z) \) is a morphism.

**Proof.** Let \( A_X = (A_{X,1}, A_{X,2}, A_{X,3}) \), \( A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3}) \), and \( A_Z = (A_{Z,1}, A_{Z,2}, A_{Z,3}) \). Then by the hypotheses, \( A_X \subset f^{-1}(A_Y) \) and \( A_Y \subset g^{-1}(A_Z) \). Thus by Definition 4.3,
\[
A_{X,1} \subset f^{-1}(A_{Y,1}), A_{X,2} \subset f^{-1}(A_{Y,2}), A_{X,3} \supset f^{-1}(A_{Y,3})
\]
and
\[
A_{Y,1} \subset g^{-1}(A_{Z,1}), A_{Y,2} \subset g^{-1}(A_{Z,2}), A_{Y,3} \supset g^{-1}(A_{Z,3}).
\]
So \( A_{X,1} \subset f^{-1}(g^{-1}(A_{Z,1})), A_{X,2} \subset f^{-1}(g^{-1}(A_{Z,2})), A_{X,3} \supset f^{-1}(g^{-1}(A_{Z,3})). \) Hence \( A_{X,1} \subset (g \circ f)^{-1}(A_{Z,1}), A_{X,2} \subset (g \circ f)^{-1}(A_{Z,2}), A_{X,3} \supset (g \circ f)^{-1}(A_{Z,3}). \) Therefore \( g \circ f \) is a morphism. \( \square \)

From Propositions 4.4 and 4.5, we can form the concrete category \( \text{NCSSet} \) [resp., \( \text{NCSet}_j \)] consisting of NCs [resp., -Type \( j, j = 1, 2, 3 \)] and morphisms between them. Every \( \text{NCSSet} \) [resp., \( \text{NCSet}_j, j = 1, 2, 3 \)]-morphism will be called a \( \text{NCSSet} \) [resp., \( \text{NCSet}_j, j = 1, 2, 3 \)]-mapping.

**Theorem 4.6.** The category \( \text{NCSSet} \) is topological over \( \text{Set} \).

**Proof.** Let \( X \) be any set and let \( \{(X_j, A_j)\}_{j \in J} \) be any families of NCs indexed by a class \( J \). Suppose \( (f_j : X \to (X_j, A_j))_j \) is a source of ordinary mappings. We define the NCS \( A_X \) in \( X \) by \( A_X = \bigcap f_j^{-1}(A_j) \) and \( A_X = (A_{X,1}, A_{X,2}, A_{X,3}). \)

Then clearly,
\[
A_{X,1} = f_j^{-1}(A_{j,1}), A_{X,2} = f_j^{-1}(A_{j,2}), A_{X,3} = f_j^{-1}(A_{j,3}).
\]

Thus \( (X, A_X) \) is an NCS and \( A_{X,1} \subset f_j^{-1}(A_{j,1}), A_{X,2} \subset f_j^{-1}(A_{j,2}) \) and \( A_{X,3} \supset f_j^{-1}(A_{j,3}). \) So each \( f_j : (X, A_X) \to (X_j, A_j) \) is an \( \text{NCSSet} \)-morphism.

Now let \( (Y, A_Y) \) be any NCSp and suppose \( g : Y \to X \) is an ordinary mapping for which \( f_j \circ g : (Y, A_Y) \to (X_j, A_j) \) is a \( \text{NCSSet} \)-mapping for each \( j \in J \). Then for each \( j \in J \), \( A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j)) \). Thus
\[
A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j)) = g^{-1}(A_X).
\]

So \( g : (Y, A_Y) \to (X, A_X) \) is an \( \text{NCSSet} \)-mapping. Hence \( (f_j : (X, A_X) \to (X_j, A_j))_j \) is an initial source in \( \text{NCSSet} \). This completes the proof. \( \square \)

**Example 4.7.** (1) Let \( X \) be a set, let \( (Y, A_Y) \) be an NCSp and let \( f : X \to Y \) be an ordinary mapping. Then clearly, there exists a unique NCS \( A_X \) in \( X \) for which \( f : (X, A_X) \to (Y, A_Y) \) is an \( \text{NCSSet} \)-mapping. In fact, \( A_X = f^{-1}(A_Y) \).

In this case, \( A_X \) is called the inverse image under \( f \) of the NCS structure \( A_Y \).
Hence, for each \((X_j, A_j)\)\(j\in J\) be any family of NCSp and let \(X = \Pi_{j\in J} X_j\). For each \(j \in J\), let \(pr_j : X \rightarrow X_j\) be the ordinary projection. Then there exists a unique NCS \(A_X\) in \(X\) for which \(pr_j : (X, A_X) \rightarrow (X_j, A_j)\) is an NCSet-mapping for each \(j \in J\).

In this case, \(A_X\) is called the product of \((A_j)_{j \in J}\), denoted by \(A_X = \Pi A_j = (\Pi A_{j,1}, \Pi A_{j,2}, \Pi A_{j,3})\) and \((\Pi X_j, \Pi A_j)\) is the product NCSp of \((X_j, A_j)_{j \in J}\).

In fact, if \(J = \{1, 2\}\), then \(A_1 \times A_2 = (A_{1,1} \times A_{2,1}, A_{1,2} \times A_{2,2}, A_{1,3} \times A_{2,3})\) and \(A_1 = (A_{1,1}, A_{1,2}, A_{1,3}) \in NCSet(X_1)\) and \(A_2 = (A_{2,1}, A_{2,2}, A_{2,3}) \in NCSet(X_2)\).

The following is obvious from Result 2.2. But we show directly it.

**Corollary 4.8.** The category NCSet is cotopological over Set.

**Proof.** Let \(X\) be any set and let \((X_j, A_j)_{j \in J}\) be any family of NCSp indexed by a class \(J\). Suppose \((f_j : X_j \rightarrow X)_{j \in J}\) is a sink of ordinary mappings. We define \(A_X\) as \(A_X = \bigcup f_j(A_j)\), where \(A_X = (A_{X,1}, A_{X,2}, A_{X,3})\) and \(A_j = (A_{j,1}, A_{j,2}, A_{j,3})\). Then clearly, \(A_X \in NCSet(X)\) and each \(f_j : (X_j, A_j) \rightarrow (X, A_X)\) is an NCSet-mapping.

Now for each NCSet \((Y, A_Y)\), let \(g : X \rightarrow Y\) be an ordinary mapping for which each \(g \circ f_j : (X_j, A_j) \rightarrow (Y, A_Y)\) is an NCSet-mapping. Then clearly for each \(j \in J\), \(A_j \subset (g \circ f_j)^{-1}(A_Y)\), i.e., \(A_j \subset f_j^{-1}(g^{-1}(A_Y))\). Thus \(\bigcup A_j \subset \bigcup f_j^{-1}(g^{-1}(A_Y))\). So \(f_j(\bigcup A_j) \subset f_j(\bigcup f_j^{-1}(g^{-1}(A_Y)))\).

By Proposition 3.7 and the definition of \(A_X\),

\[
f_j(\bigcup A_j) = \bigcup f_j(A_j) = A_X
\]

and

\[
f_j(\bigcup f_j^{-1}(g^{-1}(A_Y))) = \bigcup (f_j \circ f_j^{-1})(g^{-1}(A_Y)) = g^{-1}(A_Y).
\]

Hence \(A_X \subset g^{-1}(A_Y)\). Therefore \(g : (X, A_X) \rightarrow (Y, A_Y)\) is an NCSet-mapping. This completes the proof.

The following is proved similarly as the proof of Theorem 4.6.

**Corollary 4.9.** The category NCSet\(_j\) is topological over Set for \(j = 1, 2, 3\).

The following is proved similarly as the proof of Corollary 4.8.

**Corollary 4.10.** The category NCSet\(_j\) is cotopological over Set for \(j = 1, 2, 3\).

**Theorem 4.11.** Final episinks in NCSet are preserved by pullbacks.

**Proof.** Let \((g_j : (X_j, A_j) \rightarrow (Y, A_Y))_{j \in J}\) be any final episink in NCSet and let \(f : (W, A_W) \rightarrow (Y, A_Y)\) be any NCSet-mapping. For each \(j \in J\), let \(U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}\).

For each \(j \in J\), we define the NCS \(A_{U_j} = (A_{U,j,1}, A_{U,j,2}, A_{U,j,3})\) in \(U_j\) by:

\[
A_{U,j,1} = A_{W,1} \times A_{j,1}, \quad A_{U,j,2} = A_{W,2} \times A_{j,2}, \quad A_{U,j,3} = A_{W,3} \times A_{j,3}.
\]

For each \(j \in J\), let \(e_j : U_j \rightarrow W\) and \(p_j : U_j \rightarrow X_j\) be ordinary projections of \(U_j\). Then clearly,

\[
A_{U,j,1} \subset e_j^{-1}(A_{W,1}), \quad A_{U,j,2} \subset e_j^{-1}(A_{W,2}), \quad A_{U,j,3} \supset e_j^{-1}(A_{W,3})
\]
and

\[ A_{U_j,1} \subset p_j^{-1}(A_{j,1}), A_{U_j,2} \subset p_j^{-1}(A_{j,2}), A_{U_j,3} \supset p_j^{-1}(A_{j,3}). \]

Thus \( A_{U_j} \subset e_j^{-1}(A_W) \) and \( A_{U_j} \subset p_j^{-1}(A_j) \). So \( e_j : (U_j, A_{U_j}) \to (W, A_W) \) and \( p_j : (U_j, A_{U_j}) \to (X_j, A_j) \) are \textbf{NCSet}-mappings. Moreover, \( g_h \circ p_h = f \circ e_j \) for each \( j \in J \), i.e., the diagram is a pullback square in \textbf{NCSet}:

\[
\begin{array}{ccc}
(U_j, A_{U_j}) & \xrightarrow{p_j} & (X_j, A_j) \\
\downarrow e_j & & \downarrow g_j \\
(W, A_W) & \xrightarrow{f} & (Y, A_Y).
\end{array}
\]

Now in order to prove that \((e_j)_j\) is an episink in \textbf{NCSet}, i.e., each \( e_j \) is surjective, let \( w \in W \). Since \((g_j)_j\) is an episink, there exists \( j \in J \) such that \( g_j(x_j) = f(w) \) for some \( x_j \in X_j \). Thus \((w, x_j) \in U_j \) and \( w = e_j(w, x_j) \). So \((e_j)_j\) is an episink in \textbf{NCSet}.

Finally, let us show that \((e_j)_j\) is final in \textbf{NCSet}. Let \( A^*_W \) be the final structure in \( W \) w.r.t. \((e_j)_j\) and let \( w = (w_1, w_2, w_3) \in A_W \). Since \( f : (W, A_W) \to (Y, A_Y) \) is an \textbf{NCSet}-mapping, by Definition 3.9,

\[
w_1 \in A_{W,1} \cap f^{-1}(A_{Y,1}), w_2 \in A_{W,2} \cap f^{-1}(A_{Y,2}) \text{ and } w_3 \in A_{W,3} \cap (f^{-1}(A_{Y,3}))^c.
\]

Thus

\[
w_1 \in A_{W,1}, f(w_1) \in A_{Y,1}, w_2 \in A_{W,2}, f(w_2) \in A_{Y,2} \text{ and } w_3 \in A_{W,3}, f(w_3) \in A_{Y,3}^c.
\]

Since \((g_j)_j\) is final,

\[
w_1 \in A_{W,1}, x_{j,1} \in \bigcup_{j,1} A_{j,1},
\]

\[
w_2 \in A_{W,2}, x_{j,2} \in \bigcup_{j,2} A_{j,2},
\]

and

\[
w_3 \in A_{W,3}, x_{j,3} \in \bigcap_{j,3} A_{j,3}^c.
\]

So \((w_1, x_{j,1}) \in A_{U_{j,1}}, (w_2, x_{j,2}) \in A_{U_{j,2}} \) and \((w_3, x_{j,3}) \in A_{U_{j,3}}^c \). Since \( A^*_W \) is the final structure in \( W \) w.r.t. \((e_j)_j, w \in A^*_W \), i.e., \( A_W \subset A^*_W \). On the other hand, since \((e_j : (U_j, A_{U_j}) \to (W, A_W))_j\) is final, \( I_W : (W, A_W) \to (W, A_W) \) is an \textbf{NCSet}-mapping and thus \( A^*_W \subset A_W \). Hence \( A^*_W = A_W \). Therefore \((e_j)_j\) is final. This completes the proof. \( \square \)

The following is proved similarly as the proof of Theorem 4.9.

\textbf{Corollary 4.12.} Final episinks in \textbf{NCSet}\(_j\) are preserved by pullbacks, for \( J = 1, 2, 3 \).

For any singleton set \( \{a\} \), \textbf{NCS} \( A_{\{a\}} \) [resp., \textbf{NCSet}-Type \( j \) \( A_{\{a\}} \)] on \( \{a\} \) is not unique, the category \textbf{NCSet} [resp., \textbf{NCSet}\(_j\)] for \( j = 1, 2, 3 \) is not properly fibred over \textbf{Set}. Then by Definition 2.6, Corollary 4.8 and Theorem 4.11 [resp., Corollaries 4.10 and 4.12], we have the following result.
Theorem 4.13. The category $\text{NCSet}$ [resp., $\text{NCSet}_j$ for $j = 1, 2, 3$] satisfies all the conditions of a topological universe over $\text{Set}$ except the terminal separator property.

The following is an immediate result of Definitions 3.9 and 3.15.

Proposition 4.14. Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \text{NCP}(X)$ and let $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in \text{NCS}(X)$. Then $(p, q) \in A \times B$ if and only if $(p_1, q_1) \in A_1 \times B_1, (p_2, q_2) \in A_2 \times B_2$ and $(p_3, q_3) \in (A_2 \times B_2)^c$, i.e., $p_3 \in A_3^c$ or $q_3 \in B_3^c$.

Theorem 4.15. The category $\text{NCSet}$ is Cartesian closed over $\text{Set}$.

Proof. It is clear that $\text{NCSet}$ has products by Theorem 4.6. Then it is sufficient to see that $\text{NCSet}$ has exponential objects.

For any NCSps $X = (X, A_X)$ and $Y = (Y, A_Y)$, let $Y^X$ be the set of all ordinary mappings from $X$ to $Y$. We define the NCS $A_{Y^X} = (A_{Y^X, 1}, A_{Y^X, 2}, A_{Y^X, 3})$ in $Y^X$ by: for each $f = (f_1, f_2, f_3) \in Y^X$, $f \in A_{Y^X}$ if and only if $f(x) \in A_Y$, for each $x = (x_1, x_2, x_3) \in \text{NCP}(X)$, i.e.,

$$f_1 \in A_{Y^X, 1}, f_2 \in A_{Y^X, 2}, f_3 \not\in A_{Y^X, 3}$$

if and only if

$$f_1(x_1) \in A_{Y, 1}, f_2(x_2) \in A_{Y, 2}, f_3(x_3) \not\in A_{Y, 3}.$$ 

In fact,

$$A_{Y^X, 1} = \{ f_1 \in Y^X : f_1(x_1) \in A_{Y, 1} \text{ for each } x_1 \in X \},$$

$$A_{Y^X, 2} = \{ f_2 \in Y^X : f_2(x_2) \in A_{Y, 2} \text{ for each } x_2 \in X \},$$

$$A_{Y^X, 3} = \{ f_3 \in Y^X : f_3(x_3) \not\in A_{Y, 3} \text{ for some } x_3 \in X \}.$$ 

Then clearly, $(Y^X, A_{Y^X})$ is an NCSp.

Let $Y^X = (Y^X, A_{Y^X})$. Then by the definition of $A_{Y^X}$,

$$A_{Y^X, 1} \subseteq f^{-1}(A_{Y, 1}), A_{Y^X, 2} \subseteq f^{-1}(A_{Y, 2})$$

and

$$A_{Y^X, 3} \supseteq f^{-1}(A_{Y, 3}).$$

We define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x,f) = f(x)$, for each $(x,f) \in X \times Y^X$.

Let $(x,f) \in A_X \times A_{Y^X}$, where $x = (x_1, x_2, x_3), f = (f_1, f_2, f_3)$. Then by Proposition 4.14 and the definition of $e_{X,Y}$,

$$(x_1, f_1) \in A_{X^1} \times A_{Y^X, 1}, (x_2, f_2) \in A_{X^2} \times A_{Y^X, 2}, (x_3, f_3) \in (A_{X, 3} \times A_{Y^X, 3})^c$$

and

$$e_{X,Y}(x_1, f_1) = f_1(x_1), e_{X,Y}(x_2, f_2) = f_2(x_2), e_{X,Y}(x_3, f_3) = f_3(x_3).$$

Thus by the definition of $A_{Y^X}$,

$$(x_1, f_1) \in f^{-1}(A_{Y, 1}) \times f^{-1}(A_{Y, 1}),$$

$$(x_2, f_2) \in f^{-1}(A_{Y, 2}) \times f^{-1}(A_{Y, 2}),$$

$$(x_3, f_3) \in f^{-1}(A_{Y, 3}) \times f^{-1}(A_{Y, 3})^c.$$

So $(x_1, f_1) \in e_{X,Y}^{-1}(A_{Y, 1}), (x_2, f_2) \in e_{X,Y}^{-1}(A_{Y, 2})$ and $(x_3, f_3) \in (e_{X,Y}^{-1}(A_{Y, 3}))^c$. Hence $A_X \times A_{Y^X} \subseteq e_{X,Y}^{-1}(A_Y)$. Therefore $e_{X,Y} : X \times Y^X \to Y$ is an $\text{NCSet}$-mapping.

For any $Z = (Z, A_Z) \in \text{NCSet}$, let $h : X \times Z \to Y$ be an $\text{NCSet}$-mapping. We define $h : Z \to Y^X$ by $[h(z)](x) = h(x, z)$, for each $z \in Z$ and each $x \in X$. Let $(x, z) \in A_X \times A_Z$, where $x = (x_1, x_2, x_3)$ and $z = (z_1, z_2, z_3)$. Since $h : X \times Z \to Y$ is an $\text{NCSet}$-mapping.
Thus by the definition of \( \overline{\text{A}} \), we have
\[
A_{X,1} \times A_{Z,1} \subseteq h^{-1}(A_{Y,1}), \quad A_{X,2} \times A_{Z,2} \subseteq h^{-1}(A_{Y,2}), \quad A_{X,3} \times A_{Z,3} \subseteq h^{-1}(A_{Y,3}).
\]
Then by Proposition 4.14,
\[
(x_1, z_1) \in h^{-1}(A_{Y,1}), \quad (x_2, z_2) \in h^{-1}(A_{Y,2}), \quad (x_3, z_3) \in h^{-1}(A_{Y,3})^c.
\]
Thus \( h((x_1, z_1)) \in A_{Y,1}, \quad h((x_2, z_2)) \in A_{Y,2}, \quad h((x_3, z_3)) \in (A_{Y,3})^c. \)
By the definition of \( h \),
\[
[h(z_1)](x_1) \in A_{Y,1}, \quad [h(z_2)](x_2) \in A_{Y,2}, \quad [h(z_3)](x_3) \in (A_{Y,3})^c.
\]
By the definition of \( h \times X \),
\[
[h(z_1)](A_{Z,1}) \subseteq A_{Y \times 1}, \quad [h(z_2)](A_{Z,2}) \subseteq A_{Y \times 2}, \quad [h(z_3)](A_{Z,3}) \subseteq A_{Y \times 3}.
\]
So \( A_Z \subseteq h^{-1}(A_{Y \times 2}) \). Hence \( h : Z \to Y^X \) is an \( \text{NCSet} \)-mapping. Furthermore, \( h \) is the unique \( \text{NCSet} \)-mapping such that \( e_{X,Y} \circ (1_X \times h) = h \). This completes the proof. \( \square \)

**The following is proved similarly as the proof of Theorem 4.15.**

**Corollary 4.16.** The category \( \text{NCSet}_j \) is Cartesian closed over \( \text{Set} \) for \( j = 1, 2, 3 \).

## 5. Conclusions

For a non-empty set \( X \), by defining a neutrosophic crisp set \( A = (A_1, A_2, A_3) \) and an intuitionistic crisp set \( A = (A_1, A_2) \) in \( X \), respectively as follows:

(i) \( A_1 \subseteq X, A_2 \subseteq X, A_3 \subseteq X \),
(ii) \( A_1 \subseteq A^c_2, A_3 \subseteq A^c_2 \),
and

(i) \( A_1 \subseteq X, A_2 \subseteq X \),
(ii) \( A_1 \subseteq A^c_2 \),
we can form another categories \( \text{NCSet}_* \) and \( \text{ICSet} \). Furthermore, we will study them in view points of a topological universe and obtain some relationship between them.

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Special types of bipolar single valued neutrosophic graphs

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Abstract. Neutrosophic theory has many applications in graph theory, bipolar single valued neutrosophic graphs (BSVNGs) is the generalization of fuzzy graphs and intuitionistic fuzzy graphs, SVNGs. In this paper we introduce some types of BSVNGs, such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and bipolar single valued neutrosophic line graphs (BSVNLGs), also investigate the isomorphism, co weak isomorphism and weak isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

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1. Introduction

Neutrosophic set theory (NS) is a part of neutrosophy which was introduced by Smarandache [43] from philosophical point of view by incorporating the degree of indeterminacy or neutrality as independent component for dealing problems with indeterminate and inconsistent information. The concept of neutrosophic set theory is a generalization of the theory of fuzzy set [50], intuitionistic fuzzy sets [5], interval-valued fuzzy sets [47], interval-valued intuitionistic fuzzy sets [6]. The concept of neutrosophic set is characterized by a truth-membership degree (T), an indeterminacy-membership degree (I) and a falsity-membership degree (f) independently, which are within the real standard or nonstandard unit interval \([-0, 1]\). Therefore, if their range is restrained within the real standard unit interval \([0, 1]\): Nevertheless, NSs are hard to be apply in practical problems since the values of the functions of truth, indeterminacy and falsity lie in \([-0, 1]\]. The single valued neutrosophic set was introduced for the first time by Smarandache [43].
of single valued neutrosophic sets is a subclass of neutrosophic sets in which the value of truth-membership, indeterminacy membership and falsity-membership degrees are intervals of numbers instead of the real numbers. Later on, Wang et al. [49] studied some properties related to single valued neutrosophic sets. The concept of neutrosophic sets and its extensions such as single valued neutrosophic sets, interval neutrosophic sets, bipolar neutrosophic sets and so on have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine and economic and can be found in [9, 15, 16, 30, 31, 32, 33, 34, 35, 36, 37, 51]. Graphs are the most powerful tool used in representing information involving relationship between objects and concepts. In a crisp graphs two vertices are either related or not related to each other, mathematically, the degree of relationship is either 0 or 1. While in fuzzy graphs, the degree of relationship takes values from \([0, 1]\). Atanassov [42] defined the concept of intuitionistic fuzzy graphs (IFGs) using five types of Cartesian products. The concept fuzzy graphs, intuitionistic fuzzy graphs and their extensions such interval valued fuzzy graphs, bipolar fuzzy graph, bipolar intuitionistic fuzzy graphs, interval valued intuitionistic fuzzy graphs, hesitancy fuzzy graphs, vague graphs and so on, have been studied deeply by several researchers in the literature. When description of the object or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy intuitionistic fuzzy, bipolar fuzzy, vague and interval valued fuzzy graphs. So, for this purpose, Smarandache [45] proposed the concept of neutrosophic graphs based on literal indeterminacy (I) to deal with such situations. Later on, Smarandache [44] gave another definition for neutrosophic graph theory using the neutrosophic truth-values \((T, I, F)\) without and constructed three structures of neutrosophic graphs: neutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Recently, Smarandache [46] proposed new version of neutrosophic graphs such as neutrosophic offgraph, neutrosophic bipolar/tripola/multipolar graph. Recently several researchers have studied deeply the concept of neutrosophic vertex-edge graphs and presented several extensions neutrosophic graphs. In [1, 2, 3], Akram et al. introduced the concept of single valued neutrosophic hypergraphs, single valued neutrosophic planar graphs, neutrosophic soft graphs and intuitionistic neutrosophic soft graphs. Then, followed the work of Brouni et al. [7, 8, 9, 10, 11, 12, 13, 14, 15], Malik and Hassan [38] defined the concept of single valued neutrosophic trees and studied some of their properties. Later on, Hassan et Malik [17] introduced some classes of bipolar single valued neutrosophic graphs and studied some of their properties, also the authors generalized the concept of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs in [19, 20]. In [23, 24] Hassan et Malik gave the important types of single (interval) valued neutrosophic graphs, another important classes of single valued neutrosophic graphs have been presented in [22] and in [25] Hassan et Malik introduced the concept of m-Polar single valued neutrosophic graphs and its classes. Hassan et al. [18, 21] studied the concept on regularity and total regularity of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs. Hassan et al. [26, 27, 28] discussed the isomorphism properties on SVNHG, BSVNHG and IVNHG. Nasir et al. [40] introduced a new type of graph called neutrosophic soft graphs and established a link between graphs
and neutrosophic soft sets. The authors also studied some basic operations of neutrosophic soft graphs such as union, intersection and complement. Nasir and Broumi [41] studied the concept of irregular neutrosophic graphs and investigated some of their related properties. Ashraf et al. [4], proposed some novel concepts of edge regular, partially edge regular and full edge regular single valued neutrosophic graphs and investigated some of their properties. Also the authors, introduced the notion of single valued neutrosophic digraphs (SVNDGs) and presented an application of SVNDG in multi-attribute decision making. Mehra and Singh [39] introduced a new concept of neutrosophic graph named single valued neutrosophic Signed graphs (SVNSGs) and examined the properties of this concept with suitable illustration. Ulucay et al. [48] proposed a new extension of neutrosophic graphs called neutrosophic soft expert graphs (NSEGS) and have established a link between graphs and neutrosophic soft expert sets and studies some basic operations of neutrosophic soft expert graphs such as union, intersection and complement. The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we introduce others types of BSVNGs such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs and these are all the strong BSVNGs, also we discuss their relations based on isomorphism, co weak isomorphism and weak isomorphism.

2. Preliminaries

In this section we recall some basic concepts on BSVNG. Let $G$ denotes BSVNG and $G^* = (V, E)$ denotes its underlying crisp graph.

**Definition 2.1** ([10]). Let $X$ be a crisp set, the single valued neutrosophic set (SVNS) $Z$ is characterized by three membership functions $T_Z(x), I_Z(x)$ and $F_Z(x)$ which are truth, indeterminacy and falsity membership functions, $\forall x \in X$

\[
T_Z(x), I_Z(x), F_Z(x) \in [0, 1].
\]

**Definition 2.2** ([10]). Let $X$ be a crisp set, the bipolar single valued neutrosophic set (BSVNS) $Z$ is characterized by membership functions $T^+_Z(x), I^+_Z(x), F^+_Z(x), T^-_Z(x), I^-_Z(x), F^-_Z(x)$. That is $\forall x \in X$

\[
T^+_Z(x), I^+_Z(x), F^+_Z(x) \in [0, 1],
\]

\[
T^-_Z(x), I^-_Z(x), F^-_Z(x) \in [-1, 0].
\]

**Definition 2.3** ([10]). A bipolar single valued neutrosophic graph (BSVNG) is a pair $G = (Y, Z)$ of $G^*$, where $Y$ is BSVNS on $V$ and $Z$ is BSVNS on $E$ such that

\[
T^+_Z(\beta, \gamma) \leq \min(T^+_Y(\beta), T^+_Y(\gamma)), \quad I^+_Z(\beta, \gamma) \geq \max(I^+_Y(\beta), I^+_Y(\gamma)),
\]

\[
I^-_Z(\beta, \gamma) \leq \min(I^-_Y(\beta), I^-_Y(\gamma)), \quad F^-_Z(\beta, \gamma) \leq \min(F^-_Y(\beta), F^-_Y(\gamma)),
\]

\[
F^+_Z(\beta, \gamma) \geq \max(F^+_Y(\beta), F^+_Y(\gamma)), \quad T^-_Z(\beta, \gamma) \geq \max(T^-_Y(\beta), T^-_Y(\gamma)),
\]

where

\[
0 \leq T^+_Z(\beta, \gamma) + I^+_Z(\beta, \gamma) + F^+_Z(\beta, \gamma) \leq 3
\]

\[
-3 \leq T^-_Z(\beta, \gamma) + I^-_Z(\beta, \gamma) + F^-_Z(\beta, \gamma) \leq 0
\]

$\forall \beta, \gamma \in V$. 

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In this case, $D$ is bipolar single valued neutrosophic relation (BSVNR) on $C$. The BSVNG $G=(Y,Z)$ is complete (strong) BSVNG, if
\[ T^+_Z(\beta \gamma) = \min(T^+_Y(\beta), T^+_Y(\gamma)), \quad I^+_Z(\beta \gamma) = \max(I^+_Y(\beta), I^+_Y(\gamma)), \]
\[ I^-_Z(\beta \gamma) = \min(I^-_Y(\beta), I^-_Y(\gamma)), \quad F^-_Z(\beta \gamma) = \min(F^-_Y(\beta), F^-_Y(\gamma)), \]
\[ F^+_Z(\beta \gamma) = \max(F^+_Y(\beta), F^+_Y(\gamma)), \quad T^-_Z(\beta \gamma) = \max(T^-_Y(\beta), T^-_Y(\gamma)), \]
\[ \forall \beta, \gamma \in \mathcal{V}(\forall \beta \gamma \in \mathcal{E}). \] The order of BSVNG $G=(A, B)$ of $G^*$, denoted by $O(G)$, is defined by
\[ O(G) = (O^+_T(G), O^-_T(G), O^+_F(G), O^-_F(G), O_T^{-}(G), O^{-}_F(G)), \]
where
\[ O^+_T(G) = \sum_{\alpha \in \mathcal{V}} T^+_A(\alpha), \quad O^+_F(G) = \sum_{\alpha \in \mathcal{V}} F^+_A(\alpha), \]
\[ O^-_T(G) = \sum_{\alpha \in \mathcal{V}} T^-_A(\alpha), \quad O^-_F(G) = \sum_{\alpha \in \mathcal{V}} F^-_A(\alpha). \]
The size of BSVNG $G=(A, B)$ of $G^*$, denoted by $S(G)$, is defined by
\[ S(G) = (S^+_T(G), S^-_T(G), S^+_F(G), S^-_F(G), S^+_T(G), S^-_T(G)), \]
where
\[ S^+_T(G) = \sum_{\beta \gamma \in \mathcal{E}} T^+_B(\beta \gamma), \quad S^-_T(G) = \sum_{\beta \gamma \in \mathcal{E}} T^-_B(\beta \gamma), \]
\[ S^+_F(G) = \sum_{\beta \gamma \in \mathcal{E}} F^+_B(\beta \gamma), \quad S^-_F(G) = \sum_{\beta \gamma \in \mathcal{E}} F^-_B(\beta \gamma). \]
The degree of a vertex $\beta$ in BSVNG $G=(A, B)$ of $G^*$, denoted by $d_G(\beta)$, is defined by
\[ d_G(\beta) = (d^+_T(\beta), d^-_T(\beta), d^+_F(\beta), d^-_F(\beta), d^-_T(\beta), d^-_F(\beta)), \]
where
\[ d^+_T(\beta) = \sum_{\beta \gamma \in \mathcal{E}} T^+_B(\beta \gamma), \quad d^-_T(\beta) = \sum_{\beta \gamma \in \mathcal{E}} T^-_B(\beta \gamma), \]
\[ d^+_F(\beta) = \sum_{\beta \gamma \in \mathcal{E}} F^+_B(\beta \gamma), \quad d^-_F(\beta) = \sum_{\beta \gamma \in \mathcal{E}} F^-_B(\beta \gamma). \]
3. Types of BSVNGs

In this section we introduce the special types of BSVNGs such as subdivision, middle and total and intersection BSVNGs, for this first we give the basic definitions of homomorphism, isomorphism, weak isomorphism and co weak isomorphism of BSVNGs which are very useful to understand the relations among the types of BSVNGs.

**Definition 3.1.** Let \( G_1 = (C_1, D_1) \) and \( G_2 = (C_2, D_2) \) be two BSVNGs of \( G^*_1 = (V_1, E_1) \) and \( G^*_2 = (V_2, E_2) \), respectively. Then the homomorphism \( \chi : G_1 \rightarrow G_2 \) is a mapping \( \chi : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\[
T^+_{C_1}(p) \leq T^+_{C_2}(\chi(p)), \quad I^+_{C_1}(p) \geq I^+_{C_2}(\chi(p)), \quad F^+_{C_1}(p) \geq F^+_{C_2}(\chi(p)),
\]

\[
T^-_{C_1}(p) \geq T^-_{C_2}(\chi(p)), \quad I^-_{C_1}(p) \leq I^-_{C_2}(\chi(p)), \quad F^-_{C_1}(p) \leq F^-_{C_2}(\chi(p)),
\]

\( \forall p \in V_1, \)

\[
T^+_{D_1}(pq) \leq T^+_{D_2}(\chi(p)\chi(q)), \quad T^-_{D_1}(pq) \geq T^-_{D_2}(\chi(p)\chi(q)),
\]

\[
I^+_{D_1}(pq) \geq I^+_{D_2}(\chi(p)\chi(q)), \quad I^-_{D_1}(pq) \leq I^-_{D_2}(\chi(p)\chi(q)),
\]

\[
F^+_{D_1}(pq) \geq F^+_{D_2}(\chi(p)\chi(q)), \quad F^-_{D_1}(pq) \leq F^-_{D_2}(\chi(p)\chi(q)),
\]

\( \forall pq \in E_1. \)

**Remark 3.3.** The weak isomorphism between two BSVNGs preserves the orders.

**Remark 3.4.** The weak isomorphism between two BSVNGs is a partial order relation.

**Definition 3.2.** Let \( G_1 = (C_1, D_1) \) and \( G_2 = (C_2, D_2) \) be two BSVNGs of \( G^*_1 = (V_1, E_1) \) and \( G^*_2 = (V_2, E_2) \), respectively. Then the weak isomorphism \( v : G_1 \rightarrow G_2 \) is a bijective mapping \( v : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\( v \) is a homomorphism such that

\[
T^+_{C_1}(p) = T^+_{C_2}(v(p)), \quad I^+_{C_1}(p) = I^+_{C_2}(v(p)), \quad F^+_{C_1}(p) = F^+_{C_2}(v(p)),
\]

\[
T^-_{C_1}(p) = T^-_{C_2}(v(p)), \quad I^-_{C_1}(p) = I^-_{C_2}(v(p)), \quad F^-_{C_1}(p) = F^-_{C_2}(v(p)),
\]

\( \forall p \in V_1. \)

**Remark 3.5.** The weak isomorphism between two BSVNGs is a partial order relation.

**Definition 3.5.** Let \( G_1 = (C_1, D_1) \) and \( G_2 = (C_2, D_2) \) be two BSVNGs of \( G^*_1 = (V_1, E_1) \) and \( G^*_2 = (V_2, E_2) \), respectively. Then the co-weak isomorphism \( \kappa : G_1 \rightarrow G_2 \) is a bijective mapping \( \kappa : V_1 \rightarrow V_2 \) which satisfies the following conditions:

\( \kappa \) is a homomorphism such that

\[
T^+_{D_1}(pq) = T^+_{D_2}(\kappa(p)\kappa(q)), \quad T^-_{D_1}(pq) = T^-_{D_2}(\kappa(p)\kappa(q)),
\]

\[
I^+_{D_1}(pq) = I^+_{D_2}(\kappa(p)\kappa(q)), \quad I^-_{D_1}(pq) = I^-_{D_2}(\kappa(p)\kappa(q)),
\]

\[
F^+_{D_1}(pq) = F^+_{D_2}(\kappa(p)\kappa(q)), \quad F^-_{D_1}(pq) = F^-_{D_2}(\kappa(p)\kappa(q)),
\]

\( \forall pq \in E_1. \)

**Remark 3.6.** The co-weak isomorphism between two BSVNGs preserves the sizes.

**Remark 3.7.** The co-weak isomorphism between BSVNGs is a partial order relation.
Definition 3.8. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the isomorphism $\psi : G_1 \to G_2$ is a bijective mapping $\forall$ is a BSVNS on $C$ and the isomorphism between two BSVNGs is an equivalence relation.

Remark 3.9. $\forall$ The subdivision SVNG be $\psi$ of their vertices.

Remark 3.10. $\forall$ The isomorphism between two BSVNGs preserves the orders and sizes.

Remark 3.11. $\forall$ The isomorphism between two BSVNGs preserves the degrees of their vertices.

Definition 3.12. The subdivision SVNG be $sd(G) = (C, D)$ of $G = (A, B)$, where $C$ is a BSVNS on $V \cup E$ and $D$ is a BSVNR on $C$ such that

(i) $C = A$ on $V$ and $C = B$ on $E$;

(ii) if $v \in V$ lie on edge $e \in E$, then

\[
\begin{align*}
T_D^+(ve) &= \min(T_A^+(v), T_B^+(e)), & T_D^+(ve) &= \max(T_A^+(v), I_B^+(e)) \\
I_D^+(ve) &= \min(I_A^+(v), I_B^+(e)), & F_D^+(ve) &= \min(F_A^+(v), F_B^+(e)) \\
F_D^+(ve) &= \max(F_A^+(v), F_B^+(e)), & T_D^-(ve) &= \max(T_A^-(v), T_B^-(e))
\end{align*}
\]

else

\[
D(ve) = O = (0, 0, 0, 0, 0, 0, 0).
\]

Example 3.13. Consider the BSVNG $G = (A, B)$ of a $G^* = (V, E)$, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc, r = ac\}$, the crisp graph of $G$ is shown in Fig. 1. The BSVNSs $A$ and $B$ are defined on $V$ and $E$ respectively which are defined in Table 1. The SDBSVNG $sd(G) = (C, D)$ of a BSVNG $G$, the underlying crisp graph of $sd(G)$ is given in Fig. 2. The BSVNSs $C$ and $D$ are defined in Table 2.
Figure 1. Crisp Graph of BSVNG.

Figure 2. Crisp Graph of SDBSVNG.

Table 2. BSVNSs of SDBSVNG.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>T^+_C</td>
<td>I^+_C</td>
<td>F^+_C</td>
<td>T^-_C</td>
<td>I^-_C</td>
<td>F^-_C</td>
</tr>
<tr>
<td>a</td>
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<td>0.4</td>
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<td>-0.1</td>
<td>-0.4</td>
</tr>
<tr>
<td>p</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>b</td>
<td>0.3</td>
<td>0.2</td>
<td>0.5</td>
<td>-0.5</td>
<td>-0.4</td>
<td>-0.6</td>
</tr>
<tr>
<td>q</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>c</td>
<td>0.4</td>
<td>0.7</td>
<td>0.6</td>
<td>-0.2</td>
<td>-0.6</td>
<td>-0.2</td>
</tr>
<tr>
<td>r</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>T^+_D</td>
<td>I^+_D</td>
<td>F^+_D</td>
<td>T^-_D</td>
<td>I^-_D</td>
<td>F^-_D</td>
</tr>
<tr>
<td>ap</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>pb</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>bq</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>qc</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>cr</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
<tr>
<td>ra</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

**Proposition 3.14.** Let $G$ be a BSVNG and $sd(G)$ be the SDBSVNG of a BSVNG $G$, then $O(sd(G)) = O(G) + S(G)$ and $S(sd(G)) = 2S(G)$.

**Remark 3.15.** Let $G$ be a complete BSVNG, then $sd(G)$ need not to be complete BSVNG.
Definition 3.16. The total bipolar single valued neutrosophic graph (TBSVNG) is \( T(G) = (C, D) \) of \( G = (A, B) \), where \( C \) is a BSVNS on \( V \cup E \) and \( D \) is a BSVNR on \( C \) such that

(i) \( C = A \) on \( V \) and \( C = B \) on \( E \),

(ii) if \( v \in V \) lie on edge \( e \in E \), then

\[
T_D^+(ve) = \min(T_A^+(v), T_B^+(e)), \quad I_D^+(ve) = \max(I_A^+(v), I_B^+(e))
\]

\[
T_D^-(ve) = \min(T_A^-(v), T_B^-(e)), \quad I_D^-(ve) = \max(I_A^-(v), I_B^-(e))
\]

\[
F_D^+(ve) = \max(F_A^+(v), F_B^+(e)), \quad T_D^-(ve) = \max(T_A^-(v), T_B^-(e))
\]

(iii) if \( \alpha \beta \in E \), then

\[
T_D^+(\alpha \beta) = T_B^+(\alpha \beta), \quad I_D^+(\alpha \beta) = I_B^+(\alpha \beta), \quad F_D^+(\alpha \beta) = F_B^+(\alpha \beta),
\]

\[
T_D^-(\alpha \beta) = T_B^-(\alpha \beta), \quad I_D^-(\alpha \beta) = I_B^-(\alpha \beta), \quad F_D^-(\alpha \beta) = F_B^-(\alpha \beta),
\]

(iv) if \( e, f \in E \) have a common vertex, then

\[
T_D^+(ef) = \min(T_A^+(e), T_B^+(f)), \quad I_D^+(ef) = \max(I_A^+(e), I_B^+(f))
\]

\[
T_D^-(ef) = \min(T_A^-(e), T_B^-(f)), \quad I_D^-(ef) = \max(I_A^-(e), I_B^-(f))
\]

\[
F_D^+(ef) = \max(F_A^+(e), F_B^+(f)), \quad T_D^-(ef) = \max(T_A^-(e), T_B^-(f))
\]

Example 3.17. Consider the Example 3.13 the TBSVNG \( T(G) = (C, D) \) of underlying crisp graph as shown in Fig. 3. The BSVNS \( C \) is given in Example 3.13. The BSVNS \( D \) is given in Table 3.

Proposition 3.18. Let \( G \) be a BSVNG and \( T(G) \) be the TBSVNG of a BSVNG \( G \), then \( O(T(G)) = O(G) + S(G) = O(sd(G)) \) and \( S(sd(G)) = 2S(G) \).

Proposition 3.19. Let \( G \) be a BSVNG, then \( sd(G) \) is weak isomorphic to \( T(G) \).
Table 3. BSVNS of TBSVNG.

<table>
<thead>
<tr>
<th></th>
<th>$T_D^+$</th>
<th>$T_D^-$</th>
<th>$I_D^+$</th>
<th>$I_D^-$</th>
<th>$F_D^+$</th>
<th>$F_D^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>$bc$</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>$ca$</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
<tr>
<td>$pq$</td>
<td>0.2</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>$qr$</td>
<td>0.1</td>
<td>0.8</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.8</td>
</tr>
<tr>
<td>$rp$</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.6</td>
</tr>
<tr>
<td>$ap$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>$pb$</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>$bq$</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>$qc$</td>
<td>0.3</td>
<td>0.8</td>
<td>0.6</td>
<td>-0.1</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>$cr$</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
<tr>
<td>$ra$</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

**Definition 3.20.** The middle bipolar single valued neutrosophic graph (MBSVNG) $M(G) = (C, D)$ of $G$, where $C$ is a BSVNS on $V \cup E$ and $D$ is a BSVNR on $C$ such that

(i) $C = A$ on $V$ and $C = B$ on $E$, else $C = O = (0, 0, 0, 0, 0, 0)$,

(ii) if $v \in V$ lie on edge $e \in E$, then

\[
T_D^+(ve) = T_B^+(e), \quad I_D^+(ve) = I_B^+(e), \quad F_D^+(ve) = F_B^+(e)
\]

\[
T_D^-(ve) = T_B^-(e), \quad I_D^-(ve) = I_B^-(e), \quad F_D^-(ve) = F_B^-(e)
\]

else

\[D(ve) = O = (0, 0, 0, 0, 0, 0),\]

(iii) if $u, v \in V$, then

\[D(uv) = O = (0, 0, 0, 0, 0, 0),\]

(iv) if $e, f \in E$ and $e$ and $f$ are adjacent in $G$, then

\[
T_D^+(ef) = T_B^+(uv), \quad I_D^+(ef) = I_B^+(uv), \quad F_D^+(ef) = F_B^+(uv)
\]

\[
T_D^-(ef) = T_B^-(uv), \quad I_D^-(ef) = I_B^-(uv), \quad F_D^-(ef) = F_B^-(uv).
\]

**Example 3.21.** Consider the BSVNG $G = (A, B)$ of a $G^*$, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc\}$ the underlaying crisp graph is shown in Fig. 4. The BSVNSs $A$ and $B$ are defined in Table 4. The crisp graph of MBSVNG $M(G) = (C, D)$ is shown in Fig. 5. The BSVNSs $C$ and $D$ are given in Table 5.

**Remark 3.22.** Let $G$ be a BSVNG and $M(G)$ be the MBSVNG of a BSVNG $G$, then $O(M(G)) = O(G) + S(G)$.

**Remark 3.23.** Let $G$ be a BSVNG, then $M(G)$ is a strong BSVNG.

**Remark 3.24.** Let $G$ be complete BSVNG, then $M(G)$ need not to be complete BSVNG.

**Proposition 3.25.** Let $G$ be a BSVNG, then $sd(G)$ is weak isomorphic with $M(G)$.
Table 4. BSVNSs of BSVNG.

<table>
<thead>
<tr>
<th></th>
<th>$T^+_A$</th>
<th>$I^+_A$</th>
<th>$F^+_A$</th>
<th>$T^-_A$</th>
<th>$I^-_A$</th>
<th>$F^-_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.1</td>
<td>-0.3</td>
</tr>
<tr>
<td>$b$</td>
<td>0.7</td>
<td>0.6</td>
<td>0.3</td>
<td>-0.3</td>
<td>-0.3</td>
<td>-0.2</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9</td>
<td>0.7</td>
<td>0.2</td>
<td>-0.5</td>
<td>-0.4</td>
<td>-0.6</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
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<th>$T^+_B$</th>
<th>$I^+_B$</th>
<th>$F^+_B$</th>
<th>$T^-_B$</th>
<th>$I^-_B$</th>
<th>$F^-_B$</th>
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<td>-0.3</td>
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<tr>
<td>$q$</td>
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<td>0.7</td>
<td>-0.3</td>
<td>-0.5</td>
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</tr>
</tbody>
</table>

Table 5. BSVNSs of MBSVNG.

<table>
<thead>
<tr>
<th></th>
<th>$T^+_C$</th>
<th>$I^+_C$</th>
<th>$F^+_C$</th>
<th>$T^-_C$</th>
<th>$I^-_C$</th>
<th>$F^-_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>-0.2</td>
<td>-0.1</td>
<td>-0.3</td>
</tr>
<tr>
<td>$b$</td>
<td>0.7</td>
<td>0.6</td>
<td>0.3</td>
<td>-0.3</td>
<td>-0.3</td>
<td>-0.2</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9</td>
<td>0.7</td>
<td>0.2</td>
<td>-0.5</td>
<td>-0.4</td>
<td>-0.6</td>
</tr>
<tr>
<td>$e_1$</td>
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<td>0.6</td>
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<td>-0.3</td>
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<tr>
<td>$e_2$</td>
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<table>
<thead>
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<th></th>
<th>$T^+_D$</th>
<th>$I^+_D$</th>
<th>$F^+_D$</th>
<th>$T^-_D$</th>
<th>$I^-_D$</th>
<th>$F^-_D$</th>
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</thead>
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<td>-0.6</td>
</tr>
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<td>-0.1</td>
<td>-0.4</td>
<td>-0.3</td>
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<td>-0.1</td>
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<td>-0.3</td>
</tr>
<tr>
<td>$bq$</td>
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<td>0.6</td>
<td>0.6</td>
<td>-0.3</td>
<td>-0.5</td>
<td>-0.6</td>
</tr>
<tr>
<td>$cq$</td>
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<td>0.7</td>
<td>-0.3</td>
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<td>-0.6</td>
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</table>

Figure 4. Crisp Graph of BSVNG.

Figure 5. Crisp Graph of MBSVNG.
Proposition 3.26. Let $G$ be a BSVNG, then $M(G)$ is weak isomorphic with $T(G)$.

Proposition 3.27. Let $G$ be a BSVNG, then $T(G)$ is isomorphic with $G \cup M(G)$.

Definition 3.28. Let $P(X) = (X, Y)$ be the intersection graph of a $G^*$, let $C_1$ and $D_1$ be BSVNSs on $V$ and $E$, respectively and $C_2$ and $D_2$ be BSVNSs on $X$ and $Y$ respectively. Then bipolar single valued neutrosophic intersection graph (BSNVG) of a BSVNG $G = (C_1, D_1)$ is a BSVNG $P(G) = (C_2, D_2)$ such that,

$$T_{C_2}^+(X_i) = T_{C_1}^+(v_i), \quad T_{C_2}^-(X_i) = T_{C_1}^-(v_i), \quad F_{C_2}^+(X_i) = F_{C_1}^+(v_i),$$

$$T_{D_2}^+(X_i, X_j) = T_{D_1}^+(v_i, v_j), \quad T_{D_2}^-(X_i, X_j) = T_{D_1}^-(v_i, v_j),$$

$$I_{D_2}^+(X_i, X_j) = I_{D_1}^+(v_i, v_j), \quad I_{D_2}^-(X_i, X_j) = I_{D_1}^-(v_i, v_j),$$

$$F_{D_2}^+(X_i, X_j) = F_{D_1}^+(v_i, v_j), \quad F_{D_2}^-(X_i, X_j) = F_{D_1}^-(v_i, v_j)$$

for all $X_i, X_j \in X$ and $X_i, X_j \in Y$.

Proposition 3.29. Let $G = (A_1, B_1)$ be a BSVNG of $G^* = (V, E)$, and let $P(G) = (A_2, B_2)$ be a BSVNG of $P(S)$. Then BSNVG is a also BSVNG and BSVNG is always isomorphic to BSNVG.

Proof. By the definition of BSNVG, we have

$$T_{B_2}^+(S_i S_j) = T_{B_1}^+(v_i v_j) \leq \min(T_{A_2}^+(v_i), T_{A_1}^+(v_j)) = \min(T_{A_2}^+(S_i), T_{A_1}^+(S_j)),$$

$$I_{B_2}^+(S_i S_j) = I_{B_1}^+(v_i v_j) \geq \max(I_{A_2}^+(v_i), I_{A_1}^+(v_j)) = \max(I_{A_2}^+(S_i), I_{A_1}^+(S_j)),$$

$$F_{B_2}^+(S_i S_j) = F_{B_1}^+(v_i v_j) \geq \max(F_{A_2}^+(v_i), F_{A_1}^+(v_j)) = \max(F_{A_2}^+(S_i), F_{A_1}^+(S_j)),$$

$$T_{B_2}^-(S_i S_j) = T_{B_1}^-(v_i v_j) \geq \max(T_{A_2}^-(v_i), T_{A_1}^-(v_j)) = \max(T_{A_2}^-(S_i), T_{A_1}^-(S_j)),$$

$$I_{B_2}^-(S_i S_j) = I_{B_1}^-(v_i v_j) \leq \max(I_{A_2}^-(v_i), I_{A_1}^-(v_j)) = \max(I_{A_2}^-(S_i), I_{A_1}^-(S_j)),$$

$$F_{B_2}^-(S_i S_j) = F_{B_1}^-(v_i v_j) \leq \max(F_{A_2}^-(v_i), F_{A_1}^-(v_j)) = \max(F_{A_2}^-(S_i), F_{A_1}^-(S_j))$$

This shows that BSNVG is a BSVNG.

Next define $f : V \to S$ by $f(v_i) = S_i$ for $i = 1, 2, 3, \ldots, n$ clearly $f$ is bijective. Now $v_i v_j \in E$ if and only if $S_i S_j \in T$ and $T = \{f(v_i) f(v_j) : v_i v_j \in E\}$. Also

$$T_{A_2}^+(f(v_i)) = T_{A_1}^+(S_i), \quad I_{A_2}^+(f(v_i)) = I_{A_1}^+(S_i),$$

$$F_{A_2}^+(f(v_i)) = F_{A_1}^+(S_i), \quad T_{A_2}^-(f(v_i)) = T_{A_1}^-(S_i), \quad I_{A_2}^-(f(v_i)) = I_{A_1}^-(S_i),$$

$$F_{A_2}^-(f(v_i)) = F_{A_1}^-(S_i)$$

for all $v_i \in V$.

$$T_{B_2}^+(f(v_i) f(v_j)) = T_{B_1}^+(S_i S_j) = T_{B_1}^+(v_i v_j),$$

$$I_{B_2}^+(f(v_i) f(v_j)) = I_{B_1}^+(S_i S_j) = I_{B_1}^+(v_i v_j),$$

$$F_{B_2}^+(f(v_i) f(v_j)) = F_{B_1}^+(S_i S_j) = F_{B_1}^+(v_i v_j),$$

$$T_{B_2}^-(f(v_i) f(v_j)) = T_{B_1}^-(S_i S_j) = T_{B_1}^-(v_i v_j),$$

$$I_{B_2}^-(f(v_i) f(v_j)) = I_{B_1}^-(S_i S_j) = I_{B_1}^-(v_i v_j),$$

$$F_{B_2}^-(f(v_i) f(v_j)) = F_{B_1}^-(S_i S_j) = F_{B_1}^-(v_i v_j),$$

for all $v_i v_j \in E$. \qed
**Definition 3.30.** Let $G^* = (V, E)$ and $L(G^*) = (X, Y)$ be its line graph, where $A_1$ and $B_1$ be BSVNSs on $V$ and $E$, respectively. Let $A_2$ and $B_2$ be BSVNSs on $X$ and $Y$, respectively. The bipolar single valued neutrosophic line graph (BSVNLG) of BSVNG $G = (A_1, B_1)$ is BSVNG $L(G) = (A_2, B_2)$ such that,

\[
T_A^+(S_x) = T_{B_2}^+(x) = T_{B_1}^+(u_xv_x), \quad I_A^+(S_x) = I_{B_1}^+(x) = I_{B_1}^+(u_xv_x),
\]

\[
I_A^-(S_x) = I_{B_1}^-(x) = I_{B_1}^-(u_xv_x), \quad F_A^+(S_x) = F_{B_1}^+(x) = F_{B_1}^+(u_xv_x),
\]

\[
F_A^- (S_x) = F_{B_1}^-(x) = F_{B_1}^-(u_xv_x), \quad T_A^- (S_x) = T_{B_1}^-(x) = T_{B_1}^-(u_xv_x),
\]

\[
\forall \ x \in X \ and \ y \ in Y
\]

\[
T_{B_2}^+(S_xS_y) = \min(T_{B_1}^+(x), T_{B_1}^+(y)), \quad I_{B_2}^+(S_xS_y) = \max(I_{B_1}^+(x), I_{B_1}^+(y)),
\]

\[
F_{B_2}^+(S_xS_y) = \max(F_{B_1}^+(x), F_{B_1}^+(y)), \quad T_{B_2}^-(S_xS_y) = \max(T_{B_1}^-(x), T_{B_1}^-(y)),
\]

\[
\forall \ x, y \in X \ and \ y \ in Y
\]

**Remark 3.31.** Every BSVNLG is a strong BSVNG.

**Remark 3.32.** The $L(G) = (A_2, B_2)$ is a BSVNLG corresponding to BSVNG $G = (A_1, B_1)$.

**Example 3.33.** Consider the $G^* = (V, E)$ where $V = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $E = \{x_1 = \alpha_1 \alpha_2, x_2 = \alpha_2 \alpha_3, x_3 = \alpha_3 \alpha_4, x_4 = \alpha_4 \alpha_1\}$ and $G = (A_1, B_1)$ is BSVNG of $G^* = (V, E)$ which is defined in Table 6. Consider the $L(G^*) = (X, Y)$ such that $X = \{\Gamma_{x_1}, \Gamma_{x_2}, \Gamma_{x_3}, \Gamma_{x_4}\}$ and $Y = \{\Gamma_{x_1} \Gamma_{x_2}, \Gamma_{x_2} \Gamma_{x_3}, \Gamma_{x_3} \Gamma_{x_4}, \Gamma_{x_4} \Gamma_{x_1}\}$. Let $A_2$ and $B_2$ be BSVNSs of $X$ and $Y$ respectively, then BSVNLG $L(G)$ is given in Table 7.

**Proposition 3.34.** The $L(G) = (A_2, B_2)$ is a BSVNLG of some BSVNG $G = (A_1, B_1)$ if and only if

\[
T_{B_1}^+(S_xS_y) = \min(T_{A_2}^+(S_x), T_{A_2}^+(S_y)),
\]

\[
T_{B_1}^-(S_xS_y) = \max(T_{A_1}^-(S_x), T_{A_1}^-(S_y)),
\]

\[
I_{B_1}^+(S_xS_y) = \max(I_{A_2}^+(S_x), I_{A_2}^+(S_y)),
\]

\[
F_{B_1}^+(S_xS_y) = \min(F_{A_2}^+(S_x), F_{A_2}^+(S_y)),
\]

\[
F_{B_1}^-(S_xS_y) = \max(F_{A_1}^-(S_x), F_{A_1}^-(S_y)),
\]
∀ \ \text{will suffice. Converse is straight forward.}

\[ x \in S \]

\[ y \in S \]

Define

\[ + \]

\[ T_{A_1}^+(x) = T_{A_2}^+(S_x), \quad I_{A_1}^+(x) = I_{A_2}^+(S_x), \quad F_{A_1}^+(x) = F_{A_2}^+(S_x), \]

\[ T_{A_2}^-(x) = T_{A_2}^-(S_x), \quad I_{A_2}^-(x) = I_{A_2}^-(S_x), \quad F_{A_2}^-(x) = F_{A_2}^-(S_x) \]

∀ \ \text{in E. Then}

\[ I_{B_1}^+(S_x S_y) = \max(I_{A_1}^+(S_x), I_{A_2}^+(S_y)) = \max(I_{A_1}^+(x), I_{A_2}^+(y)), \]

\[ I_{B_2}^-(S_x S_y) = \min(I_{A_2}^-(S_x), I_{A_2}^-(S_y)) = \min(I_{A_2}^-(x), I_{A_2}^-(y)), \]

\[ T_{B_2}^+(S_x S_y) = \min(T_{A_2}^+(S_x), T_{A_2}^+(S_y)) = \min(T_{A_2}^+(x), T_{A_2}^+(y)), \]

\[ T_{B_2}^-(S_x S_y) = \max(T_{A_2}^-(S_x), T_{A_2}^-(S_y)) = \max(T_{A_2}^-(x), T_{A_2}^-(y)), \]

\[ F_{B_2}^+(S_x S_y) = \max(F_{A_2}^-(S_x), F_{A_2}^-(S_y)) = \max(F_{A_2}^-(x), F_{A_2}^-(y)), \]

\[ F_{B_2}^-(S_x S_y) = \min(F_{A_2}^+(S_x), F_{A_2}^+(S_y)) = \min(F_{A_2}^+(x), F_{A_2}^+(y)). \]

A BSVNS \text{ A_1 that yields the property}

\[ T_{B_1}^+(xy) \leq \min(T_{A_1}^-(x), T_{A_1}^-(y)), \quad I_{B_1}^-(xy) \leq \max(I_{A_1}^+(x), I_{A_1}^+(y)), \]

\[ I_{B_1}^+(xy) \leq \min(I_{A_1}^-(x), I_{A_1}^-(y)), \quad F_{B_1}^-(xy) \leq \min(F_{A_1}^+(x), F_{A_1}^+(y)), \]

\[ F_{B_1}^+(xy) \geq \max(F_{A_1}^-(x), F_{A_1}^-(y)), \quad T_{B_1}^-(xy) \geq \max(T_{A_1}^-(x), T_{A_1}^-(y)) \]

will suffice. Converse is straight forward.

\[ \square\]

Table 7. BSVNSs of BSVNLG.

<table>
<thead>
<tr>
<th>A_1</th>
<th>T_{A_1}^+</th>
<th>I_{A_1}^+</th>
<th>F_{A_1}^+</th>
<th>T_{A_1}^-</th>
<th>I_{A_1}^-</th>
<th>F_{A_1}^-</th>
</tr>
</thead>
<tbody>
<tr>
<td>\Gamma_{x_1}</td>
<td>0.1</td>
<td>0.6</td>
<td>0.7</td>
<td>-0.1</td>
<td>-0.4</td>
<td>-0.5</td>
</tr>
<tr>
<td>\Gamma_{x_2}</td>
<td>0.3</td>
<td>0.6</td>
<td>0.7</td>
<td>-0.2</td>
<td>-0.3</td>
<td>-0.6</td>
</tr>
<tr>
<td>\Gamma_{x_3}</td>
<td>0.2</td>
<td>0.7</td>
<td>0.8</td>
<td>-0.3</td>
<td>-0.2</td>
<td>-0.6</td>
</tr>
<tr>
<td>\Gamma_{x_4}</td>
<td>0.1</td>
<td>0.7</td>
<td>0.8</td>
<td>-0.1</td>
<td>-0.4</td>
<td>-0.5</td>
</tr>
<tr>
<td>B_1</td>
<td>T_{B_1}^+</td>
<td>I_{B_1}^+</td>
<td>F_{B_1}^+</td>
<td>T_{B_1}^-</td>
<td>I_{B_1}^-</td>
<td>F_{B_1}^-</td>
</tr>
<tr>
<td>\Gamma_{x_1} \Gamma_{x_2}</td>
<td>0.1</td>
<td>0.6</td>
<td>0.7</td>
<td>-0.1</td>
<td>-0.4</td>
<td>-0.6</td>
</tr>
<tr>
<td>\Gamma_{x_2} \Gamma_{x_3}</td>
<td>0.2</td>
<td>0.7</td>
<td>0.8</td>
<td>-0.2</td>
<td>-0.3</td>
<td>-0.6</td>
</tr>
<tr>
<td>\Gamma_{x_3} \Gamma_{x_4}</td>
<td>0.1</td>
<td>0.7</td>
<td>0.8</td>
<td>-0.1</td>
<td>-0.4</td>
<td>-0.6</td>
</tr>
<tr>
<td>\Gamma_{x_4} \Gamma_{x_4}</td>
<td>0.1</td>
<td>0.7</td>
<td>0.8</td>
<td>-0.1</td>
<td>-0.4</td>
<td>-0.5</td>
</tr>
</tbody>
</table>
Proposition 3.35. If \( L(G) \) be a BSVNLG of BSVNG \( G \), then \( L(G^*) = (X,Y) \) is the crisp line graph of \( G^* \).

Proof. Since \( L(G) \) is a BSVNLG,
\[
T_{A_2}(S_x) = T_{B_1}^+(x), \quad I_{A_2}^+(S_x) = I_{B_1}^+(x), \quad F_{A_2}^+(S_x) = F_{B_1}^+(x),
\]
\[
T_{A_2}^-(S_x) = T_{B_1}^-(x), \quad I_{A_2}^-(S_x) = I_{B_1}^-(x), \quad F_{A_2}^-(S_x) = F_{B_1}^-(x)
\]
\( \forall x \in E, S_x \in X \) if and only if \( x \in E \), also
\[
T_{B_2}(S_x S_y) = \min(T_{B_1}^+(x), T_{B_1}^+(y)), \quad I_{B_2}^+(S_x S_y) = \max(I_{B_1}^+(x), I_{B_1}^+(y)),
\]
\[
I_{B_2}^-(S_x S_y) = \min(I_{B_1}^-(x), I_{B_1}^-(y)), \quad F_{B_2}^-(S_x S_y) = \min(F_{B_1}^-(x), F_{B_1}^-(y)),
\]
\[
F_{B_2}^+(S_x S_y) = \max(F_{B_1}^+(x), F_{B_1}^+(y)), \quad T_{B_2}^-(S_x S_y) = \max(T_{B_1}^-(x), T_{B_1}^-(y)),
\]
\( \forall S_x S_y \in Y \). Then \( Y = \{S_x S_y : S_x \cap S_y = \emptyset, x, y \in E, x \neq y \} \).

Proposition 3.36. The \( L(G) = (A_2, B_2) \) be a BSVNLG of BSVNG \( G \) if and only if \( L(G^*) = (X,Y) \) is the line graph and
\[
T_{B_3}(xy) = \min(T_{A_3}^+(x), T_{A_3}^+(y)), \quad I_{B_3}^+(xy) = \max(I_{A_3}^+(x), I_{A_3}^+(y)),
\]
\[
I_{B_3}^-(xy) = \min(I_{A_3}^-(x), I_{A_3}^-(y)), \quad F_{B_3}^-(xy) = \min(F_{A_3}^-(x), F_{A_3}^-(y)),
\]
\[
F_{B_3}^+(xy) = \max(F_{A_3}^+(x), F_{A_3}^+(y)), \quad T_{B_3}^+(xy) = \max(T_{A_3}^+(x), T_{A_3}^+(y)),
\]
\( \forall xy \in Y \).

Proof. It follows from propositions 3.34 and 3.35.

Proposition 3.37. Let \( G \) be a BSVNG, then \( M(G) \) is isomorphic with \( sd(G) \cup L(G) \).

Theorem 3.38. Let \( L(G) = (A_2, B_2) \) be BSVNLG corresponding to BSVNG \( G = (A_1, B_1) \).

(1) If \( G \) is weak isomorphic onto \( L(G) \) if and only if \( \forall v \in V, x \in E \) and \( G^* \) to be a cycle, such that
\[
T_{A_1}^+(v) = T_{B_1}^+(x), \quad I_{A_1}^+(v) = I_{B_1}^+(x), \quad F_{A_1}^+(v) = F_{B_1}^+(x),
\]
\[
T_{A_1}^-(v) = T_{B_1}^-(x), \quad I_{A_1}^-(v) = I_{B_1}^-(x), \quad F_{A_1}^-(v) = F_{B_1}^-(x).
\]

(2) If \( G \) is weak isomorphic onto \( L(G) \), then \( G \) and \( L(G) \) are isomorphic.

Proof. By hypothesis, \( G^* \) is a cycle. Let \( V = \{v_1, v_2, v_3, \ldots, v_n\} \) and \( E = \{x_1 = v_1 v_2, x_2 = v_2 v_3, \ldots, x_n = v_n v_1\} \), where \( P : v_1 v_2 v_3 \ldots v_n \) is a cycle, characterize a BSVNS \( A_1 \) by \( A_1(v_i) = (p_i, q_i, r_i, p'_i, q'_i, r'_i) \) and \( B_1 \) by \( B_1(x_i) = (a_i, b_i, c_i, a'_i, b'_i, c'_i) \) for \( i = 1, 2, 3, \ldots, n \) and \( v_n+1 = v_1 \). Then for \( p_{n+1} = p_1, q_{n+1} = q_1, r_{n+1} = r_1 \),
\[
a_i \leq \min(p_i, p_{i+1}), \quad b_i \geq \max(q_i, q_{i+1}), \quad c_i = \max(r_i, r_{i+1}),
\]
\[
a'_i \geq \max(p'_i, p'_{i+1}), \quad b'_i \leq \min(q'_i, q'_{i+1}), \quad c'_i \leq \min(r'_i, r'_{i+1}),
\]
for \( i = 1, 2, 3, \ldots, n \).

Now let \( X = \{\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_n}\} \) and \( Y = \{\Gamma_{x_1}, \Gamma_{x_2}, \Gamma_{x_2} \Gamma_{x_3}, \ldots, \Gamma_{x_n} \Gamma_{x_1}\} \). Then for \( a_{n+1} = a_1 \), we obtain
\[
A_2(\Gamma_{x_i}) = B_1(x_i) = (a_i, b_i, c_i, a'_i, b'_i, c'_i)
\]
and $B_2(\Gamma_x, \Gamma_{x,i+1}) = (\min(a_i, a_{i+1}), \max(b_i, b_{i+1}), \max(c_i, c_{i+1}), \max(a'_i, a'_{i+1}), \min(b'_i, b'_{i+1}), \min(c'_i, c'_{i+1}))$ for $i = 1, 2, 3, \ldots, n$ and $v_{n+1} = v_1$. Since $f$ preserves adjacency, it induce permutation $\pi$ of $\{1, 2, 3, \ldots, n\}$,

$$f(v_i) = \Gamma_{v_{\pi(i)}}v_{\pi(i)+1}$$

and

$$v_iv_{i+1} \rightarrow f(v_i)f(v_{i+1}) = \Gamma_{v_{\pi(i)}}v_{\pi(i)+1}\Gamma_{v_{\pi(i)+1}}v_{\pi(i)+1+1},$$

for $i = 1, 2, 3, \ldots, n - 1$. Thus

$$p_i = T^+_{A_v}(v_i) \leq T^+_{A_v}(f(v_i)) = T^+_{A_v}(\Gamma_{v_{\pi(i)}}v_{\pi(i)+1}) = T^+_{B_v}(v_{\pi(i)}v_{\pi(i)+1}) = a_{\pi(i)}.$$

Similarly, $p'_i \geq a'_{\pi(i)}$, $q_i \geq b_{\pi(i)}$, $r_i \geq c_{\pi(i)}$, $q'_i \leq b'_{\pi(i)}$, $r'_i \leq c'_{\pi(i)}$ and

$$a_i = T^+_{B_v}(v_iv_{i+1}) \leq T^+_{B_v}(f(v_i)f(v_{i+1})) = T^+_{B_v}(\Gamma_{v_{\pi(i)}}v_{\pi(i)+1}\Gamma_{v_{\pi(i)+1}}v_{\pi(i)+1+1}) = \min(T^+_{B_v}(v_{\pi(i)}v_{\pi(i)+1}), T^+_{B_v}(v_{\pi(i)+1}v_{\pi(i)+1+1})) = \min(a_{\pi(i)}, a_{\pi(i)+1}).$$

Similarly, $b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1}), c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1}), a'_i \geq \max(a'_{\pi(i)}, a'_{\pi(i)+1}), b'_i \leq \min(b'_{\pi(i)}, b'_{\pi(i)+1})$ and $c'_i \leq \min(c'_{\pi(i)}, c'_{\pi(i)+1})$ for $i = 1, 2, 3, \ldots, n$. Therefore

$$p_i \leq a_{\pi(i)}, q_i \geq b_{\pi(i)}, r_i \geq c_{\pi(i)}, p_i \geq a'_{\pi(i)}, q'_i \leq b'_{\pi(i)}, r'_i \leq c'_{\pi(i)}$$

and

$$a_i \leq \min(a_{\pi(i)}, a_{\pi(i)+1}), a'_i \geq \max(a'_{\pi(i)}, a'_{\pi(i)+1}),$$

$$b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1}), b'_i \leq \min(b'_{\pi(i)}, b'_{\pi(i)+1}),$$

$$c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1}), c'_i \leq \min(c'_{\pi(i)}, c'_{\pi(i)+1}).$$

Thus

$$a_i \leq a_{\pi(i)}, b_i \geq b_{\pi(i)}, c_i \geq c_{\pi(i)}, a'_i \geq a'_{\pi(i)}, b'_i \leq b'_{\pi(i)}, c'_i \leq c'_{\pi(i)}$$

and so

$$a_{\pi(i)} \leq a_{\pi(\pi(i))}, b_{\pi(i)} \geq b_{\pi(\pi(i))}, c_{\pi(i)} \geq c_{\pi(\pi(i))}$$

$$a_{\pi(i)} \geq a'_{\pi(\pi(i))}, b'_{\pi(i)} \leq b'_{\pi(\pi(i))}, c'_{\pi(i)} \leq c'_{\pi(\pi(i))}$$

$\forall i = 1, 2, 3, \ldots, n$. Next to extend,

$$a_i \leq a_{\pi(i)} \leq \ldots \leq a_{\pi^i(i)} \leq \ldots \leq a_{\pi^i(i)} \leq a'_i$$

$$b_i \geq b_{\pi(i)} \geq \ldots \geq b_{\pi^i(i)} \geq b'_i$$

$$c_i \leq c_{\pi(i)} \leq \ldots \leq c_{\pi^i(i)} \leq c'_i$$

where $\pi^{i+1}$ identity. Hence

$$a_i = a_{\pi(i)}, b_i = b_{\pi(i)}, c_i = c_{\pi(i)}, a'_i = a'_{\pi(i)}, b'_i = b'_{\pi(i)}, c'_i = c'_{\pi(i)}$$

$\forall i = 1, 2, 3, \ldots, n$. Thus we conclude that

$$a_i = a_{\pi(i+1)} = a_{i+1}, b_i = b_{\pi(i+1)} = b_{i+1}, c_i = c_{\pi(i+1)} = c_{i+1}$$

$$a'_i = a'_{\pi(i+1)} = a'_{i+1}, b'_i = b'_{\pi(i+1)} = b'_{i+1}, c'_i = c'_{\pi(i+1)} = c'_{i+1}$$

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which together with
\[ a_{n+1} = a_1, \quad b_{n+1} = b_1, \quad c_{n+1} = c_1, \quad a'_{n+1} = a'_1, \quad b'_{n+1} = b'_1, \quad c'_{n+1} = c'_1 \]
which implies that
\[ a_i = a_1, \quad b_i = b_1, \quad c_i = c_1, \quad a'_i = a'_1, \quad b'_i = b'_1, \quad c'_i = c'_1 \]
\[ \forall i = 1, 2, 3, \ldots, n. \]
Thus we have
\[ a_1 = a_2 = \ldots = a_n = p_1 = p_2 = \ldots = p_n \]
\[ a'_1 = a'_2 = \ldots = a'_n = p'_1 = p'_2 = \ldots = p'_n \]
\[ b_1 = b_2 = \ldots = b_n = q_1 = q_2 = \ldots = q_n \]
\[ b'_1 = b'_2 = \ldots = b'_n = q'_1 = q'_2 = \ldots = q'_n \]
\[ c_1 = c_2 = \ldots = c_n = r_1 = r_2 = \ldots = r_n \]
\[ c'_1 = c'_2 = \ldots = c'_n = r'_1 = r'_2 = \ldots = r'_n \]
Therefore (a) and (b) holds, since converse of result (a) is straightforward. \(\Box\)

4. Conclusion

The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we discussed the special types of BSVNGs, subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs of the given BSVNGs. We investigated isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

References


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Neutrosophic subalgebras of several types in $BCK/BCI$-algebras

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ABSTRACT. Given $\Phi, \Psi \in \{\in, q, \in \lor q\}$, the notion of $(\Phi, \Psi)$-neutrosophic subalgebras of a $BCK/BCI$-algebra are introduced, and related properties are investigated. Characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \lor q)$-neutrosophic subalgebra are provided. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic $q$-subsets and neutrosophic $\in \lor q$-subsets, conditions for the neutrosophic $\in$-subsets, neutrosophic $q$-subsets and neutrosophic $\in \lor q$-subsets to be subalgebras are discussed. Conditions for a neutrosophic set to be a $(q, \in \lor q)$-neutrosophic subalgebra are considered.

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1. INTRODUCTION

The concept of neutrosophic set (NS) developed by Smarandache [5, 6, 7] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts. For further particulars I refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Agboola et al. [1] studied neutrosophic ideals of neutrosophic $BCI$-algebras. Agboola et al. [2] also introduced the concept of neutrosophic $BCI/BCK$-algebras, and presented elementary properties of neutrosophic $BCI/BCK$-algebras.

In this paper, we introduce the notion of $(\Phi, \Psi)$-neutrosophic subalgebra of a $BCK/BCI$-algebra $X$ for $\Phi, \Psi \in \{\in, q, \in \lor q\}$, and investigate related properties.
We provide characterizations of an \((\varepsilon, \varepsilon\)-neutrosophic subalgebra and an \((\varepsilon, \varepsilon \lor q)\)-neutrosophic subalgebra. Given special sets, so called neutrosophic \(\varepsilon\)-subsets, neutrosophic \(q\)-subsets and neutrosophic \(\varepsilon \lor q\)-subsets, we provide conditions for the neutrosophic \(\varepsilon\)-subsets, neutrosophic \(q\)-subsets and neutrosophic \(\varepsilon \lor q\)-subsets to be subalgebras. We consider conditions for a neutrosophic set to be a \((q, \varepsilon \lor q)\)-neutrosophic subalgebra.

2. Preliminaries

By a \(BCI\)-algebra we mean an algebra \((X, \ast, 0)\) of type \((2, 0)\) satisfying the axioms:

\begin{enumerate}
\item[(a1)] \((x \ast y) \ast (x \ast z) \ast (z \ast y) = 0,\)
\item[(a2)] \((x \ast (x \ast y)) \ast y = 0,\)
\item[(a3)] \(x \ast x = 0,\)
\item[(a4)] \(x \ast y = y \ast x = 0 \Rightarrow x = y,\)
\end{enumerate}

for all \(x, y, z \in X\). If a \(BCI\)-algebra \(X\) satisfies the axiom

\item[(a5)] \(0 \ast x = 0\) for all \(x \in X\),

then we say that \(X\) is a \(BCK\)-algebra. A nonempty subset \(S\) of a \(BCK/BCI\)-algebra \(X\) is called a subalgebra of \(X\) if \(x \ast y \in S\) for all \(x, y \in S\).

We refer the reader to the books [3] and [4] for further information regarding \(BCK/BCI\)-algebras.

Let \(X\) be a non-empty set. A neutrosophic set (NS) in \(X\) (see [6]) is a structure of the form:

\[ A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \} \]

where \(A_T : X \to [0,1]\) is a truth membership function, \(A_I : X \to [0,1]\) is an indeterminate membership function, and \(A_F : X \to [0,1]\) is a false membership function. For the sake of simplicity, we shall use the symbol \(A = (A_T, A_I, A_F)\) for the neutrosophic set

\[ A := \{ \langle x; A_T(x), A_I(x), A_F(x) \rangle \mid x \in X \}. \]

3. Neutrosophic subalgebras of several types

Given a neutrosophic set \(A = (A_T, A_I, A_F)\) in a set \(X\), \(\alpha, \beta \in (0,1]\) and \(\gamma \in [0,1]\), we consider the following sets:

\begin{align*}
T_\varepsilon(A; \alpha) &:= \{ x \in X \mid A_T(x) \geq \alpha \}, \\
I_\varepsilon(A; \beta) &:= \{ x \in X \mid A_I(x) \geq \beta \}, \\
F_\varepsilon(A; \gamma) &:= \{ x \in X \mid A_F(x) \leq \gamma \}, \\
T_\varepsilon(A; \alpha) &:= \{ x \in X \mid A_T(x) + \alpha > 1 \}, \\
I_\varepsilon(A; \beta) &:= \{ x \in X \mid A_I(x) + \beta > 1 \}, \\
F_\varepsilon(A; \gamma) &:= \{ x \in X \mid A_F(x) + \gamma < 1 \}, \\
T_{\varepsilon \lor q}(A; \alpha) &:= \{ x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1 \}, \\
I_{\varepsilon \lor q}(A; \beta) &:= \{ x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1 \}, \\
F_{\varepsilon \lor q}(A; \gamma) &:= \{ x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1 \}. 
\end{align*}

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We say $T_\varepsilon(A; \alpha)$, $I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are neutrosophic $\varepsilon$-subsets; $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are neutrosophic $q$-subsets; and $T_{\vee q}(A; \alpha)$, $I_{\vee q}(A; \beta)$ and $F_{\vee q}(A; \gamma)$ are neutrosophic $\vee q$-subsets. For $\Phi \in \{\varepsilon, q, \varepsilon \vee q\}$, the element of $T_\Phi(A; \alpha)$ (resp., $I_\Phi(A; \beta)$ and $F_\Phi(A; \gamma)$) is called a neutrosophic $T_\Phi$-point (resp., neutrosophic $I_\Phi$-point and neutrosophic $F_\Phi$-point) with value $\alpha$ (resp., $\beta$ and $\gamma$). It is clear that

\begin{align}
(3.1) & \quad T_{\vee q}(A; \alpha) = T_\varepsilon(A; \alpha) \cup T_q(A; \alpha), \\
(3.2) & \quad I_{\vee q}(A; \beta) = I_\varepsilon(A; \beta) \cup I_q(A; \beta), \\
(3.3) & \quad F_{\vee q}(A; \gamma) = F_\varepsilon(A; \gamma) \cup F_q(A; \gamma).
\end{align}

**Proposition 3.1.** For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we have

\begin{align}
(3.4) & \quad \alpha \in [0, 0.5] \implies T_{\vee q}(A; \alpha) = T_\varepsilon(A; \alpha), \\
(3.5) & \quad \beta \in [0, 0.5] \implies I_{\vee q}(A; \beta) = I_\varepsilon(A; \beta), \\
(3.6) & \quad \gamma \in [0.5, 1] \implies F_{\vee q}(A; \gamma) = F_\varepsilon(A; \gamma), \\
(3.7) & \quad \alpha \in [0.5, 1] \implies T_{\vee q}(A; \alpha) = T_q(A; \alpha), \\
(3.8) & \quad \beta \in [0.5, 1] \implies I_{\vee q}(A; \beta) = I_q(A; \beta), \\
(3.9) & \quad \gamma \in [0, 0.5] \implies F_{\vee q}(A; \gamma) = F_q(A; \gamma).
\end{align}

**Proof.** If $\alpha \in [0, 0.5]$, then $1 - \alpha \in [0.5, 1]$ and $\alpha \leq 1 - \alpha$. It is clear that $T_\varepsilon(A; \alpha) \subseteq T_{\vee q}(A; \alpha)$ by (3.1). If $x \notin T_\varepsilon(A; \alpha)$, then $A_T(x) < \alpha \leq 1 - \alpha$, i.e., $x \notin T_q(A; \alpha)$. Hence $x \notin T_{\vee q}(A; \alpha)$, and so $T_{\vee q}(A; \alpha) \subseteq T_\varepsilon(A; \alpha)$ by (3.4). Similarly, we have the result (3.5). If $\gamma \in [0.5, 1]$, then $1 - \gamma \in [0, 0.5]$ and $\gamma \geq 1 - \gamma$. It is clear that $F_\varepsilon(A; \gamma) \subseteq F_{\vee q}(A; \gamma)$ by (3.3). Let $z \in F_{\vee q}(A; \gamma)$. Then $z \in F_\varepsilon(A; \gamma)$ or $z \in F_q(A; \gamma)$. If $z \notin F_\varepsilon(A; \gamma)$, then $A_F(z) \geq 1 - \gamma$, i.e., $A_F(z) + \gamma > 1$. Thus $z \notin F_q(A; \gamma)$, and therefore $z \notin F_{\vee q}(A; \gamma)$. This is a contradiction. Hence $z \in F_\varepsilon(A; \gamma)$, and therefore $F_{\vee q}(A; \gamma) \subseteq F_\varepsilon(A; \gamma)$. Let $\beta \in (0.5, 1]$. Then $\beta > 1 - \beta$. Note that $I_q(A; \beta) \subseteq I_{\vee q}(A; \beta)$ by (3.2). Let $y \in I_{\vee q}(A; \beta)$. Then $y \in I_\varepsilon(A; \beta)$ or $y \in I_q(A; \beta)$. If $y \notin I_\varepsilon(A; \beta)$, then $A_I(y) + \beta \leq 1$ and so $A_F(y) \leq 1 - \beta < \beta$, i.e., $y \notin I_q(A; \beta)$. Thus $y \notin I_{\vee q}(A; \beta)$, a contradiction. Hence $y \in I_q(A; \beta)$. Therefore $I_{\vee q}(A; \beta) \subseteq I_q(A; \beta)$. This shows that (3.8) is true. The result (3.7) is proved by the similar way. Let $\gamma \in [0, 0.5)$ and $z \in F_{\vee q}(A; \gamma)$. Then $1 - \gamma > \gamma$ and $z \in F_\varepsilon(A; \gamma)$ or $z \in F_q(A; \gamma)$. If $z \notin F_\varepsilon(A; \gamma)$, then $A_F(z) + \gamma \geq 1$ and so $A_F(z) \geq 1 - \gamma > \gamma$, i.e., $z \notin F_q(A; \gamma)$. Thus $z \notin F_{\vee q}(A; \gamma)$, which is a contradiction. Hence $F_{\vee q}(A; \gamma) \subseteq F_q(A; \gamma)$. The reverse inclusion is by (3.3). 

**Definition 3.2.** Given $\Phi, \Psi \in \{\varepsilon, q, \varepsilon \vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is called a $(\Phi, \Psi)$-neutrosophic subalgebra of $X$ if the following assertions are valid.

\begin{align}
(3.10) & \quad x \in T_\Phi(A; \alpha_x), \; y \in T_\Phi(A; \alpha_y) \implies x * y \in T_\Phi(A; \alpha_x \wedge \alpha_y), \\
& \quad x \in I_\Phi(A; \beta_x), \; y \in I_\Phi(A; \beta_y) \implies x * y \in I_\Phi(A; \beta_x \wedge \beta_y), \\
& \quad x \in F_\Phi(A; \gamma_x), \; y \in F_\Phi(A; \gamma_y) \implies x * y \in F_\Phi(A; \gamma_x \vee \gamma_y)
\end{align}

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.
Theorem 3.3. A neutrosophic set $A = (A_T, A_I, A_F)$ in a $BCK/BCI$-algebra $X$ is an $(\varepsilon, \varepsilon)$-neutrosophic subalgebra of $X$ if and only if it satisfies:

\[
\begin{align*}
(\forall x, y \in X) \quad & A_T(x \ast y) \geq A_T(x) \land A_T(y) \\
& A_I(x \ast y) \geq A_I(x) \land A_I(y) \\
& A_F(x \ast y) \leq A_F(x) \lor A_F(y)
\end{align*}
\]

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \varepsilon)$-neutrosophic subalgebra of $X$. If there exist $x, y \in X$ such that $A_T(x \ast y) < A_T(x) \land A_T(y)$, then $A_T(x \ast y) < A_T(x) \land A_T(y)$ for some $\alpha \in (0, 1]$. It follows that $x, y \in T_\varepsilon(A; \alpha)$ but $x \ast y \notin T_\varepsilon(A; \alpha)$. Hence $A_T(x \ast y) \geq A_T(x) \land A_T(y)$ for all $x, y \in X$. Similarly, we show that $A_I(x \ast y) \geq A_I(x) \land A_I(y)$ for all $x, y \in X$. Suppose that there exist $a, b \in X$ and $\gamma, \gamma_f \in [0, 1]$ such that $A_F(a \ast b) > \gamma \geq A_F(a) \lor A_F(b)$. Then $a, b \in F_\varepsilon(A; \gamma_f)$ and $a \ast b \notin F_\varepsilon(A; \gamma_f)$, which is a contradiction. Therefore $A_F(x \ast y) \leq A_F(x) \lor A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ which satisfies the condition (3.11). Let $x, y \in X$ be such that $x \in T_\varepsilon(A; \alpha_x)$ and $y \in T_\varepsilon(A; \alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$, which imply that $A_T(x \ast y) \geq A_T(x) \land A_T(y) \geq \alpha_x \land \alpha_y$, that is, $x \ast y \in T_\varepsilon(A; \alpha_x \land \alpha_y)$. Similarly, if $x \in I_\varepsilon(A; \beta_x)$ and $y \in I_\varepsilon(A; \beta_y)$ then $x \ast y \in I_\varepsilon(A; \beta_x \lor \beta_y)$. Now, let $x \in F_\varepsilon(A; \gamma_x)$ and $y \in F_\varepsilon(A; \gamma_y)$ for $x, y \in X$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$, and so $A_F(x \ast y) \leq A_F(x) \lor A_F(y) \leq \gamma_x \lor \gamma_y$. Hence $x \ast y \in F_\varepsilon(A; \gamma_x \lor \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \varepsilon)$-neutrosophic subalgebra of $X$.

Theorem 3.4. If $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \varepsilon)$-neutrosophic subalgebra of a $BCK/BCI$-algebra $X$, then neutrosophic $q$-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1]$ whenever they are nonempty.

Proof. Let $x, y \in T_q(A; \alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows that $A_T(x \ast y) + \alpha \geq A_T(x) \land A_T(y) + \alpha = (A_T(x) + \alpha) \land (A_T(y) + \alpha) > 1$ and so that $x \ast y \in T_q(A; \alpha)$. Hence $T_q(A; \alpha)$ is a subalgebra of $X$. Similarly, we can prove that $I_q(A; \beta)$ is a subalgebra of $X$. Now let $x, y \in F_q(A; \gamma)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$, which imply that $A_F(x \ast y) + \gamma \geq A_F(x) \lor A_F(y) + \gamma = (A_F(x) + \alpha) \lor (A_F(y) + \alpha) < 1$.

Hence $x \ast y \in F_q(A; \gamma)$ and $F_q(A; \gamma)$ is a subalgebra of $X$.

Theorem 3.5. If $A = (A_T, A_I, A_F)$ is a $(\varepsilon, \varepsilon)$-neutrosophic subalgebra of a $BCK/BCI$-algebra $X$, then neutrosophic $q$-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5]$ whenever they are nonempty.
Proof. Let \( x, y \in T_q(A; \alpha) \). Then \( x \ast y \in T_{\vee \wedge q}(A; \alpha) \), and so \( x \ast y \in T_q(A; \alpha) \) or \( x \ast y \in T_q(A; \alpha) \). If \( x \ast y \in T_q(A; \alpha) \), then \( A_T(x \ast y) \geq \alpha > 1 - \alpha \) since \( \alpha > 0.5 \). Hence \( x \ast y \in T_q(A; \alpha) \). Therefore \( T_q(A; \alpha) \) is a subalgebra of \( X \). Similarly, we prove that \( I_q(A; \beta) \) is a subalgebra of \( X \). Let \( x, y \in F_q(A; \gamma) \). Then \( x \ast y \in F_{\vee \wedge q}(A; \gamma) \), and so \( x \ast y \in F_q(A; \gamma) \) or \( x \ast y \in F_q(A; \gamma) \). If \( x \ast y \in F_q(A; \gamma) \), then \( A_F(x \ast y) \leq \gamma < 1 - \gamma \) since \( \gamma \in [0, 0.5) \). Hence \( x \ast y \in F_q(A; \gamma) \), and therefore \( F_q(A; \gamma) \) is a subalgebra of \( X \).

We provide characterizations of an \((\varepsilon, \in \vee q)\)-neutrosophic subalgebra.

**Theorem 3.6.** A neutrosophic set \( A = (A_T, A_I, A_F) \) in a \( BCK/BCI \)-algebra \( X \) is an \((\varepsilon, \in \vee q)\)-neutrosophic subalgebra of \( X \) if and only if it satisfies:

\[
\begin{align*}
(A_T(x \ast y) \geq \land \{A_T(x), A_T(y), 0.5\} & , \\
A_I(x \ast y) \geq \land \{A_I(x), A_I(y), 0.5\} & , \\
A_F(x \ast y) \leq \lor \{A_F(x), A_F(y), 0.5\} &
\end{align*}
\]

(3.12) \( \forall x, y \in X \)

Proof. Suppose that \( A = (A_T, A_I, A_F) \) is an \((\varepsilon, \in \vee q)\)-neutrosophic subalgebra of \( X \) and let \( x, y \in X \). If \( A_T(x \ast y) \land A_I(y) < 0.5 \), then \( A_T(x \ast y) \geq A_T(x) \land A_T(y) \). For, assume that \( A_T(x \ast y) < A_T(x) \land A_T(y) \) and choose \( \alpha_t \) such that

\( A_T(x \ast y) < \alpha_t < A_T(x) \land A_T(y) \).

Then \( x \in T_q(A; \alpha_t) \) and \( y \in T_q(A; \alpha_t) \) but \( x \ast y \notin T_q(A; \alpha_t) \). Also \( A_T(x \ast y) + \alpha_t < 1 \), i.e., \( x \ast y \notin T_{\vee \wedge q}(A; \alpha_t) \), a contradiction. Therefore \( A_T(x \ast y) \geq \land \{A_T(x), A_T(y), 0.5\} \) whenever \( A_T(x) \land A_T(y) < 0.5 \). Now suppose that \( A_T(x) \land A_T(y) \geq 0.5 \). Then \( x \in T_q(A; 0.5) \) and \( y \in T_q(A; 0.5) \), which imply that \( x \ast y \in T_{\vee \wedge q}(A; 0.5) \). Hence \( A_T(x \ast y) \geq 0.5 \). Otherwise, \( A_T(x \ast y) + 0.5 < 0.5 + 0.5 = 1 \), a contradiction. Consequently, \( A_T(x \ast y) \geq \land \{A_T(x), A_T(y), 0.5\} \) for all \( x, y \in X \).

Similarly, we know that \( A_I(x \ast y) \geq \land \{A_I(x), A_I(y), 0.5\} \) for all \( x, y \in X \). Suppose that \( A_F(x \lor A_F(y) > 0.5 \). If \( A_F(x \lor y) > A_F(x) \lor A_F(y) =: \gamma_f \), then \( x, y \in F_q(A; \gamma_f) \), \( x \ast y \notin F_q(A; \gamma_f) \), and \( A_F(x \ast y) + \gamma_f > 2 \gamma_f > 1 \), i.e., \( x \ast y \notin F_q(A; \gamma_f) \). This is a contradiction. Hence \( A_F(x \ast y) \leq \lor \{A_F(x), A_F(y), 0.5\} \) whenever \( A_F(x) \lor A_F(y) > 0.5 \). Now, assume that \( A_F(x) \lor A_F(y) < 0.5 \). Then \( x, y \in F_q(A; 0.5) \) and so \( x \ast y \notin F_{\vee \wedge q}(A; 0.5) \). Thus \( A_F(x \lor y) \leq 0.5 \) or \( A_F(x \ast y) + 0.5 < 1 \). If \( A_F(x \ast y) > 0.5 \), then \( A_F(x \ast y) + 0.5 > 0.5 + 0.5 = 1 \), a contradiction. Thus \( A_F(x \ast y) \leq 0.5 \), and so \( A_F(x \ast y) \leq \lor \{A_F(x), A_F(y), 0.5\} \) whenever \( A_F(x) \lor A_F(y) \leq 0.5 \). Therefore \( A_F(x \ast y) \leq \lor \{A_F(x), A_F(y), 0.5\} \) for all \( x, y \in X \).

Conversely, let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in \( X \) which satisfies the condition (3.12). Let \( x, y \in X \) and \( \alpha_x, \alpha_y, \beta_x, \beta_y, \gamma_x, \gamma_y \in [0, 1] \). If \( x \in T_q(A; \alpha_x) \) and \( y \in T_q(A; \alpha_y) \), then \( A_T(x) \geq \alpha_x \) and \( A_T(y) \geq \alpha_y \). If \( A_T(x \ast y) < \alpha_x \land \alpha_y \), then \( A_T(x) \land A_T(y) \geq 0.5 \). Otherwise, we have

\[
A_T(x \ast y) \geq \land \{A_T(x), A_T(y), 0.5\} = A_T(x) \land A_T(y) \geq \alpha_x \land \alpha_y,
\]

a contradiction. It follows that

\[
A_T(x \ast y) + \alpha_x \land \alpha_y > 2 A_T(x \ast y) \geq 2 \land \{A_T(x), A_T(y), 0.5\} = 1
\]

and so that \( x \ast y \in T_q(A; \alpha_x \land \alpha_y) \subseteq T_{\vee \wedge q}(A; \alpha_x \land \alpha_y) \). Similarly, if \( x \in I_q(A; \beta_x) \) and \( y \in I_q(A; \beta_y) \), then \( x \ast y \in I_{\vee \wedge q}(A; \beta_x \land \beta_y) \). Now, let \( x \in F_q(A; \gamma_x) \) and
It follows from Theorem 3.6 that

\[ A_F(x \ast y) = \bigvee \{A_F(x), A_F(y), 0.5\} \leq A_F(x) \lor A_F(y) \leq \gamma_x \lor \gamma_y, \]

which is a contradiction. Hence

\[ A_F(x \ast y) + \gamma_x \lor \gamma_y < 2A_F(x \ast y) \leq 2 \bigvee \{A_F(x), A_F(y), 0.5\} = 1, \]

and so \( x \ast y \in F_5(A; \gamma_x \lor \gamma_y) \subseteq F_5(A; \gamma_x \lor \gamma_y) \). Therefore \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q)\)-neutrosophic subalgebra of \( X \).

Theorem 3.7. If \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q)\)-neutrosophic subalgebra of a \( BCK/BCI \)-algebra \( X \), then neutrosophic \( q\)-subsets \( T_q(A; \alpha), I_q(A; \beta) \) and \( F_q(A; \gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in (0.5, 1) \) and \( \gamma \in [0, 0.5) \) whenever they are nonempty.

Proof. Assume that \( T_q(A; \alpha), I_q(A; \beta) \) and \( F_q(A; \gamma) \) are nonempty for all \( \alpha, \beta \in (0.5, 1) \) and \( \gamma \in [0, 0.5) \). Let \( x, y \in T_q(A; \alpha) \). Then \( A_T(x) + \alpha > 1 \) and \( A_T(y) + \alpha > 1 \). It follows from Theorem 3.6 that

\[ A_T(x \ast y) + \alpha \geq \bigwedge \{A_T(x), A_T(y), 0.5\} + \alpha \]

\[ = \bigwedge \{A_T(x) + \alpha, A_T(y) + \alpha, 0.5 + \alpha\} \]

\[ > 1, \]

that is, \( x \ast y \in T_q(A; \alpha) \). Hence \( T_q(A; \alpha) \) is a subalgebra of \( X \). By the similar way, we can induce that \( I_q(A; \beta) \) is a subalgebra of \( X \). Now, let \( x, y \in F_q(A; \gamma) \). Then \( A_F(x) + \gamma < 1 \) and \( A_F(y) + \gamma < 1 \). Using Theorem 3.6, we have

\[ A_F(x \ast y) + \gamma \leq \bigvee \{A_F(x), A_F(y), 0.5\} + \gamma \]

\[ = \bigvee \{A_F(x) + \gamma, A_F(y) + \gamma, 0.5 + \gamma\} \]

\[ < 1, \]

and so \( x \ast y \in F_q(A; \gamma) \). Therefore \( F_q(A; \gamma) \) is a subalgebra of \( X \).

Theorem 3.8. For a neutrosophic set \( A = (A_T, A_I, A_F) \) in a \( BCK/BCI \)-algebra \( X \), if the nonempty neutrosophic \( \in \lor q\)-subsets \( T_{\in \lor q}(A; \alpha), I_{\in \lor q}(A; \beta) \) and \( F_{\in \lor q}(A; \gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in (0.1, 1) \) and \( \gamma \in [0, 1) \), then \( A = (A_T, A_I, A_F) \) is an \((\in, \in \lor q)\)-neutrosophic subalgebra of \( X \).

Proof. Let \( T_{\in \lor q}(A; \alpha) \) be a subalgebra of \( X \) and assume that

\[ A_T(x \ast y) < \bigwedge \{A_T(x), A_T(y), 0.5\} \]

for some \( x, y \in X \). Then there exists \( \alpha \in (0, 0.5] \) such that

\[ A_T(x \ast y) < \alpha \leq \bigwedge \{A_T(x), A_T(y), 0.5\}. \]

It follows that \( x, y \in T_\alpha(A; \alpha) \subseteq T_{\in \lor q}(A; \alpha) \), and so that \( x \ast y \in T_{\in \lor q}(A; \alpha) \). Hence \( A_T(x \ast y) \geq \alpha \) or \( A_T(x \ast y) + \alpha > 1 \). This is a contradiction, and so

\[ A_T(x \ast y) \geq \bigwedge \{A_T(x), A_T(y), 0.5\} \]
for all \( x, y \in X \). Similarly, we show that
\[
A_I(x \ast y) \geq \bigwedge \{A_I(x), A_I(y), 0.5\}
\]
for all \( x, y \in X \). Now let \( F_{\vee \gamma}(A; \gamma) \) be a subalgebra of \( X \) and assume that
\[
A_F(x \ast y) > \bigvee \{A_F(x), A_F(y), 0.5\}
\]
for some \( x, y \in X \). Then
\[
A_F(x \ast y) > \gamma \geq \bigvee \{A_F(x), A_F(y), 0.5\},
\]
for some \( \gamma \in [0.5, 1) \), which implies that \( x, y \in F_{\vee \gamma}(A; \gamma) \subseteq F_{\vee \gamma}(A; \gamma) \). Thus \( x \ast y \in F_{\vee \gamma}(A; \gamma) \). From (3.13), we have \( x \ast y \notin F_{\vee \gamma}(A; \gamma) \) and \( A_F(x \ast y) + \gamma > 2\gamma \geq 1 \), i.e., \( x \ast y \notin F_{\vee \gamma}(A; \gamma) \). This is a contradiction, and hence
\[
A_F(x \ast y) \leq \bigvee \{A_F(x), A_F(y), 0.5\}
\]
for all \( x, y \in X \). Using Theorem 3.6, we know that \( A = (A_T, A_I, A_F) \) is an \((\bar{\in}, \in, \vee)\)-neutrosophic subalgebra of \( X \).

**Theorem 3.9.** If \( A = (A_T, A_I, A_F) \) is an \((\bar{\in}, \in, \vee)\)-neutrosophic subalgebra of a BCK/BCI-algebra \( X \), then nonempty neutrosophic \((\bar{\in}, \in, \vee)\)-subsets \( T_{\vee \gamma \alpha}(A; \alpha) \), \( I_{\vee \gamma \alpha}(A; \beta) \) and \( F_{\vee \gamma \alpha}(A; \gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in (0, 0.5] \) and \( \gamma \in [0.5, 1) \).

**Proof.** Assume that \( T_{\vee \gamma \alpha}(A; \alpha) \), \( I_{\vee \gamma \alpha}(A; \beta) \) and \( F_{\vee \gamma \alpha}(A; \gamma) \) are nonempty for all \( \alpha, \beta \in (0, 0.5] \) and \( \gamma \in [0.5, 1) \). Let \( x, y \in I_{\vee \gamma \alpha}(A; \beta) \). Then
\[
x \in I_{\vee \gamma \alpha}(A; \beta) \text{ or } x \in F_{\vee \gamma \alpha}(A; \gamma),
\]
and
\[
y \in I_{\vee \gamma \alpha}(A; \beta) \text{ or } y \in F_{\vee \gamma \alpha}(A; \gamma).
\]

Hence we have the following four cases:

(i) \( x \in I_{\vee \gamma \alpha}(A; \beta) \) and \( y \in I_{\vee \gamma \alpha}(A; \beta) \),

(ii) \( x \in I_{\vee \gamma \alpha}(A; \beta) \) and \( y \in I_{\vee \gamma \alpha}(A; \beta) \),

(iii) \( x \in I_{\vee \gamma \alpha}(A; \beta) \) and \( y \in I_{\vee \gamma \alpha}(A; \beta) \),

(iv) \( x \in I_{\vee \gamma \alpha}(A; \beta) \) and \( y \in I_{\vee \gamma \alpha}(A; \beta) \).

The first case implies that \( x \ast y \in I_{\vee \gamma \alpha}(A; \beta) \). For the second case, \( y \in I_{\vee \gamma \alpha}(A; \beta) \) induces \( A_I(y) > 1 - \beta \geq \beta \), that is, \( y \in I_{\vee \gamma \alpha}(A; \beta) \). Thus \( x \ast y \in I_{\vee \gamma \alpha}(A; \beta) \). Similarly, the third case implies \( x \ast y \in I_{\vee \gamma \alpha}(A; \beta) \). The last case induces \( A_I(x) > 1 - \beta \geq \beta \) and \( A_I(y) > 1 - \beta \geq \beta \), that is, \( x \in I_{\vee \gamma \alpha}(A; \beta) \) and \( y \in I_{\vee \gamma \alpha}(A; \beta) \). Hence \( x \ast y \in I_{\vee \gamma \alpha}(A; \beta) \). Therefore \( I_{\vee \gamma \alpha}(A; \beta) \) is a subalgebra of \( X \) for all \( \beta \in (0, 0.5) \). By the similar way, we show that \( T_{\vee \gamma \alpha}(A; \alpha) \) is a subalgebra of \( X \) for all \( \alpha \in (0, 0.5] \). Let \( x, y \in F_{\vee \gamma \alpha}(A; \gamma) \). Then
\[
A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1,
\]
and
\[
A_F(y) \leq \gamma \text{ or } A_F(y) + \gamma < 1.
\]
If \( A_F(x) \leq \gamma \) and \( A_F(y) \leq \gamma \), then
\[
A_F(x \ast y) \leq \bigvee \{A_F(x), A_F(y), 0.5\} \leq \bigvee \{\gamma, 0.5\} = \gamma
\]
by Theorem 3.6, and so \(x \ast y \in F_\varepsilon(A; \gamma) \subseteq F_{\vee q}(A; \gamma)\). If \(A_F(x) \leq \gamma\) and \(A_F(y) + \gamma < 1\), then
\[
A_F(x \ast y) \leq \sqrt{\{A_F(x), A_F(y), 0.5\}} \leq \sqrt{\{\gamma, 1 - \gamma, 0.5\}} = \gamma
\]
by Theorem 3.6. Thus \(x \ast y \in F_\varepsilon(A; \gamma) \subseteq F_{\vee q}(A; \gamma)\). Similarly, if \(A_F(x) + \gamma < 1\) and \(A_F(y) \leq \gamma\), then \(x \ast y \in F_{\vee q}(A; \gamma)\). Finally, assume that \(A_F(x) + \gamma < 1\) and \(A_F(y) + \gamma < 1\). Then
\[
A_F(x \ast y) \leq \sqrt{\{A_F(x), A_F(y), 0.5\}} \leq \sqrt{\{1 - \gamma, 0.5\}} = 0.5 < \gamma
\]
by Theorem 3.6. Hence \(x \ast y \in F_\varepsilon(A; \gamma) \subseteq F_{\vee q}(A; \gamma)\). Consequently, \(F_{\vee q}(A; \gamma)\) is a subalgebra of \(X\) for all \(\gamma \in [0.5, 1]\).

**Theorem 3.10.** If \(A = (A_T, A_I, A_F)\) is a \((q, \in \vee q)\)-neutrosophic subalgebra of a BCK/BCI-algebra \(X\), then nonempty neutrosophic \(\in \vee q\)-subsets \(T_{\vee q}(A; \alpha), I_{\vee q}(A; \beta)\) and \(F_{\vee q}(A; \gamma)\) are subalgebras of \(X\) for all \(\alpha, \beta \in (0.5, 1)\) and \(\gamma \in [0, 0.5]\).

**Proof.** Assume that \(T_{\vee q}(A; \alpha), I_{\vee q}(A; \beta)\) and \(F_{\vee q}(A; \gamma)\) are nonempty for all \(\alpha, \beta \in (0.5, 1)\) and \(\gamma \in [0, 0.5]\). Let \(x, y \in T_{\vee q}(A; \alpha)\). Then
\[
x \in T_\varepsilon(A; \alpha) \text{ or } x \in T_q(A; \alpha),
\]
and
\[
y \in T_\varepsilon(A; \alpha) \text{ or } y \in T_q(A; \alpha).
\]
If \(x \in T_q(A; \alpha)\) and \(y \in T_q(A; \alpha)\), then obviously \(x \ast y \in T_{\vee q}(A; \alpha)\). Suppose that \(x \in T_\varepsilon(A; \alpha)\) and \(y \in T_q(A; \alpha)\). Then \(A_T(x) + \alpha \geq 2\alpha > 1\), i.e., \(x \in T_q(A; \alpha)\). It follows that \(x \ast y \in T_{\vee q}(A; \alpha)\). Similarly, if \(x \in T_q(A; \alpha)\) and \(y \in T_\varepsilon(A; \alpha)\), then \(x \ast y \in T_{\vee q}(A; \alpha)\). Now, let \(x, y \in F_{\vee q}(A; \gamma)\). Then
\[
x \in F_\varepsilon(A; \gamma) \text{ or } x \in F_q(A; \gamma),
\]
and
\[
y \in F_\varepsilon(A; \gamma) \text{ or } y \in F_q(A; \gamma).
\]
If \(x \in F_q(A; \gamma)\) and \(y \in F_q(A; \gamma)\), then clearly \(x \ast y \in F_{\vee q}(A; \gamma)\). If \(x \in F_\varepsilon(A; \gamma)\) and \(y \in F_q(A; \gamma)\), then \(A_F(x) + \gamma \leq 2\gamma < 1\), i.e., \(x \in F_q(A; \gamma)\). It follows that \(x \ast y \in F_{\vee q}(A; \gamma)\). Similarly, if \(x \in F_q(A; \gamma)\) and \(y \in F_\varepsilon(A; \gamma)\), then \(x \ast y \in F_{\vee q}(A; \gamma)\). Finally, assume that \(x \in F_\varepsilon(A; \gamma)\) and \(y \in F_\varepsilon(A; \gamma)\). Then \(A_F(x) + \gamma \leq 2\gamma < 1\) and \(A_F(y) + \gamma \leq 2\gamma < 1\), that is, \(x \in F_\varepsilon(A; \gamma)\) and \(y \in F_\varepsilon(A; \gamma)\). Therefore \(x \ast y \in F_{\vee q}(A; \gamma)\). Consequently, \(T_{\vee q}(A; \alpha), I_{\vee q}(A; \beta)\) and \(F_{\vee q}(A; \gamma)\) are subalgebras of \(X\) for all \(\alpha, \beta \in (0.5, 1)\) and \(\gamma \in [0, 0.5]\).

Given a neutrosophic set \(A = (A_T, A_I, A_F)\) in a set \(X\), we consider:
\[
X^3_0 := \{x \in X \mid A_T(x) > 0, A_I(x) > 0, A_F(x) < 1\}.
\]

**Theorem 3.11.** If a neutrosophic set \(A = (A_T, A_I, A_F)\) in a BCK/BCI-algebra \(X\) is an \((\varepsilon, \in)\)-neutrosophic subalgebra of \(X\), then the set \(X^3_0\) is a subalgebra of \(X\).
Proof. Let $x, y \in X_1^0$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Suppose that $A_T(x \ast y) = 0$. Note that $x \in T_\epsilon(A; A_T(x))$ and $y \in T_\epsilon(A; A_T(y))$. But $x \ast y \notin T_\epsilon(A; A_T(x) \land A_T(y))$ because $A_T(x \ast y) = 0 < A_T(x) \land A_T(y)$. This is a contradiction, and thus $A_T(x \ast y) > 0$. By the similar way, we show that $A_I(x \ast y) > 0$. Note that $x \in F_\epsilon(A; A_F(x))$ and $y \in F_\epsilon(A; A_F(y))$. If $A_F(x \ast y) = 1$, then $A_F(x \ast y) = 1 > A_F(x) \lor A_F(y)$, and so $x \ast y \notin F_\epsilon(A; A_F(x) \lor A_F(y))$. This is impossible. Hence $x \ast y \in X_0^1$, and therefore $X_0^1$ is a subalgebra of $X$.

**Theorem 3.12.** If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is an $(\in, q)$-neutrosophic subalgebra of $X$, then the set $X_0^1$ is a subalgebra of $X$.

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. If $A_T(x \ast y) = 0$, then

$$A_T(x \ast y) + A_T(x) \land A_T(y) = A_T(x) \land A_T(y) \leq 1.$$  

Hence $x \ast y \notin T_\epsilon(A; A_T(x) \land A_T(y))$, which is a contradiction since $x \in T_\epsilon(A; A_T(x))$ and $y \in T_\epsilon(A; A_T(y))$. Thus $A_T(x \ast y) > 0$. Similarly, we get $A_I(x \ast y) > 0$. Assume that $A_F(x \ast y) = 1$. Then

$$A_F(x \ast y) + A_F(x) \lor A_F(y) = 1 + A_F(x) \lor A_F(y) \geq 1,$$  

that is, $x \ast y \notin F_\epsilon(A; A_F(x) \lor A_F(y))$. This is a contradiction because of $x \in F_\epsilon(A; A_F(x))$ and $y \in F_\epsilon(A; A_F(y))$. Hence $A_F(x \ast y) < 1$. Consequently, $x \ast y \in X_0^1$ and $X_0^1$ is a subalgebra of $X$.

**Theorem 3.13.** If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is a $(q, \in)$-neutrosophic subalgebra of $X$, then the set $X_0^1$ is a subalgebra of $X$.

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. It follows that $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_\epsilon(A; 1) \cap I_\epsilon(A; 1) \cap F_\epsilon(A; 0)$. If $A_T(x \ast y) = 0$ or $A_I(x \ast y) = 0$, then $A_T(x \ast y) < 1 \land 1$ or $A_I(x \ast y) < 1 \land 1$. Thus $x \ast y \notin T_\epsilon(A; 1 \land 1)$ or $x \ast y \notin I_\epsilon(A; 1 \land 1)$, a contradiction. Hence $A_T(x \ast y) > 0$ and $A_I(x \ast y) > 0$. If $A_F(x \ast y) = 1$, then $x \ast y \notin F_\epsilon(A; 0 \lor 0)$ which is a contradiction. Thus $A_F(x \ast y) < 1$. Therefore $x \ast y \in X_0^1$ and the proof is complete.

**Theorem 3.14.** If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is a $(q, q)$-neutrosophic subalgebra of $X$, then the set $X_0^1$ is a subalgebra of $X$.

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Hence $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_\epsilon(A; 1) \cap I_\epsilon(A; 1) \cap F_\epsilon(A; 0)$. If $A_T(x \ast y) = 0$ or $A_I(x \ast y) = 0$, then

$$A_T(x \ast y) + 1 \land 1 = 0 + 1 = 1$$  

or

$$A_I(x \ast y) + 1 \land 1 = 0 + 1 = 1,$$  

and so $x \ast y \notin T_\epsilon(A; 1 \land 1)$ or $x \ast y \notin I_\epsilon(A; 1 \land 1)$. This is impossible, and thus $A_T(x \ast y) > 0$ and $A_I(x \ast y) > 0$. If $A_F(x \ast y) = 1$, then $A_F(x \ast y) + 0 \lor 0 = 1$, that
is, \(x * y \notin F_q(A; 0 \lor 0)\). This is a contradiction, and so \(A_F(x * y) < 1\). Therefore \(x * y \in X^1\) and the proof is complete. \(\Box\)

**Theorem 3.15.** If a neutrosophic set \(A = (A_T, A_I, A_F)\) in a BCK/BCI-algebra \(X\) is a \((q, q)\)-neutrosophic subalgebra of \(X\), then \(A = (A_T, A_I, A_F)\) is neutrosophic constant on \(X^1\), that is, \(A_T\), \(A_I\) and \(A_F\) are constants on \(X^1\).

**Proof.** Assume that \(A_T\) is not constant on \(X^1\). Then there exist \(y \in X^1\) such that \(A_T(y) \neq A_T(0) = a_0\). Then either \(a_y > a_0\) or \(a_y < a_0\). Suppose \(a_y < a_0\) and choose \(a_1, a_2 \in (0, 1]\) such that \(1 - a_0 < a_1 \leq 1 - a_y < a_2\). Then \(A_T(0) + a_1 = a_0 + a_1 > 1\) and \(A_T(y) + a_2 = a_y + a_2 > 1\), which imply that \(0 \in T_q(A; a_1)\) and \(y \in T_q(A; a_2)\). Since
\[
A_T(y * 0) + a_1 \wedge a_2 = A_T(y) + a_1 = a_y + a_1 \leq 1,
\]
we get \(y * 0 \notin T_q(A; a_1 \wedge a_2)\), which is a contradiction. Next assume that \(a_y > a_0\). Then \(A_T(y) + (1 - a_0) = a_y + 1 - a_0 > 1\) and so \(y \in T_q(A; 1 - a_0)\). Since
\[
A_T(y * y) + (1 - a_0) = A_T(0) + 1 - a_0 = a_0 + 1 - a_0 = 1,
\]
we have \(y * y \notin T_q(A; (1 - a_0) \wedge (1 - a_0))\). This is impossible. Therefore \(A_T\) is constant on \(X^1\). Similarly, \(A_I\) is constant on \(X^1\). Finally, suppose that \(A_F\) is not constant on \(X^1\). Then \(A_F(y) \neq A_F(0) = \gamma_0\) for some \(y \in X^1\), and we have two cases:

(i) \(\gamma_y < \gamma_0\) and (ii) \(\gamma_y > \gamma_0\).

The first case implies that \(A_F(y) + 1 - \gamma_0 = \gamma_y + 1 - \gamma_0 < 1\), that is, \(y \in F_q(A; 1 - \gamma_0)\). Hence \(y * y \in F_q(A; (1 - \gamma_0) \lor (1 - \gamma_0))\), i.e., \(0 \in F_q(A; 1 - \gamma_0)\), which is a contradiction since \(A_F(0) + 1 - \gamma_0 = 1\). For the second case, there exist \(\gamma_1, \gamma_2 \in (0, 1)\) such that
\[
1 - \gamma_0 > \gamma_1 > 1 - \gamma_y > \gamma_2.
\]
Then \(A_F(y) + \gamma_2 = \gamma_y + \gamma_2 < 1\), i.e., \(y \in F_q(A; \gamma_2)\), and \(A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 < 1\), i.e., \(0 \in F_q(A; \gamma_1)\). It follows that \(y * 0 \notin F_q(A; \gamma_1 \lor \gamma_2)\). But
\[
A_F(y * 0) + \gamma_1 \lor \gamma_2 = A_F(y) + \gamma_1 = \gamma_y + \gamma_1 > 1,
\]
and so \(y * 0 \notin F_q(A; \gamma_1 \lor \gamma_2)\). This is a contradiction. Therefore \(A_F\) is constant on \(X^1\). This completes the proof. \(\Box\)

We provide conditions for a neutrosophic set to be a \((q, \lor \land q)\)-neutrosophic subalgebra.

**Theorem 3.16.** For a subalgebra \(S\) of a BCK/BCI-algebra \(X\), let \(A = (A_T, A_I, A_F)\) be a neutrosophic set in \(X\) such that
\[
\begin{align*}
(\forall x \in S) (A_T(x) & \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5) , \\
(\forall x \in X \setminus S) (A_T(x) &= 0, A_I(x) = 0, A_F(x) = 1).
\end{align*}
\]
Then \(A = (A_T, A_I, A_F)\) is a \((q, \lor \land q)\)-neutrosophic subalgebra of \(X\).

**Proof.** Assume that \(x \in I_q(A; \beta_x)\) and \(y \in I_q(A; \beta_y)\) for all \(x, y \in X\) and \(\beta_x, \beta_y \in [0, 1]\). Then \(A_I(x) + \beta_x > 1\) and \(A_I(y) + \beta_y > 1\). If \(x * y \notin S\), then \(x \in X \setminus S\) or \(y \in X \setminus S\) since \(S\) is a subalgebra of \(X\). Hence \(A_I(x) = 0\) or \(A_I(y) = 0\), which imply that \(\beta_x > 1\) or \(\beta_y > 1\). This is a contradiction, and so \(x * y \in S\). If \(\beta_x \land \beta_y > 0.5\),
then $A_I(x \ast y) + \beta_x \wedge \beta_y > 1$, i.e., $x \ast y \in I_q(A; \beta_x \wedge \beta_y)$. If $\beta_x \wedge \beta_y \leq 0.5$, then $A_I(x \ast y) \geq 0.5 \geq \beta_x \wedge \beta_y$, i.e., $x \ast y \in I_q(A; \beta_x \wedge \beta_y)$. Hence $x \ast y \in I_{\text{evq}}(A; \beta_x \wedge \beta_y)$.

Similarly, if $x \in T_q(A; \alpha_x)$ and $y \in T_q(A; \alpha_y)$ for all $x, y \in X$ and $\alpha_x, \alpha_y \in [0, 1]$, then $x \ast y \in T_{\text{evq}}(A; \alpha_x \wedge \alpha_y)$. Now let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1]$ be such that $x \in F_q(A; \gamma_x)$ and $y \in F_q(A; \gamma_y)$. Then $A_F(x) + \gamma_x < 1$ and $A_F(y) + \gamma_y < 1$. It follows that $x \ast y \in S$. In fact, if not then $x \in X \setminus S$ or $y \in X \setminus S$ since $S$ is a subalgebra of $X$. Hence $A_F(x) = 1$ or $A_F(y) = 1$, which imply that $\gamma_x < 0$ or $\gamma_y < 0$. This is a contradiction, and so $x \ast y \in S$. If $\gamma_x \vee \gamma_y > 0.5$, then $A_F(x \ast y) \leq 0.5 \leq \gamma_x \vee \gamma_y$, that is, $x \ast y \in F_q(A; \gamma_x \vee \gamma_y)$. Hence $x \ast y \in F_{\text{evq}}(A; \gamma_x \vee \gamma_y)$, and consequently $A = (A_T, A_I, A_F)$ is a $(q, \in, \notin)$-neutrosophic subalgebra of $X$.

Combining Theorems 3.5 and 3.16, we have the following corollary.

**Corollary 3.17.** For a subalgebra $S$ of $X$, if $A = (A_T, A_I, A_F)$ is a neutrosophic set in $X$ satisfying conditions (3.14) and (3.15), then $T_q(A; \alpha), I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.

**Theorem 3.18.** Let $A = (A_T, A_I, A_F)$ be a $(q, \in, \notin)$-neutrosophic subalgebra of $X$ in which $A_T, A_I$ and $A_F$ are not constant on $X_0^1$. Then there exist $x, y, z \in X$ such that $A_T(x) \geq 0.5$, $A_I(y) \geq 0.5$ and $A_F(z) \leq 0.5$.

In particular, $A_T(x) \geq 0.5$, $A_I(y) \geq 0.5$ and $A_F(z) \leq 0.5$ for all $x, y, z \in X_0^1$.

**Proof.** Assume that $A_T(x) < 0.5$ for all $x \in X$. Since there exists $a \in X_0^1$ such that $\alpha_a = A_T(a) \neq A_T(0) = \alpha_0$, we have $\alpha_a > \alpha_0$ or $\alpha_a < \alpha_0$. If $\alpha_a > \alpha_0$, then we can choose $\delta > 0.5$ such that $\alpha_0 + \delta < 1 < \alpha_a + \delta$. It follows that $a \in T_q(A; \delta)$, $A_T(a * a) = A_T(0) = \alpha_0 < \delta = \delta \wedge \delta$ and $A_T(a * a) + \delta \wedge \delta = A_T(0) + \delta = \alpha_0 + \delta < 1$ so that $a * a \notin T_{\text{evq}}(A; \delta \wedge \delta)$. This is a contradiction. Now if $\alpha_a < \alpha_0$, we can take $\delta > 0.5$ such that $\alpha_a + \delta < 1 < \alpha_0 + \delta$. Then $0 \in T_q(A; \delta)$ and $a \in T_q(A; 1)$, but $a * 0 \notin T_{\text{evq}}(A; 1 \wedge \delta)$ since $A_T(a) < 0.5 < \delta$ and $A_T(a) + \delta = \alpha_0 + \delta < 1$. This is also a contradiction. Thus $A_T(x) \geq 0.5$ for some $x \in X$. Similarly, we know that $A_I(y) \geq 0.5$ for some $y \in X$. Finally, suppose that $A_F(z) < 0.5$ for all $z \in X$. Note that $\gamma_c = A_F(c) \neq A_F(0) = \gamma_0$ for some $c \in X_0^1$. It follows that $\gamma_c < \gamma_0$ or $\gamma_c > \gamma_0$. We first consider the case $\gamma_c < \gamma_0$. Then $\gamma_0 + \varepsilon > 1 > \gamma_c + \varepsilon$ for some $\varepsilon \in (0, 0.5]$, and so $c \in F_q(A; \varepsilon)$. Also $A_F(c * c) = A_F(0) = \gamma_0 > \delta$ and $A_F(c * c) + \varepsilon \wedge \varepsilon = A_F(0) + \varepsilon \wedge \varepsilon = \gamma_0 + \varepsilon > 1$ which shows that $c * c \notin F_{\text{evq}}(A; \varepsilon \wedge \varepsilon)$. This is impossible. Now, if $\gamma_c > \gamma_0$, then we can take $\varepsilon \in (0, 0.5]$ and so that $\gamma_0 + \varepsilon < 1 < \gamma_c + \varepsilon$. It follows that $0 \notin F_q(A; \varepsilon)$ and $c \notin F_q(A; 0)$. Since $A_F(c * 0) = A_F(c) = \gamma_c > \varepsilon$ and $A_F(c * 0) + \varepsilon = A_F(c) + \varepsilon = \gamma_c + \varepsilon > 1$, we have $c * 0 \notin F_{\text{evq}}(A; \varepsilon)$. This is a contradiction, and therefore $A_F(z) < 0.5$ for some $z \in X$. We now show that $A_T(0) \geq 0.5$, $A_I(0) \geq 0.5$ and $A_F(0) \leq 0.5$. Suppose that $A_T(0) = \alpha_0 < 0.5$. Since there exists $x \in X$ such that $A_T(x) = \alpha_x \geq 0.5$, it follows that $\alpha_0 < \alpha_x$. Choose $\alpha_1 \in [0, 1]$ such that $\alpha_1 > \alpha_0$ and $\alpha_0 + \alpha_1 < 1 < \alpha_x + \alpha_1$. Then $A_T(x) + \alpha_1 = \alpha_x + \alpha_1 > 1$, and so $x \in T_q(A; \alpha_1)$. Now we have $A_F(x * x) + \alpha_1 = A_F(0) + \alpha_1 = \alpha_0 + \alpha_1 < 1$ and $A_T(x * x) = A_T(0) = \alpha_0 < \alpha_1 = \alpha_x + \alpha_1$. Thus $x * x \notin F_{\text{evq}}(A; \alpha_1 \wedge \alpha_1)$, a contradiction. Hence $A_T(0) \geq 0.5$. Similarly, we have $A_I(0) \geq 0.5$. Assume that $A_F(0) = \gamma_0 > 0.5$. Note that $A_F(z) = \gamma_z \leq 0.5$ for some $z \in X$. Hence $\gamma_z < \gamma_0$, and

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so we can take \( \gamma_1 \in [0, 1] \) such that \( \gamma_1 < \gamma_0 \) and \( \gamma_0 + \gamma_1 > 1 > \gamma_2 + \gamma_1 \). It follows that \( A_F(z) + \gamma_1 = \gamma_2 + \gamma_1 < 1 \), that is, \( z \in F_q(A; \gamma_1) \). Also \( A_F(z + z) = A_F(0) = \gamma_0 > \gamma_1 = \gamma_1 \vee \gamma_1 \), i.e., \( z \neq F_q(A; \gamma_1 \vee \gamma_1) \), and \( A_F(z \vee z) = A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 > 1 \), i.e., \( z \neq F_q(A; \gamma_1 \vee \gamma_1) \). Thus \( z \neq F_q(A; \gamma_1 \vee \gamma_1) \), a contradiction. Hence \( A_F(0) \leq 0.5 \). We finally show that \( A_T(x) \geq 0.5 \), \( A_T(y) \geq 0.5 \) and \( A_F(z) \leq 0.5 \) for all \( x, y, z \in X_0 \). We first assume that \( A_T(y) = \beta_y < 0.5 \) for some \( y \in X_0 \), and take \( \beta > 0 \) such that \( \beta_y + \beta < 0.5 \). Then \( A_T(y) + 1 = \beta_y + 1 > 1 \) and \( A_T(0) + \beta = 0.5 > 1 \), which imply that \( y \in I_q(A; 1) \) and \( 0 \in I_q(A; \beta + 0.5) \). But \( y \neq 0 \neq I_q(A; \beta + 0.5) \) since \( A_T(y * 0) = A_T(y) < \beta + 0.5 < 1 \) and \( A_T(0) + 1 \leq (\beta + 0.5) \). This is a contradiction. Hence \( A_T(y) \geq 0.5 \) for all \( y \in X_1 \). Similarly, we induces \( A_T(x) \geq 0.5 \) for all \( x \in X_1 \). Suppose \( A_T(z) = \gamma_z > 0.5 \) for some \( z \in X_0 \), and take \( \gamma \in (0, 0.5) \) such that \( \gamma_z > 0.5 + \gamma \). Then \( z \neq F_q(A; 0) \) and \( A_F(0) = 0.5 - \gamma \leq 1 - 1 - \gamma < 1 \), i.e., \( 0 \neq F_q(A; 0.5 - \gamma) \). But \( A_F(z * 0) = A_F(z) > 0.5 - \gamma + \gamma = A_F(z) + 0.5 - \gamma = \gamma_z + 0.5 - \gamma > 1 \), which imply that \( z \neq F_q(A; 0.5 - \gamma) \). This is a contradiction, and the proof is complete. \( \square \)

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References


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Neutrosophic subalgebras of $BCK/BCI$-algebras based on neutrosophic points

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Abstract. Properties on neutrosophic $\in \lor q$-subsets and neutrosophic $q$-subsets are investigated. Relations between an $(\in, \in \lor q)$-neutrosophic subalgebra and a $(q, \in \lor q)$-neutrosophic subalgebra are considered. Characterization of an $(\in, \in \lor q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets are discussed. Conditions for an $(\in, \in \lor q)$-neutrosophic subalgebra to be a $(q, \in \lor q)$-neutrosophic subalgebra are provided.

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1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [17, 18, 19] is a more general platform which extends the concepts of the classic set and fuzzy set (see [20], [21]), intuitionistic fuzzy set (see [1]) and interval valued intuitionistic fuzzy set (see [2]). Neutrosophic set theory is applied to various part (see [4], [5], [8], [9], [10], [11], [12], [13], [15], [16]). For further particulars, we refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Barbhuiya [3] introduced and studied the concept of $(\in, \in \lor q)$-intuitionistic fuzzy ideals of $BCK/BCI$-algebras. Jun [7] introduced the notion of neutrosophic subalgebras in $BCK/BCI$-algebras with several types. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \lor q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic $q$-subsets and neutrosophic $\in \lor q$-subsets, he considered conditions for the neutrosophic $\in$-subsets, neutrosophic $q$-subsets and neutrosophic $\in \lor q$-subsets to be subalgebras. He discussed conditions for a neutrosophic set to be a $(q, \in \lor q)$-neutrosophic subalgebra.
In this paper, we give relations between an \((\in, \in \lor q)\)-neutrosophic subalgebra and a \((q, \in \lor q)\)-neutrosophic subalgebra. We discuss characterization of an \((\in, \in \lor q)\)-neutrosophic subalgebra by using neutrosophic \(\in\)-subsets. We provide conditions for an \((\in, \in \lor q)\)-neutrosophic subalgebra to be a \((q, \in \lor q)\)-neutrosophic subalgebra. We investigate properties on neutrosophic \(q\)-subsets and neutrosophic \(\in \lor q\)-subsets.

2. Preliminaries

By a \(BCI\)-algebra we mean an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the axioms:

\[
\begin{align*}
(a1) \quad & (x * y) * (x * z) * (z * y) = 0, \\
(a2) \quad & x * (x * y) * y = 0, \\
(a3) \quad & x * x = 0, \\
(a4) \quad & x * y = y * x = 0 \Rightarrow x = y,
\end{align*}
\]

for all \(x, y, z \in X\). If a \(BCI\)-algebra \(X\) satisfies the axiom

\[
(a5) \quad 0 * x = 0 \text{ for all } x \in X,
\]

then we say that \(X\) is a \(BCK\)-algebra. A nonempty subset \(S\) of a \(BCK/BCI\)-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\).

We refer the reader to the books [6] and [14] for further information regarding \(BCK/BCI\)-algebras.

For any family \(\{a_i \mid i \in \Lambda\}\) of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} 
\max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\sup \{a_i \mid i \in \Lambda\} & \text{otherwise}.
\end{cases}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} 
\min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\inf \{a_i \mid i \in \Lambda\} & \text{otherwise}.
\end{cases}
\]

If \(\Lambda = \{1, 2\}\), we will also use \(a_1 \lor a_2\) and \(a_1 \land a_2\) instead of \(\bigvee \{a_i \mid i \in \Lambda\}\) and \(\bigwedge \{a_i \mid i \in \Lambda\}\), respectively.

Let \(X\) be a non-empty set. A neutrosophic set (NS) in \(X\) (see [18]) is a structure of the form:

\[
A := \{<x; A_T(x), A_I(x), A_F(x)> \mid x \in X\}
\]

where \(A_T: X \to [0, 1]\) is a truth membership function, \(A_I: X \to [0, 1]\) is an indeterminate membership function, and \(A_F: X \to [0, 1]\) is a false membership function. For the sake of simplicity, we shall use the symbol \(A = (A_T, A_I, A_F)\) for the neutrosophic set

\[
A := \{<x; A_T(x), A_I(x), A_F(x)> \mid x \in X\}.
\]
3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set $X$, $\alpha, \beta \in (0, 1)$ and $\gamma \in [0, 1)$, we consider the following sets:

$$
T_\varepsilon(A; \alpha) := \{ x \in X \mid A_T(x) \geq \alpha \}, \\
I_\varepsilon(A; \beta) := \{ x \in X \mid A_I(x) \geq \beta \}, \\
F_\varepsilon(A; \gamma) := \{ x \in X \mid A_F(x) \leq \gamma \}, \\
T_\eta(A; \alpha) := \{ x \in X \mid A_T(x) + \alpha > 1 \}, \\
I_\eta(A; \beta) := \{ x \in X \mid A_I(x) + \beta > 1 \}, \\
F_\eta(A; \gamma) := \{ x \in X \mid A_F(x) + \gamma < 1 \}, \\
T_{\vee \eta}(A; \alpha) := \{ x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1 \}, \\
I_{\vee \eta}(A; \beta) := \{ x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1 \}, \\
F_{\vee \eta}(A; \gamma) := \{ x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1 \}.
$$

We say $T_\varepsilon(A; \alpha)$, $I_\varepsilon(A; \beta)$ and $F_\varepsilon(A; \gamma)$ are neutrosophic $\varepsilon$-subsets; $T_\eta(A; \alpha)$, $I_\eta(A; \beta)$ and $F_\eta(A; \gamma)$ are neutrosophic $\eta$-subsets; and $T_{\vee \eta}(A; \alpha)$, $I_{\vee \eta}(A; \beta)$ and $F_{\vee \eta}(A; \gamma)$ are neutrosophic $\vee \eta$-subsets. For $\Phi \in \{\varepsilon, \eta, \vee \eta\}$, the element of $T_{\Phi}(A; \alpha)$ (resp., $I_{\Phi}(A; \beta)$ and $F_{\Phi}(A; \gamma)$) is called a neutrosophic $T_{\Phi}$-point (resp., neutrosophic $I_{\Phi}$-point and neutrosophic $F_{\Phi}$-point) with value $\alpha$ (resp., $\beta$ and $\gamma$) (see [7]).

It is clear that

$$(3.1) \quad T_{\vee \eta}(A; \alpha) = T_{\varepsilon}(A; \alpha) \cup T_{\eta}(A; \alpha),$$

$$(3.2) \quad I_{\vee \eta}(A; \beta) = I_{\varepsilon}(A; \beta) \cup I_{\eta}(A; \beta),$$

$$(3.3) \quad F_{\vee \eta}(A; \gamma) = F_{\varepsilon}(A; \gamma) \cup F_{\eta}(A; \gamma).$$

**Definition 3.1** ([7]). Given $\Phi, \Psi \in \{\varepsilon, \eta, \vee \eta\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is called a $(\Phi, \Psi)$-neutrosophic subalgebra of $X$ if the following assertions are valid.

$$(3.4) \quad x \in T_{\Phi}(A; \alpha_x), \ y \in T_{\Phi}(A; \alpha_y) \Rightarrow x \ast y \in T_{\Phi}(A; \alpha_x \land \alpha_y),$$

$$(3.5) \quad x \in I_{\Phi}(A; \beta_x), \ y \in I_{\Phi}(A; \beta_y) \Rightarrow x \ast y \in I_{\Phi}(A; \beta_x \land \beta_y),$$

$$(3.6) \quad x \in F_{\Phi}(A; \gamma_x), \ y \in F_{\Phi}(A; \gamma_y) \Rightarrow x \ast y \in F_{\Phi}(A; \gamma_x \lor \gamma_y)$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

**Lemma 3.2** ([7]). A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is an $(\varepsilon, \eta, \vee \eta)$-neutrosophic subalgebra of $X$ if and only if it satisfies:

$$
(\forall x, y \in X) \left( \begin{array}{c}
A_T(x \ast y) \geq \bigwedge \{A_T(x), A_T(y), 0.5\} \\
A_I(x \ast y) \geq \bigwedge \{A_I(x), A_I(y), 0.5\} \\
A_F(x \ast y) \leq \bigvee \{A_F(x), A_F(y), 0.5\}
\end{array} \right).
$$

**Theorem 3.3.** A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$ is an $(\varepsilon, \eta, \vee \eta)$-neutrosophic subalgebra of $X$ if and only if the neutrosophic $\varepsilon$-subsets $T_{\varepsilon}(A; \alpha)$, $I_{\varepsilon}(A; \beta)$ and $F_{\varepsilon}(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. 

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Assume that $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \in \vee q)$-neutrosophic subalgebra of $X$. For any $x, y \in X$, let $\alpha \in (0, 0.5]$ be such that $x, y \in T_{\varepsilon}(A; \alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from (3.5) that

$$A_T(x * y) \geq \bigwedge \{A_T(x), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha$$

and so that $x * y \in T_{\varepsilon}(A; \alpha)$. Thus $T_{\varepsilon}(A; \alpha)$ is a subalgebra of $X$ for all $\alpha \in (0, 0.5]$. Similarly, $I_{\varepsilon}(A; \beta)$ is a subalgebra of $X$ for all $\beta \in (0, 0.5]$. Now, let $\gamma \in [0.5, 1)$ be such that $x, y \in F_{\varepsilon}(A; \gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Hence

$$A_F(x * y) \leq \bigvee \{A_F(x), A_F(y), 0.5\} \leq \gamma \vee 0.5 = \gamma$$

by (3.5), and so $x * y \in F_{\varepsilon}(A; \gamma)$. Thus $F_{\varepsilon}(A; \gamma)$ is a subalgebra of $X$ for all $\gamma \in [0.5, 1)$.

Conversely, let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_{\varepsilon}(A; \alpha), I_{\varepsilon}(A; \beta)$ and $F_{\varepsilon}(A; \gamma)$ are subalgebras of $X$. If there exist $a, b \in X$ such that

$$A_I(a * b) < \bigwedge \{A_I(a), A_I(b), 0.5\},$$

then we can take $\beta \in (0, 1)$ such that

$$A_I(a * b) < \beta < \bigwedge \{A_I(a), A_I(b), 0.5\}.$$  \hspace{1cm} (3.6)

Thus $a, b \in I_{\varepsilon}(A; \beta)$ and $\beta < 0.5$, and so $a * b \in I_{\varepsilon}(A; \beta)$. But, the left inequality in (3.6) induces $a * b \notin I_{\varepsilon}(A; \beta)$, a contradiction. Hence

$$A_I(x * y) \geq \bigwedge \{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Similarly, we can show that

$$A_T(x * y) \geq \bigwedge \{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Now suppose that

$$A_F(a * b) > \bigvee \{A_F(a), A_F(b), 0.5\}$$

for some $a, b \in X$. Then there exists $\gamma \in (0, 1)$ such that

$$A_F(a * b) > \gamma > \bigvee \{A_F(a), A_F(b), 0.5\}.$$  \hspace{1cm} (3.7)

It follows that $\gamma \in (0.5, 1)$ and $a, b \in F_{\varepsilon}(A; \gamma)$. Since $F_{\varepsilon}(A; \gamma)$ is a subalgebra of $X$, we have $a * b \in F_{\varepsilon}(A; \gamma)$ and so $A_F(a * b) \leq \gamma$. This is a contradiction, and thus

$$A_F(x * y) \leq \bigvee \{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Lemma 3.2, $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \in \vee q)$-neutrosophic subalgebra of $X$.

Using Theorem 3.3 and [7, Theorem 3.8], we have the following corollary.

**Corollary 3.4.** For a neutrosophic set $A = (A_T, A_I, A_F)$ in a $BCK/BCI$-algebra $X$, if the nonempty neutrosophic $\varepsilon \vee q$-subsets $T_{\varepsilon q}(A; \alpha)$, $I_{\varepsilon q}(A; \beta)$ and $F_{\varepsilon q}(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the neutrosophic $\varepsilon$-subsets $T_{\varepsilon q}(A; \alpha)$, $I_{\varepsilon q}(A; \beta)$ and $F_{\varepsilon q}(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.  \hspace{1cm} $\square$
Theorem 3.5. Given neutrosophic set \( A = (A_T, A_I, A_F) \) in a BCK/BCI-algebra \( X \), the nonempty neutrosophic \( \varepsilon \)-subsets \( T_\varepsilon(A; \alpha) \), \( I_\varepsilon(A; \beta) \) and \( F_\varepsilon(A; \gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in (0, 1) \) and \( \gamma \in [0, 1] \) if and only if the following assertion is valid.

\[
(3.7) \quad \begin{cases} 
A_T(x*y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \\
A_I(x*y) \vee 0.5 \geq A_I(x) \wedge A_I(y) \\
A_F(x*y) \wedge 0.5 \leq A_F(x) \vee A_F(y)
\end{cases}
\]

Proof. Assume that the nonempty neutrosophic \( \varepsilon \)-subsets \( T_\varepsilon(A; \alpha) \), \( I_\varepsilon(A; \beta) \) and \( F_\varepsilon(A; \gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in (0, 1) \) and \( \gamma \in [0, 1] \). Suppose that there are \( a, b \in X \) such that \( A_T(a*b) \vee 0.5 < A_T(a) \wedge A_T(b) := \alpha \). Then \( \alpha \in (0, 1) \) and \( a, b \in T_\varepsilon(A; \alpha) \). Since \( T_\varepsilon(A; \alpha) \) is a subalgebra of \( X \), it follows that \( a*b \in T_\varepsilon(A; \alpha) \), that is, \( A_T(a*b) \geq \alpha \) which is a contradiction. Thus

\[
A_T(x*y) \vee 0.5 \geq A_T(x) \wedge A_T(y)
\]

for all \( x, y \in X \). Similarly, we know that \( A_I(x*y) \vee 0.5 \geq A_I(x) \wedge A_I(y) \) for all \( x, y \in X \). Now, if \( A_F(x*y) \wedge 0.5 > A_F(x) \vee A_F(y) \) for some \( x, y \in X \), then \( x, y \in F_\varepsilon(A; \gamma) \) and \( \gamma \in [0, 0.5) \) where \( \gamma = A_F(x) \vee A_F(y) \). But, \( x*y \notin F_\varepsilon(A; \gamma) \) which is a contradiction. Hence \( A_F(x*y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \) for all \( x, y \in X \).

Conversely, let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in \( X \) satisfying the condition \((3.7)\). Let \( x, y, a, b \in X \) and \( \alpha, \beta \in (0, 1) \) be such that \( x, y \in T_\varepsilon(A; \alpha) \) and \( a, b \in I_\varepsilon(A; \beta) \). Then

\[
A_T(x*y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \geq \alpha > 0.5,
\]

\[
A_I(a*b) \vee 0.5 \geq A_I(a) \wedge A_I(b) \geq \beta > 0.5.
\]

It follows that \( A_T(x*y) \geq \alpha \) and \( A_I(a*b) \geq \beta \), that is, \( x*y \in T_\varepsilon(A; \alpha) \) and \( a*b \in I_\varepsilon(A; \beta) \). Now, let \( x, y \in X \) and \( \gamma \in [0, 0.5) \) be such that \( x, y \in F_\varepsilon(A; \gamma) \). Then \( A_F(x*y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \leq \gamma < 0.5 \) and so \( A_F(x*y) \leq \gamma \), i.e., \( x*y \in F_\varepsilon(A; \gamma) \). This completes the proof. \( \square \)

We consider relations between a \((q, \in \vee q)\)-neutrosophic subalgebra and an \((\in, \varepsilon \vee q)\)-neutrosophic subalgebra.

Theorem 3.6. In a BCK/BCI-algebra, every \((q, \in \vee q)\)-neutrosophic subalgebra is an \((\in, \varepsilon \vee q)\)-neutrosophic subalgebra.

Proof. Let \( A = (A_T, A_I, A_F) \) be a \((q, \in \vee q)\)-neutrosophic subalgebra of a BCK/BCI-algebra \( X \) and let \( x, y \in X \). Let \( \alpha_x, \alpha_y \in (0, 1) \) be such that \( x \in T_{\varepsilon \vee q}(A; \alpha_x) \) and \( y \in T_{\varepsilon \vee q}(A; \alpha_y) \). Then \( A_T(x) \geq \alpha_x \) and \( A_T(y) \geq \alpha_y \). Suppose \( x*y \notin T_{\varepsilon \vee q}(A; \alpha_x \wedge \alpha_y) \). Then

\[
A_T(x*y) < \alpha_x \wedge \alpha_y,
\]

\[
A_T(x*y) + (\alpha_x \wedge \alpha_y) \leq 1.
\]

It follows that

\[
A_T(x*y) < 0.5.
\]
Combining (3.8) and (3.10), we have

$$A_T(x * y) < \bigwedge \{\alpha_x, \alpha_y, 0.5\}$$

and so

$$1 - A_T(x * y) > 1 - \bigwedge \{\alpha_x, \alpha_y, 0.5\}$$

$$= \bigvee \{1 - \alpha_x, 1 - \alpha_y, 0.5\}$$

$$\geq \bigvee \{1 - A_T(x), 1 - A_T(y), 0.5\}.$$ 

Hence there exists \(\alpha \in (0, 1]\) such that

$$1 - A_T(x * y) \geq \alpha > \bigvee \{1 - A_T(x), 1 - A_T(y), 0.5\}. \quad (3.11)$$

The right inequality in (3.11) induces \(A_T(x) + \alpha > 1\) and \(A_T(y) + \alpha > 1\), that is, \(x, y \in T_q(A; \alpha)\). Since \(A = (A_T, A_I, A_P)\) is a \((\exists \in \vee q)\)-neutrosophic subalgebra of \(X\), we have \(x * y \in T_{\vee q}(A; \alpha)\). But, the left inequality in (3.11) implies that \(A_T(x * y) + \alpha \leq 1\), i.e., \(x * y \notin T_q(A; \alpha)\), and \(A_T(x * y) \leq 1 - \alpha < 1 - 0.5 = 0.5 < \alpha\), i.e., \(x * y \notin T_{\vee q}(A; \alpha)\). Hence \(x * y \notin T_{\vee q}(A; \alpha)\), a contradiction. Thus \(x * y \in T_{\vee q}(A; \alpha_x \wedge \alpha_y)\). Similarly, we can show that if \(x \in I_{\vee}(A; \beta_x)\) and \(y \in I_{\vee}(A; \beta_y)\) for \(\beta_x, \beta_y \in (0, 1]\), then \(x * y \in I_{\vee q}(A; \beta_x \wedge \beta_y)\). Now, let \(\gamma_x, \gamma_y \in (0, 1]\) be such that \(x \in F_{\vee}(A; \gamma_x)\) and \(y \in F_{\vee}(A; \gamma_y)\). \(A_F(x) \leq \gamma_x\) and \(A_F(y) \leq \gamma_y\). If \(x * y \notin F_{\vee q}(A; \gamma_x \vee \gamma_y)\), then

\[
\begin{align*}
A_F(x * y) & > \gamma_x \vee \gamma_y, \quad (3.12) \\
A_F(x * y) + (\gamma_x \vee \gamma_y) & \geq 1. \quad (3.13)
\end{align*}
\]

It follows that

$$A_F(x * y) > \bigvee \{\gamma_x, \gamma_y, 0.5\}$$

and so that

$$1 - A_F(x * y) < 1 - \bigvee \{\gamma_x, \gamma_y, 0.5\}$$

$$= \bigwedge \{1 - \gamma_x, 1 - \gamma_y, 0.5\}$$

$$\leq \bigwedge \{1 - A_F(x), 1 - A_F(y), 0.5\}.$$ 

Thus there exists \(\gamma \in (0, 1]\) such that

$$1 - A_F(x * y) \leq \gamma < \bigwedge \{1 - A_F(x), 1 - A_F(y), 0.5\}. \quad (3.14)$$

It follows from the right inequality in (3.14) that \(A_F(x) + \gamma < 1\) and \(A_F(y) + \gamma < 1\), that is, \(x, y \in F_q(A; \gamma)\), which implies that \(x * y \in F_{\vee q}(A; \gamma)\). But, we have \(x * y \notin F_{\vee q}(A; \gamma)\) by the left inequality in (3.14). This is a contradiction, and so \(x * y \in F_{\vee q}(A; \gamma_x \vee \gamma_y)\). Therefore \(A = (A_T, A_I, A_P)\) is an \((\exists \in \vee q)\)-neutrosophic subalgebra of \(X\). \(\square\)

The following example shows that the converse of Theorem 3.6 is not true.
Example 3.7. Consider a $BCK$-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X$ defined by

$T_\varepsilon(A; \alpha) = \begin{cases} 
\{0, 3\} & \text{if } \alpha \in (0.4, 0.5], \\
\{0, 3, 4\} & \text{if } \alpha \in (0.2, 0.4], \\
X & \text{if } \alpha \in (0, 0.2], 
\end{cases} 
$I_\varepsilon(A; \beta) = \begin{cases} 
\{0\} & \text{if } \beta \in (0.4, 0.5], \\
\{0, 4\} & \text{if } \beta \in (0.3, 0.4], \\
\{0, 1, 2, 4\} & \text{if } \beta \in (0.1, 0.3], \\
X & \text{if } \beta \in (0, 0.1], 
\end{cases} 
$F_\varepsilon(A; \gamma) = \begin{cases} 
X & \text{if } \gamma \in (0.9, 1), \\
\{0, 1, 2, 3\} & \text{if } \gamma \in [0.7, 0.9), \\
\{0, 1, 2\} & \text{if } \gamma \in [0.6, 0.7), \\
\{0\} & \text{if } \gamma \in (0.5, 0.6), 
\end{cases} 

which are subalgebras of $X$ for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Using Theorem 3.3, $A = (A_T, A_I, A_F)$ is an $(\varepsilon, \in \vee q)$-neutrosophic subalgebra of $X$. But it is not a $(q, \in \vee q)$-neutrosophic subalgebra of $X$ since $2 \in T_q(A; 0.83)$ and $3 \in T_q(A; 0.4)$, but $2 \ast 3 = 2 \notin T_{\varepsilon \vee q}(A; 0.4)$.

We provide conditions for an $(\varepsilon, \in \vee q)$-neutrosophic subalgebra to be a $(q, \in \vee q)$-neutrosophic subalgebra.

**Theorem 3.8.** Assume that any neutrosophic $T_\Phi$-point and neutrosophic $I_\Phi$-point has the value $\alpha$ and $\beta$ in $[0, 0.5]$, respectively, and any neutrosophic $F_\Phi$-point has the value $\gamma$ in $[0.5, 1)$ for $\Phi \in \{\varepsilon, q, \in \vee q\}$. Then every $(\varepsilon, \in \vee q)$-neutrosophic subalgebra is a $(q, \in \vee q)$-neutrosophic subalgebra.

**Proof.** Let $X$ be a $BCK/BCI$-algebra and let $A = (A_T, A_I, A_F)$ be an $(\varepsilon, \in \vee q)$-neutrosophic subalgebra of $X$. For $x, y, a, b \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$ be...
such that \( x \in T_q(A;\alpha_x), y \in T_q(A;\alpha_y), a \in I_q(A;\beta_a) \) and \( b \in T_q(A;\beta_b) \). Then \( A_T(x) + a_x > 1, A_T(y) + \alpha_y > 1, A_T(a) + \beta_a > 1 \) and \( A_T(b) + \beta_b > 1 \). Since \( \alpha_x, \gamma_y, \beta_a, \beta_b \in (0, 0.5) \), it follows that \( A_T(x) > 1 - \alpha_x \geq \alpha_y, A_T(y) > 1 - \alpha_y \geq \alpha_x, A_T(a) > 1 - \beta_a \geq \beta_b \) and \( A_T(b) > 1 - \beta_b \geq \beta_a \), that is, \( x \in T_\gamma(A;\alpha_x), y \in T_\gamma(A;\alpha_y), a \in I_\gamma(A;\beta_a) \) and \( b \in I_\gamma(A;\beta_b) \). Also, let \( x \in F_q(A;\gamma_x) \) and \( y \in F_q(A;\gamma_y) \) for \( x, y \in X \) and \( \gamma_x, \gamma_y \in [0, 1) \). Then \( A_F(x) + \gamma_x < 1 \) and \( A_F(y) + \gamma_y < 1 \), and so \( A_F(x) < 1 - \gamma_x \leq \gamma_x \) and \( A_F(y) < 1 - \gamma_y \leq \gamma_y \). Consequently, \( A = (A_T, A_I, A_F) \) is a \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra of \( X \). 

**Theorem 3.9.** Both \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra and \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra are an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra.

**Proof.** It is clear that \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra is an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra. Let \( A = (A_T, A_I, A_F) \) be an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra of \( X \). For any \( x, y, a, b \in X \), let \( \alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 1) \) be such that \( x \in T_\varepsilon(A;\alpha_x), y \in T_\varepsilon(A;\alpha_y), a \in I_\varepsilon(A;\beta_a) \) and \( b \in I_\varepsilon(A;\beta_b) \). Then \( x \in T_q(A;\alpha_x), y \in T_q(A;\alpha_y), a \in I_q(A;\beta_a) \) and \( b \in I_q(A;\beta_b) \) by (3.1) and (3.2). It follows that \( x * y \in T_q(A;\alpha_x \wedge \alpha_y) \) and \( a * b \in I_q(A;\beta_a \wedge \beta_b) \). Now, let \( x, y \in X \) and \( \gamma_x, \gamma_y \in [0, 1) \) be such that \( x \in F_q(A;\gamma_x) \) and \( y \in F_q(A;\gamma_y) \). Then \( x \in F_q(A;\gamma_x) \) and \( y \in F_q(A;\gamma_y) \) by (3.3). Hence \( x * y \in F_q(A;\gamma_x \vee \gamma_y) \). Therefore \( A = (A_T, A_I, A_F) \) is an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra of \( X \).

The converse of Theorem 3.9 is not true in general. In fact, the \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra \( A = (A_T, A_I, A_F) \) in Example 3.7 is neither an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra nor an \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra.

**Theorem 3.10.** For a neutrosophic set \( A = (A_T, A_I, A_F) \) in a BCK/BCI-algebra \( X \), if the nonempty neutrosophic \( q \)-subsets \( T_q(A;\alpha), I_q(A;\beta) \) and \( F_q(A;\gamma) \) are subalgebras of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( \gamma \in (0, 0.5) \), then

\[
\begin{align*}
\text{(3.15)} & \quad x \in T_\varepsilon(A;\alpha_x), y \in T_\varepsilon(A;\alpha_y) \Rightarrow x * y \in T_q(A;\alpha_x \vee \alpha_y), \\
& \quad x \in I_\varepsilon(A;\beta_x), y \in I_\varepsilon(A;\beta_y) \Rightarrow x * y \in I_q(A;\beta_x \vee \beta_y), \\
& \quad x \in F_\varepsilon(A;\gamma_x), y \in F_\varepsilon(A;\gamma_y) \Rightarrow x * y \in F_q(A;\gamma_x \wedge \gamma_y)
\end{align*}
\]

for all \( x, y \in X \), \( \alpha_x, \alpha_y, \beta_x, \beta_y \in [0, 1] \) and \( \gamma_x, \gamma_y \in (0, 0.5) \).

**Proof.** Let \( x, y, a, b, u, v \in X \) and \( \alpha_x, \alpha_y, \beta_a, \beta_b \in [0, 1] \) and \( \gamma_u, \gamma_v \in (0, 0.5) \) be such that \( x \in T_\varepsilon(A;\alpha_x), y \in T_\varepsilon(A;\alpha_y), a \in I_\varepsilon(A;\beta_a), b \in I_\varepsilon(A;\beta_b), u \in F_\varepsilon(A;\gamma_u) \) and \( v \in F_\varepsilon(A;\gamma_v) \). Then \( A_T(x) \geq \alpha_x > 1 - \alpha_x, A_T(y) \geq \alpha_y > 1 - \alpha_y, A_T(a) \geq \beta_a > 1 - \beta_a, A_T(b) \geq \beta_b > 1 - \beta_b, A_F(u) \leq \gamma_u < 1 - \gamma_u \) and \( A_F(v) \leq \gamma_v < 1 - \gamma_v \). It follows that \( x, y \in T_q(A;\alpha_x), a, b \in I_q(A;\beta_a \vee \beta_b) \) and \( u, v \in F_q(A;\gamma_u \wedge \gamma_v) \). Since \( \alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in [0, 1] \) and \( \gamma_u \wedge \gamma_v \in (0, 0.5) \), we have \( x * y \in T_q(A;\alpha_x \vee \alpha_y), a * b \in I_q(A;\beta_a \vee \beta_b) \) and \( u * v \in F_q(A;\gamma_u \wedge \gamma_v) \) by hypothesis. This completes the proof.

The following corollary is by Theorem 3.10 and [7, Theorem 3.7].

**Corollary 3.11.** Every \((\varepsilon, \in \in \forall q, \in \in \forall q)\)-neutrosophic subalgebra \( A = (A_T, A_I, A_F) \) in a BCK/BCI-algebra \( X \) satisfies the condition (3.15).
Corollary 3.12. Every \((q, \in \vee q)\)-neutrosophic subalgebra \(A = (A_T, A_I, A_F)\) in a BCK/BCI-algebra \(X\) satisfies the condition (3.15).

Proof. It is by Theorem 3.6 and Corollary 3.11. □

Theorem 3.13. For a neutrosophic set \(A = (A_T, A_I, A_F)\) in a BCK/BCI-algebra \(X\), if the nonempty neutrosophic \(q\)-subsets \(T_q(A; \alpha), I_q(A; \beta)\) and \(F_q(A; \gamma)\) are subalgebras of \(X\) for all \(\alpha, \beta \in (0, 0.5]\) and \(\gamma \in (0.5, 1]\), then

\[
x \in T_q(A; \alpha_x), \ y \in T_q(A; \alpha_y) \Rightarrow x \star y \in T_{\varepsilon}(A; \alpha_x \vee \alpha_y),
\]

(3.16)

\[
x \in I_q(A; \beta_x), \ y \in I_q(A; \beta_y) \Rightarrow x \star y \in I_{\varepsilon}(A; \beta_x \vee \beta_y),
\]

\[
x \in F_q(A; \gamma_x), \ y \in F_q(A; \gamma_y) \Rightarrow x \star y \in F_{\varepsilon}(A; \gamma_x \wedge \gamma_y)
\]

for all \(x, y \in X, \alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 0.5]\) and \(\gamma_x, \gamma_y \in (0.5, 1]\).

Proof. Let \(x, y, a, b, u, v \in X\) and \(\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]\) and \(\gamma_u, \gamma_v \in (0.5, 1]\) be such that \(x \in T_q(A; \alpha_x), y \in T_q(A; \alpha_y), a \in I_q(A; \beta_a), b \in I_q(A; \beta_b), u \in F_q(A; \gamma_u)\) and \(v \in F_q(A; \gamma_v)\). Then \(x, y \in T_q(A; \alpha_x \vee \alpha_y), a, b \in I_q(A; \beta_a \vee \beta_b)\) and \(u, v \in F_q(A; \gamma_u \wedge \gamma_v)\). Since \(\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0, 0.5]\) and \(\gamma_u \wedge \gamma_v \in (0.5, 1]\), it follows from the hypothesis that \(x \star y \in T_{\varepsilon}(A; \alpha_x \vee \alpha_y), a \star b \in I_{\varepsilon}(A; \beta_a \vee \beta_b)\) and \(u \star v \in F_{\varepsilon}(A; \gamma_u \wedge \gamma_v)\). Hence

\[
A_T(x \star y) > 1 - (\alpha_x \vee \alpha_y) \geq \alpha_x \vee \alpha_y,
\]

that is, \(x \star y \in T_{\varepsilon}(A; \alpha_x \vee \alpha_y)\),

\[
A_I(a \star b) > 1 - (\beta_a \vee \beta_b) \geq \beta_a \vee \beta_b,
\]

that is, \(a \star b \in I_{\varepsilon}(A; \beta_a \vee \beta_b)\),

\[
A_F(u \star v) < 1 - (\gamma_u \wedge \gamma_v) \leq \gamma_u \wedge \gamma_v,
\]

that is, \(u \star v \in F_{\varepsilon}(A; \gamma_u \wedge \gamma_v)\).

Consequently, the condition (3.16) is valid for all \(x, y \in X, \alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 0.5]\) and \(\gamma_x, \gamma_y \in (0.5, 1]\). □

Theorem 3.14. Given a neutrosophic set \(A = (A_T, A_I, A_F)\) in a BCK/BCI-algebra \(X\), if the nonempty neutrosophic \(\in \vee q\)-subsets \(T_{\varepsilon q}(A; \alpha), I_{\varepsilon q}(A; \beta)\) and \(F_{\varepsilon q}(A; \gamma)\) are subalgebras of \(X\) for all \(\alpha, \beta \in (0, 0.5]\) and \(\gamma \in (0.5, 1]\), then the following assertions are valid.

\[
x \in T_{\varepsilon q}(A; \alpha_x), \ y \in T_{\varepsilon q}(A; \alpha_y) \Rightarrow x \star y \in T_{\varepsilon q}(A; \alpha_x \vee \alpha_y),
\]

(3.17)

\[
x \in I_{\varepsilon q}(A; \beta_x), \ y \in I_{\varepsilon q}(A; \beta_y) \Rightarrow x \star y \in I_{\varepsilon q}(A; \beta_x \vee \beta_y),
\]

\[
x \in F_{\varepsilon q}(A; \gamma_x), \ y \in F_{\varepsilon q}(A; \gamma_y) \Rightarrow x \star y \in F_{\varepsilon q}(A; \gamma_x \wedge \gamma_y)
\]

for all \(x, y \in X, \alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 0.5]\) and \(\gamma_x, \gamma_y \in (0.5, 1]\).

Proof. Let \(x, y, a, b, u, v \in X\) and \(\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]\) and \(\gamma_u, \gamma_v \in (0.5, 1]\) be such that \(x \in T_{\varepsilon q}(A; \alpha_x), y \in T_{\varepsilon q}(A; \alpha_y), a \in I_{\varepsilon q}(A; \beta_a), b \in I_{\varepsilon q}(A; \beta_b), u \in F_{\varepsilon q}(A; \gamma_u)\) and \(v \in F_{\varepsilon q}(A; \gamma_v)\). Then \(x, y \in T_{\varepsilon q}(A; \alpha_x \vee \alpha_y), a \in I_{\varepsilon q}(A; \beta_a), b \in I_{\varepsilon q}(A; \beta_b), u \in F_{\varepsilon q}(A; \gamma_u)\) and \(v \in F_{\varepsilon q}(A; \gamma_v)\). It follows that \(x, y \in T_{\varepsilon q}(A; \alpha_x \vee \alpha_y), a, b \in I_{\varepsilon q}(A; \beta_a \vee \beta_b)\) and \(u, v \in F_{\varepsilon q}(A; \gamma_u \wedge \gamma_v)\) which imply from the hypothesis that \(x \star y \in T_{\varepsilon q}(A; \alpha_x \vee \alpha_y), a \star b \in I_{\varepsilon q}(A; \beta_a \vee \beta_b)\) and \(u \star v \in F_{\varepsilon q}(A; \gamma_u \wedge \gamma_v)\). This completes the proof. □

Corollary 3.15. Every \((\in, \vee q)\)-neutrosophic subalgebra \(A = (A_T, A_I, A_F)\) of a BCK/BCI-algebra \(X\) satisfies the condition (3.17).

Proof. It is by Theorem 3.14 and [7, Theorem 3.9]. □
Proof. It is by Theorem 3.16 and [7, Theorem 3.10].

Corollary 3.18. Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$, if the nonempty neutrosophic $\in \vee q$-subsets $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 0.5)$, then the following assertions are valid.

\begin{equation}
\begin{aligned}
  x &\in T_{q}(A; \alpha_x), \ y \in T_{q}(A; \alpha_y) \Rightarrow x \ast y \in T_{\in \vee q}(A; \alpha_x \vee \alpha_y), \\
  x &\in I_{q}(A; \beta_x), \ y \in I_{q}(A; \beta_y) \Rightarrow x \ast y \in I_{\in \vee q}(A; \beta_x \vee \beta_y), \\
  x &\in F_{q}(A; \gamma_x), \ y \in F_{q}(A; \gamma_y) \Rightarrow x \ast y \in F_{\in \vee q}(A; \gamma_x \wedge \gamma_y)
\end{aligned}
\end{equation}

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in [0, 0.5)$.

Proof. It is similar to the proof Theorem 3.14. □

Corollary 3.17. Every $(q, \in \vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra $X$ satisfies the condition (3.18).

Proof. It is by Theorem 3.16 and [7, Theorem 3.10]. □

Combining Theorems 3.14 and 3.16, we have the following corollary.

Corollary 3.18. Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra $X$, if the nonempty neutrosophic $\in \vee q$-subsets $T_{\in \vee q}(A; \alpha)$, $I_{\in \vee q}(A; \beta)$ and $F_{\in \vee q}(A; \gamma)$ are subalgebras of $X$ for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the following assertions are valid.

\begin{equation}
\begin{aligned}
  x &\in T_{q}(A; \alpha_x), \ y \in T_{q}(A; \alpha_y) \Rightarrow x \ast y \in T_{\in \vee q}(A; \alpha_x \vee \alpha_y), \\
  x &\in I_{q}(A; \beta_x), \ y \in I_{q}(A; \beta_y) \Rightarrow x \ast y \in I_{\in \vee q}(A; \beta_x \vee \beta_y), \\
  x &\in F_{q}(A; \gamma_x), \ y \in F_{q}(A; \gamma_y) \Rightarrow x \ast y \in F_{\in \vee q}(A; \gamma_x \wedge \gamma_y)
\end{aligned}
\end{equation}

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Conclusions

We have considered relations between an $(\in, \in \vee q)$-neutrosophic subalgebra and a $(q, \in \vee q)$-neutrosophic subalgebra. We have discussed characterization of an $(\in, \in \vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets, and have provided conditions for an $(\in, \in \vee q)$-neutrosophic subalgebra to be a $(q, \in \vee q)$-neutrosophic subalgebra. We have investigated properties on neutrosophic $q$-subsets and neutrosophic $\in \vee q$-subsets. Our future research will be focused on the study of generalization of this paper.

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References


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Interval-valued neutrosophic competition graphs

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Abstract. We first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including $k$-competition interval-valued neutrosophic graphs, $p$-competition interval-valued neutrosophic graphs and $m$-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of $m$-step interval-valued neutrosophic neighbourhood graphs.

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Keywords: Interval-valued neutrosophic digraphs, Interval-valued neutrosophic competition graphs.

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1. Introduction

In 1975, Zadeh [35] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [34] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [19]. Atanassov [12] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. Smarandache [26, 27] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership ($t$), indeterminacy-membership ($i$) and falsity-membership ($f$), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $[0^-, 1^+]$ and there is no restriction on their sum. Smarandache [28] and Wang et al. [29] presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0, 1]$. Wang et al. [30] presented the concept of interval-valued neutrosophic
sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership \((t, i, f)\) functions are independent, and their values belong to the unit interval \([0, 1]\).

Kauffman [18] gave the definition of a fuzzy graph. Fuzzy graphs were narrated by Rosenfeld [22]. After that, some remarks on fuzzy graphs were represented by Bhattacharya [13]. He showed that all the concepts on crisp graph theory do not have similarities in fuzzy graphs. Wu [32] discussed fuzzy digraphs. The concept of fuzzy \(k\)-competition graphs and \(p\)-competition fuzzy graphs was first developed by Samanta and Pal in [23], it was further studied in [11, 21, 25]. Samanta et al. [24] introduced the generalization of fuzzy competition graphs, called \(m\)-step fuzzy competition graphs. Samanta et al. [24] also introduced the concepts of fuzzy \(m\)-step neighbourhood graphs, fuzzy economic competition graphs, and \(m\)-step economic competition graphs. The concepts of bipolar fuzzy competition graphs and intuitionistic fuzzy competition graphs are discussed in [21, 25]. Hongmei and Lianhua [16], gave definition of interval-valued fuzzy graphs. Akram et al. [1, 2, 3, 4] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notion of neutrosophic soft graphs with applications. Akram [7] introduced the notion of single-valued neutrosophic planar graphs. Akram and Shahzadi [8] studied properties of single-valued neutrosophic graphs by level graphs. Recently, Akram and Nasir [5] have discussed some concepts of interval-valued neutrosophic graphs. In this paper, we first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including \(k\)-competition interval-valued neutrosophic graphs, \(p\)-competition interval-valued neutrosophic graphs and \(m\)-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of \(m\)-step interval-valued neutrosophic neighbourhood graphs.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [6, 9, 10, 13, 14, 15, 17, 20, 26, 33, 36].

2. **Interval-Valued Neutrosophic Competition Graphs**

**Definition 2.1** ([35]). The interval-valued fuzzy set \(A\) in \(X\) is defined by

\[
A = \{(s, [t^l_A(s), t^u_A(s)]): s \in X\},
\]

where, \(t^l_A(s)\) and \(t^u_A(s)\) are fuzzy subsets of \(X\) such that \(t^l_A(s) \leq t^u_A(s)\) for all \(x \in X\). An interval-valued fuzzy relation on \(X\) is an interval-valued fuzzy set \(B\) in \(X \times X\).

**Definition 2.2** ([30, 31]). The interval-valued neutrosophic set (IVN-set) \(A\) in \(X\) is defined by

\[
A = \{(s, [t^l_A(s), t^u_A(s), i^l_A(s), i^u_A(s), f^l_A(s), f^u_A(s)]): s \in X\},
\]

where, \(t^l_A(s), t^u_A(s), i^l_A(s), i^u_A(s), f^l_A(s), f^u_A(s)\) are neutrosophic subsets of \(X\) such that \(t^l_A(s) \leq t^u_A(s), i^l_A(s) \leq i^u_A(s), f^l_A(s) \leq f^u_A(s)\) for all \(s \in X\). An interval-valued neutrosophic relation (IVN-relation) on \(X\) is an interval-valued neutrosophic set \(B\) in \(X \times X\).
Definition 2.3 ([5]). An interval-valued neutrosophic digraph (IVN-digraph) on a non-empty set $X$ is a pair $G = (A, \overrightarrow{B})$, (in short, $G$), where $A = ([t_A^l, t_A^u], [i_A^l, i_A^u], [f_A^l, f_A^u])$ is an IVN-set on $X$ and $B = ([t_B^l, t_B^u], [i_B^l, i_B^u], [f_B^l, f_B^u])$ is an IVN-relation on $X$, such that:

(i) $t_B^l(s, w) \leq t_A^l(s) \land t_A^l(w)$, $t_B^u(s, w) \leq t_A^u(s) \land t_A^u(w)$,
(ii) $i_B^l(s, w) \leq i_A^l(s) \land i_A^l(w)$, $i_B^u(s, w) \leq i_A^u(s) \land i_A^u(w)$,
(iii) $f_B^l(s, w) \leq f_A^l(s) \land f_A^l(w)$, $f_B^u(s, w) \leq f_A^u(s) \land f_A^u(w)$, for all $s, w \in X$.

Example 2.4. We construct an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 1.

![Figure 1. IVN-digraph](image-url)

Definition 2.5. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic out-neighborhoods (IVN-out-neighborhoods) of a vertex $s$ is an IVN-set

$$N^+(s) = (X^+_s, [t^+_s, t^+_s], [i^+_s, i^+_s], [f^+_s, f^+_s]),$$

where

$$X^+_s = \{w | t_B^l(s, w) > 0, t_B^u(s, w) > 0, i_B^l(s, w) > 0, i_B^u(s, w) > 0, f_B^l(s, w) > 0, f_B^u(s, w) > 0\},$$

such that $t^+_s : X^+_s \rightarrow [0, 1]$, defined by $t^+_s(w) = t_B^l(s, w)$, $t^+_s : X^+_s \rightarrow [0, 1]$, defined by $t^+_s(w) = t_B^u(s, w)$, $i^+_s : X^+_s \rightarrow [0, 1]$, defined by $i^+_s(w) = i_B^l(s, w)$, $i^+_s : X^+_s \rightarrow [0, 1]$, defined by $i^+_s(w) = i_B^u(s, w)$, $f^+_s : X^+_s \rightarrow [0, 1]$, defined by $f^+_s(w) = f_B^l(s, w)$, $f^+_s : X^+_s \rightarrow [0, 1]$, defined by $f^+_s(w) = f_B^u(s, w)$.

Definition 2.6. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic in-neighborhoods (IVN-in-neighborhoods) of a vertex $s$ is an IVN-set

$$N^-(s) = (X^-_s, [t^-_s, t^-_s], [i^-_s, i^-_s], [f^-_s, f^-_s]),$$

where

$$X^-_s = \{w | t_B^l(w, s) > 0, t_B^u(w, s) > 0, i_B^l(w, s) > 0, i_B^u(w, s) > 0, f_B^l(w, s) > 0, f_B^u(w, s) > 0\},$$

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such that \( t_s^{(l)^-} : X_s^- \to [0,1] \), defined by \( t_s^{(l)^-} (w) = t_B^l(w, s) \), \( t_s^{(u)^-} : X_s^+ \to [0,1] \), defined by \( t_s^{(u)^-} (w) = t_B^u(w, s) \), \( i_s^{(l)^-} : X_s^- \to [0,1] \), defined by \( i_s^{(l)^-} (w) = i_B^l(w, s) \), \( i_s^{(u)^-} : X_s^- \to [0,1] \), defined by \( i_s^{(u)^-} (w) = i_B^u(w, s) \), \( f_s^{(l)^-} : X_s^- \to [0,1] \), defined by \( f_s^{(l)^-} (w) = f_B^l(w, s) \), \( f_s^{(u)^-} : X_s^- \to [0,1] \), defined by \( f_s^{(u)^-} (w) = f_B^u(w, s) \).

**Example 2.7.** Consider an IVN-digraph \( G = (A, B) \) on \( X = \{a, b, c\} \) as shown in Fig. 2.

![IVN-digraph](image)

**Figure 2.** IVN-digraph

We have Table 1 and Table 2 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

**Table 1.** IVN-out-neighbourhoods

<table>
<thead>
<tr>
<th>s</th>
<th>( N^+(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>{b, [0.1,0.2],[0.2,0.3],[0.1,0.6]}, {c, [0.1,0.2],[0.1,0.3],[0.2,0.6]}</td>
</tr>
<tr>
<td>b</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>c</td>
<td>{b, [0.1,0.2],[0.2,0.3],[0.2,0.5]}</td>
</tr>
</tbody>
</table>

**Table 2.** IVN-in-neighbourhoods

<table>
<thead>
<tr>
<th>s</th>
<th>( N^-(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>b</td>
<td>{a, [0.1,0.2],[0.2,0.3],[0.1,0.6]}, {c, [0.1,0.2],[0.2,0.3],[0.2,0.5]}</td>
</tr>
<tr>
<td>c</td>
<td>{a, [0.1,0.2],[0.1,0.3],[0.2,0.6]}</td>
</tr>
</tbody>
</table>

**Definition 2.8.** The height of IVN-set \( A = (s, [l_A, l_A^u], [u_A, u_A^u], [f_A, f_A^u]) \) in universe of discourse \( X \) is defined as: for all \( s \in X \),

\[
h(A) = ([h_1^l(A), h_1^u(A)], [h_2^l(A), h_2^u(A)], [h_3^l(A), h_3^u(A)]),
\]

\[
= ([\sup_{s \in X} l_A(s), \sup_{s \in X} l_A^u(s)], [\inf_{s \in X} u_A(s), \inf_{s \in X} u_A^u(s)]).
\]
Definition 2.9. An interval-valued neutrosophic competition graph (IVNC-graph) of an interval-valued neutrosophic graph (IVN-graph) \( \overrightarrow{G} = (A, \overrightarrow{B}) \) is an undirected IVN-graph \( C(\overrightarrow{G}) = (A, W) \) which has the same vertex set as in \( \overrightarrow{G} \) and there is an edge between two vertices \( s \) and \( w \) if and only if \( N^+(s) \cap N^+(w) \neq \emptyset \). The truth-membership, indeterminacy-membership and falsity-membership values of the edge \( (s, w) \) are defined as: for all \( s, w \in X \),

(i) \( t_{IVN}^w(s, w) = (t_A^w(s) \land t_A^w(w))h_1^w(N^+(s) \cap N^+(w)) \),
(ii) \( i_{IVN}^w(s, w) = (i_A^w(s) \land i_A^w(w))h_2^w(N^+(s) \cap N^+(w)) \),
(iii) \( f_{IVN}^w(s, w) = (f_A^w(s) \land f_A^w(w))h_3^w(N^+(s) \cap N^+(w)) \).

Example 2.10. Consider an IVN-digraph \( G = (A, \overrightarrow{B}) \) on \( X = \{a, b, c\} \) as shown in Fig. 3.

![IVN-digraph](image)

We have Table 3 and Table 4 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

### Table 3. IVN-out-neighbourhoods

<table>
<thead>
<tr>
<th>s</th>
<th>( N^+(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>{ (b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6]) }</td>
</tr>
<tr>
<td>b</td>
<td>\emptyset</td>
</tr>
<tr>
<td>c</td>
<td>{ (b, [0.1,0.2],[0.2,0.3],[0.2,0.5]) }</td>
</tr>
</tbody>
</table>

### Table 4. IVN-in-neighbourhoods

<table>
<thead>
<tr>
<th>s</th>
<th>( N^-(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>\emptyset</td>
</tr>
<tr>
<td>b</td>
<td>{ (a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5]) }</td>
</tr>
<tr>
<td>c</td>
<td>{ (a, [0.1,0.2],[0.1,0.3],[0.2,0.6]) }</td>
</tr>
</tbody>
</table>
Then IVNC-graph of Fig. 3 is shown in Fig. 4.

**Figure 4.** IVNC-graph

**Definition 2.11.** Consider an IVN-graph $G = (A, B)$, where $A = ([A_1^1, A_1^u], [A_2^1, A_2^u], [A_3^1, A_3^u])$ and $B = ([B_1^1, B_1^u], [B_2^1, B_2^u], [B_3^1, B_3^u])$. Then, an edge $(s, w), s, w \in X$ is called independent strong, if

\[
\begin{align*}
\frac{1}{2}[A_1^1(s) \land A_1^u(w)] < B_1^1(s, w), & \quad \frac{1}{2}[A_1^u(s) \land A_1^u(w)] < B_1^u(s, w), \\
\frac{1}{2}[A_2^1(s) \land A_2^u(w)] < B_2^1(s, w), & \quad \frac{1}{2}[A_2^u(s) \land A_2^u(w)] < B_2^u(s, w), \\
\frac{1}{2}[A_3^1(s) \land A_3^u(w)] > B_3^1(s, w), & \quad \frac{1}{2}[A_3^u(s) \land A_3^u(w)] > B_3^u(s, w).
\end{align*}
\]

Otherwise, it is called weak.

We state the following theorems without their proofs.

**Theorem 2.12.** Suppose $\overrightarrow{G}$ is an IVN-digraph. If $N^+(s) \cap N^+(w)$ contains only one element of $\overrightarrow{G}$, then the edge $(s, w)$ of $C(\overrightarrow{G})$ is independent strong if and only if

\[
\begin{align*}
||N^+(s) \cap N^+(w)||_d > 0.5, & \quad ||N^+(s) \cap N^+(w)||_{lw} > 0.5, \\
||N^+(s) \cap N^+(w)||_l > 0.5, & \quad ||N^+(s) \cap N^+(w)||_{lw} > 0.5, \\
||N^+(s) \cap N^+(w)||_l > 0.5, & \quad ||N^+(s) \cap N^+(w)||_{lw} > 0.5.
\end{align*}
\]

**Theorem 2.13.** If all the edges of an IVN-digraph $\overrightarrow{G}$ are independent strong, then

\[
\begin{align*}
\frac{B_1^1(s, w)}{(A_1^1(s) \land A_1^u(w))^2} > 0.5, & \quad \frac{B_1^u(s, w)}{(A_1^u(s) \land A_1^u(w))^2} > 0.5, \\
\frac{B_2^1(s, w)}{(A_2^1(s) \land A_2^u(w))^2} > 0.5, & \quad \frac{B_2^u(s, w)}{(A_2^u(s) \land A_2^u(w))^2} > 0.5, \\
\frac{B_3^1(s, w)}{(A_3^1(s) \land A_3^u(w))^2} < 0.5, & \quad \frac{B_3^u(s, w)}{(A_3^u(s) \land A_3^u(w))^2} < 0.5,
\end{align*}
\]

for all edges $(s, w)$ in $C(\overrightarrow{G})$.

**Definition 2.14.** The interval-valued neutrosophic open-neighbourhood (IVN-open-neighbourhood) of a vertex $s$ of an IVN-graph $G = (A, B)$ is IVN-set $N(s) = (X_s, [t_s^l, t_s^u], [i_s^l, i_s^u], [f_s^l, f_s^u])$, where
\[ X_s = \{ w | [B_1^s(s, w) > 0, B_2^w(s, w) > 0, B_3^w(s, w) > 0], [B_1^s(s, w) > 0, B_3^w(s, w) > 0], [B_2^w(s, w) > 0] \}, \]

and \( t_s^1 : X_s \to [0, 1] \) defined by \( t_s^1(w) = B_1^s(s, w) \), \( t_s^2 : X_s \to [0, 1] \) defined by \( t_s^2(w) = B_1^w(s, w) \), \( t_s^3 : X_s \to [0, 1] \) defined by \( t_s^3(w) = B_3^w(s, w) \), \( t_s^4 : X_s \to [0, 1] \) defined by \( t_s^4(w) = B_3^w(s, w) \), \( f_s^1 : X_s \to [0, 1] \) defined by \( f_s^1(w) = B_2^w(s, w) \), \( f_s^2 : X_s \to [0, 1] \) defined by \( f_s^2(w) = B_2^w(s, w) \), \( f_s^3 : X_s \to [0, 1] \) defined by \( f_s^3(w) = B_2^w(s, w) \). For every vertex \( s \in X \), the interval-valued neutrosophic singleton set, \( A_s = (s, [A_1^s, A_1^w], [A_2^s, A_2^w], [A_3^s, A_3^w]) \) such that: \( A_1^u : \{ s \} \to [0, 1], A_1^w : \{ s \} \to [0, 1], A_2^u : \{ s \} \to [0, 1], A_2^w : \{ s \} \to [0, 1], A_3^u : \{ s \} \to [0, 1], A_3^w : \{ s \} \to [0, 1], \) defined by \( A_1^u(s) = A_1^w(s) \), \( A_2^u(s) = A_2^w(s) \), \( A_3^u(s) = A_3^w(s) \) and \( A_3^u(s) = A_3^w(s) \), respectively. The interval-valued neutrosophic closed-neighbourhood (IVN-closed-neighbourhood) of a vertex \( s \) is \( N[s] = N(s) \cup A_s \).

**Definition 2.15.** Suppose \( G = (A, B) \) is an IVN-graph. Interval-valued neutrosophic open-neighbourhood graph (IVN-open-neighbourhood graph) of \( G \) is an IVN-graph \( N(G) = (A, B') \) which has the same IVN-set of vertices in \( G \) and has an interval-valued neutrosophic edge between two vertices \( s, w \in X \) in \( N(G) \) if and only if \( N[s] \cap N[w] \) is a non-empty IVN-set in \( G \). The truth-membership, indeterminacy-membership, falsity-membership values of the edge \((s, w)\) are given by:

\[
\begin{align*}
B_1^{\text{IV}}(s, w) &= [A_1^s(s) \cup A_1^w(w)]h_1(N(s) \cap N(w)), \\
B_2^{\text{IV}}(s, w) &= [A_2^s(s) \cup A_2^w(w)]h_2(N(s) \cap N(w)), \\
B_3^{\text{IV}}(s, w) &= [A_3^s(s) \cup A_3^w(w)]h_3(N(s) \cap N(w)), \\
B_4^{\text{IV}}(s, w) &= [A_4^s(s) \cup A_4^w(w)]h_4(N(s) \cap N(w)), \\
B_5^{\text{IV}}(s, w) &= [A_5^s(s) \cup A_5^w(w)]h_5(N(s) \cap N(w)), \\
B_6^{\text{IV}}(s, w) &= [A_6^s(s) \cup A_6^w(w)]h_6(N(s) \cap N(w)),
\end{align*}
\]

**Definition 2.16.** Suppose \( G = (A, B) \) is an IVN-graph. Interval-valued neutrosophic closed-neighbourhood graph (IVN-closed-neighbourhood graph) of \( G \) is an IVN-graph \( N(G) = (A, B') \) which has the same IVN-set of vertices in \( G \) and has an interval-valued neutrosophic edge between two vertices \( s, w \in X \) in \( N(G) \) if and only if \( N[s] \cap N[w] \) is a non-empty IVN-set in \( G \). The truth-membership, indeterminacy-membership, falsity-membership values of the edge \((s, w)\) are given by:

\[
\begin{align*}
B_1^{\text{IC}}(s, w) &= [A_1^s(s) \cup A_1^w(w)]h_1(N[s] \cap N[w]), \\
B_2^{\text{IC}}(s, w) &= [A_2^s(s) \cup A_2^w(w)]h_2(N[s] \cap N[w]), \\
B_3^{\text{IC}}(s, w) &= [A_3^s(s) \cup A_3^w(w)]h_3(N[s] \cap N[w]), \\
B_4^{\text{IC}}(s, w) &= [A_4^s(s) \cup A_4^w(w)]h_4(N[s] \cap N[w]), \\
B_5^{\text{IC}}(s, w) &= [A_5^s(s) \cup A_5^w(w)]h_5(N[s] \cap N[w]), \\
B_6^{\text{IC}}(s, w) &= [A_6^s(s) \cup A_6^w(w)]h_6(N[s] \cap N[w]),
\end{align*}
\]

We now discuss the method of construction of interval-valued neutrosophic completion graph of the Cartesian product of IVN-digraph in following theorem which can be proof using similar method as used in [21], hence we omit its proof.
Theorem 2.17. Let $\mathcal{C}(G_1) = (A_1, B_1)$ and $\mathcal{C}(G_2) = (A_2, B_2)$ be two IVNC-graphs of IVN-digraphs $G_1 = (A_1, L_1)$ and $G_2 = (A_2, L_2)$, respectively. Then $\mathcal{C}(G_1 \mathcal{C}(G_2))^* = G_1 \mathcal{C}(G_2)$, where $G_1 \mathcal{C}(G_2)$ is an IVN-graph on the crisp graph $(X_1 \times X_2, E_{\mathcal{C}(G_1)}) \mathcal{C}(G_2))$, $\mathcal{C}(G_1)^*$ and $\mathcal{C}(G_2)^*$ are the crisp competition graphs of $G_1$ and $G_2$, respectively. $G^0$ is an IVN-graph on $(X_1 \times X_2, E^0)$ such that:

(1) $E^0 = \{(s_1, s_2)(w_1, w_2) : w_1 \in N^-(s_1)^*, w_2 \in N^+(s_2)^*\}$

$E_{\mathcal{C}(G_1)}^\bot \mathcal{C}(G_2)^* = \{(s_1, s_2)(s_1, w_2) : s_1 \in X_1, s_2 w_2 \in E_{\mathcal{C}(G_1)}\}$

$\cup \{(s_1, s_2)(w_1, s_2) : s_2 \in X_2, s_1 w_1 \in E_{\mathcal{C}(G_1)}\}^*.$

(2) $t_{A_1 \mathcal{C}(A_2)} = t_{A_1}(s_1) \land t_{A_2}(s_2), \quad t_{A_1 \mathcal{C}(A_2)} = t_{A_1}(s_1) \land t_{A_2}(s_2), \quad t_{A_1 \mathcal{C}(A_2)} = f_{A_1}(s_1) \land f_{A_2}(s_2),

(3) $t_{B_1}(s_1, s_2)(s_1, w_2) = [t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2) \times v_{A_2}(t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2)(a_2)]$

$s_1, s_2)(s_1, w_2) \in E_{\mathcal{C}(G_1)} \mathcal{C}(G_2)^*, \quad a_2 \in (N^+(s_2) \cap N^+(w_2)).$

(4) $t_{B_1}(s_1, s_2)(s_1, w_2) = [t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2) \times v_{A_2}(t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2) \land t_{L_2}(a_2)]$

$s_1, s_2)(s_1, w_2) \in E_{\mathcal{C}(G_1)} \mathcal{C}(G_2)^*, \quad a_2 \in (N^+(s_2) \cap N^+(w_2)).$

(5) $f_{B_1}(s_1, s_2)(s_1, w_2) = [f_{A_1}(s_1) \land f_{A_2}(s_2) \land f_{A_2}(w_2) \times v_{A_2}(f_{A_1}(s_1) \land f_{A_2}(s_2) \land f_{A_2}(w_2)(a_2)]$

$s_1, s_2)(s_1, w_2) \in E_{\mathcal{C}(G_1)} \mathcal{C}(G_2)^*, \quad a_2 \in (N^+(s_2) \cap N^+(w_2)).$

(6) $t_{B_1}(s_1, s_2)(s_1, w_2) = [t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2) \times v_{A_2}(t_{A_1}(s_1) \land t_{A_2}(s_2) \land t_{A_2}(w_2) \land t_{L_2}(a_2)]$

$s_1, s_2)(s_1, w_2) \in E_{\mathcal{C}(G_1)} \mathcal{C}(G_2)^*, \quad a_2 \in (N^+(s_2) \cap N^+(w_2)).$
Definition 2.18. The cardinality of an IVN-set $A$ is denoted by

$$|A| = \left( |A|^t, |A|^\mu, |A|^\iota, |A|^{l_t}, |A|^{l_\mu}, |A|^{l_\iota}, |A|_t, |A|_\mu, |A|_\iota, |A|_{l_t}, |A|_{l_\mu}, |A|_{l_\iota} \right).$$

Where $|A|^t$, $|A|^\mu$, $|A|^\iota$, $|A|_t$, $|A|_\mu$, $|A|_\iota$ represent the sum of truth-membership values, indeterminacy-membership values and falsity-membership values, respectively, of all the elements of $A$.

A. k-competition interval-valued neutrosophic graphs

We now discuss an extension of IVNC-graphs, called k-competition IVN-graphs.
Example 2.19. The cardinality of an IVN-set \( A = \{ (a, [0.5, 0.7]), [0.2, 0.8], [0.1, 0.3])\), \( (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9])\), \( (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9])\) in \( X = \{ a, b, c \} \) is

\[
|A| = \left( |A|_{\mu}, |A|_{\nu} \right), |A|_{\mu}, |A|_{\nu}
\]

\[
= (0.9, 1.4, 0.6, 2.1, 1.4, 2.1).
\]

We now discuss \( k \)-competition IVN-graphs.

**Definition 2.20.** Let \( k \) be a non-negative number. Then \( k \)-competition IVN-graph \( \overrightarrow{G} \) of an IVN-digraph \( \overrightarrow{G} = (A, \overrightarrow{B}) \) is an undirected IVN-graph \( G = (A, B) \) which has same IVN-set of vertices as in \( \overrightarrow{G} \) and has an interval-valued neutrosophic edge between two vertices \( s, w \in X \) in \( \overrightarrow{G} \) if and only if \( (|N^+(s) \cap N^+(w)|) > k \), \( (|N^-(s) \cap N^-(w)|) > k \), \( (|N(s) \cap N(w)|) > k \) and \( (|\overrightarrow{N}(s) \cap \overrightarrow{N}(w)|) > k \). The interval-valued truth-membership value of edge \( (s, w) \) in \( \overrightarrow{G} \) is \( t_{\overrightarrow{B}}(s, w) = \frac{k_i - k}{k_i} [t_A(s) \wedge t_A(w)] h_1 \), where \( k_i = (|N(s) \cap N(w)|) \), and \( t_B(s, w) \) is the interval-valued falsity-membership value of edge \( (s, w) \) in \( \overrightarrow{G} \) is \( f_{\overrightarrow{B}}(s, w) = \frac{k_i - k}{k_i} [t_A(s) \wedge t_A(w)] h_2 \), where \( k_i = (|N^+(s) \cap N^+(w)|) \).

**Example 2.21.** Consider an IVN-digraph \( G = (A, \overrightarrow{B}) \) on \( X = \{ s, w, a, b, c \} \), such that \( A = \{ (s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]) \} \), and \( B = \{ (s, a), [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (s, b), [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (s, c), [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), ((s, a), [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (w, b), [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), ((w, c), [0.2, 0.7], [0.3, 0.5], [0.2, 0.3]) \} \), as shown in Fig. 5.

We calculate \( N^+(s) = \{ (a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]) \}, (a, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]) \) and \( N^+(w) = \{ (a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3]) \}. Therefore, \( N^+(s) \cap N^+(w) = \{ (a, [0.1, 0.4], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3]) \}. \) So, \( k_i = 0.5, k_i^+ = 1.3, k_i^- = 0.6, k_i^+ = 1.5, k_i^- = 0.6 \) and \( k_i = 0.9 \). Let \( k = 0.4 \), then, \( t_A(s, w) = 0.02, f_A(s, w) = 0.06, t_A(s, w) = 0.06, f_A(s, w) = 0.02, f_A(s, w) = 0.02 \) and \( f_A(s, w) = 0.11 \). This graph is depicted in Fig. 6.
\begin{center}
\textbf{Figure 5.} IVN-digraph
\end{center}

\begin{center}
\textbf{Figure 6.} 0.4-Competition IVN-graph
\end{center}

**Theorem 2.22.** Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. If

\[
\begin{align*}
 h_1^*(N^+(s) \cap N^+(w)) &= 1, & h_2^*(N^+(s) \cap N^+(w)) &= 1, & h_3^*(N^+(s) \cap N^+(w)) &= 1, \\
 h_4^*(N^+(s) \cap N^+(w)) &= 1, & h_5^*(N^+(s) \cap N^+(w)) &= 1, & h_6^*(N^+(s) \cap N^+(w)) &= 1,
\end{align*}
\]

and

\[
\begin{align*}
 |(N^+(s) \cap N^+(w))|_{u'} &> 2k, & |(N^+(s) \cap N^+(w))|_{v'} &> 2k, & |(N^+(s) \cap N^+(w))|_{f'} &< 2k, \\
 |(N^+(s) \cap N^+(w))|_{u''} &> 2k, & |(N^+(s) \cap N^+(w))|_{v''} &> 2k, & |(N^+(s) \cap N^+(w))|_{f''} &< 2k,
\end{align*}
\]

Then the edge $(s, w)$ is independent strong in $C_k(\overrightarrow{G})$.

**Proof.** Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. Let $C_k(\overrightarrow{G})$ be the corresponding $k$-competition IVN-graph.
If $h_1^t(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{tl} > 2k$, then $k_1^t > 2k$. Thus,

$$t_B^l(s, w) = \frac{k_1^t - k}{k_1^t} [t_A^l(s) \wedge t_A^l(w)] h_1^t(N^+(s) \cap N^+(w))$$

or,

$$t_B^l(s, w) = \frac{k_1^t - k}{k_1^t} [t_A^l(s) \wedge t_A^l(w)]$$

$$\frac{t_B^l(s, w)}{[t_A^l(s) \wedge t_A^l(w)]} = \frac{k_1^t - k}{k_1^t} > 0.5.$$ 

If $h_2^u(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{lu} > 2k$, then $k_2^u > 2k$. Thus,

$$t_B^u(s, w) = \frac{k_2^u - k}{k_2^u} [t_A^u(s) \wedge t_A^u(w)] h_2^u(N^+(s) \cap N^+(w))$$

or,

$$t_B^u(s, w) = \frac{k_2^u - k}{k_2^u} [t_A^u(s) \wedge t_A^u(w)]$$

$$\frac{t_B^u(s, w)}{[t_A^u(s) \wedge t_A^u(w)]} = \frac{k_2^u - k}{k_2^u} > 0.5.$$ 

If $h_2^l(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{lt} > 2k$, then $k_2^l > 2k$. Thus,

$$i_B^l(s, w) = \frac{k_2^l - k}{k_2^l} [i_A^l(s) \wedge i_A^l(w)] h_2^l(N^+(s) \cap N^+(w))$$

or,

$$i_B^l(s, w) = \frac{k_2^l - k}{k_2^l} [i_A^l(s) \wedge i_A^l(w)]$$

$$\frac{i_B^l(s, w)}{[i_A^l(s) \wedge i_A^l(w)]} = \frac{k_2^l - k}{k_2^l} > 0.5.$$ 

If $h_2^u(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{uv} > 2k$, then $k_2^u > 2k$. Thus,

$$i_B^u(s, w) = \frac{k_2^u - k}{k_2^u} [i_A^u(s) \wedge i_A^u(w)] h_2^u(N^+(s) \cap N^+(w))$$

or,

$$i_B^u(s, w) = \frac{k_2^u - k}{k_2^u} [i_A^u(s) \wedge i_A^u(w)]$$

$$\frac{i_B^u(s, w)}{[i_A^u(s) \wedge i_A^u(w)]} = \frac{k_2^u - k}{k_2^u} > 0.5.$$ 

If $h_3^l(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{lt} < 2k$, then $k_3^l < 2k$. Thus,

$$f_B^l(s, w) = \frac{k_3^l - k}{k_3^l} [f_A^l(s) \wedge f_A^l(w)] h_3^l(N^+(s) \cap N^+(w))$$

or,

$$f_B^l(s, w) = \frac{k_3^l - k}{k_3^l} [f_A^l(s) \wedge f_A^l(w)]$$

$$\frac{f_B^l(s, w)}{[f_A^l(s) \wedge f_A^l(w)]} = \frac{k_3^l - k}{k_3^l} < 0.5.$$
If \( h_3^u(N^+(s) \cap N^+(w)) = 1 \) and \(|(N^+(s) \cap N^+(w))|_{f^u} < 2k \), then \( k_3^u < 2k \). Thus,
\[
f_B^u(s, w) = \frac{k_3^u - k_3}{k_3^u}[f_A^u(s) \land f_A^u(w)]h_3^u(N^+(s) \cap N^+(w))
\]
or,
\[
f_B^u(s, w) = \frac{k_3^u - k_3}{k_3^u}[f_A^u(s) \land f_A^u(w)]
\]
\[
\frac{f_B^u(s, w)}{[f_A^u(s) \land f_A^u(w)]} = \frac{k_3^u - k_3}{k_3^u} < 0.5.
\]
So, the edge \((s, w)\) is independent strong in \(\mathbb{C}_k(G)\).

\[\square\]

**B. \(p\)-competition interval-valued neutrosophic graphs**

We now define another extension of IVNC-graphs, called \(p\)-competition IVN-graphs.

**Definition 2.23.** The support of an IVN-set \(A = (s, [t_A^l, t_A^r], [i_A^l, i_A^r], [f_A^l, f_A^r])\) in \(X\) is the subset of \(X\) defined by
\[
supp(A) = \{s \in X : [t_A^l(s) \neq 0, t_A^r(s) \neq 0], [i_A^l(s) \neq 0, i_A^r(s) \neq 0], [f_A^l(s) \neq 1, f_A^r(s) \neq 1]\}
\]
and \(|supp(A)|\) is the number of elements in the set.

**Example 2.24.** The support of an IVN-set \(A = \{(a, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9]), (d, [0, 0], [0, 0], [1, 1])\} in \(X = \{a, b, c, d\}\) is \(supp(A) = \{a, b, c\}\) and \(|supp(A)| = 3\).

We now define \(p\)-competition IVN-graphs.

**Definition 2.25.** Let \(p\) be a positive integer. Then \(p\)-competition IVN-graph \(\mathbb{C}^p(G)\) of the IVN-digraph \(G = (A, B)\) is an undirected IVN-graph \(G = (A, B)\) which has same IVN-set of vertices as in \(G\) and has an interval-valued neutrosophic edge between two vertices \(s, w \in X\) in \(\mathbb{C}^p(G)\) if and only if \(|supp(N^+(s) \cap N^+(w))| \geq p\). The interval-valued truth-membership value of edge \((s, w)\) in \(\mathbb{C}^p(G)\) is \(t_B(s, w) = (\frac{i-p+1}{i})t_A^l(s) \land t_A^r(w)h_1^u(N^+(s) \cap N^+(w))\), and \(t_B(s, w) = (\frac{i-p+1}{i})i_A^l(s) \land i_A^r(w)h_2^u(N^+(s) \cap N^+(w))\), the interval-valued indeterminacy-membership value of edge \((s, w)\) in \(\mathbb{C}^p(G)\) is \(i_B(s, w) = (\frac{i-p+1}{i})i_A^l(s) \land i_A^r(w)h_1^u(N^+(s) \cap N^+(w))\), and \(i_B(s, w) = (\frac{i-p+1}{i})i_A^l(s) \land i_A^r(w)h_2^u(N^+(s) \cap N^+(w))\), the interval-valued falsity-membership value of edge \((s, w)\) in \(\mathbb{C}^p(G)\) is \(f_B(s, w) = (\frac{i-p+1}{i})f_A^l(s) \land f_A^r(w)h_3^u(N^+(s) \cap N^+(w))\), where \(i = |supp(N^+(s) \cap N^+(w))|\).

**Example 2.26.** Consider an IVN-digraph \(G = (A, B)\) on \(X = \{s, w, a, b, c\}\), such that \(A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\} and \(B = \{(s, a), [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (s, b), [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (s, c), [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (w, a), [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), ([w, b], [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), ([w, c], [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}, as shown in Fig. 7.
Theorem 2.27. We state the following theorem without its proof.

We calculate $N^+(s) = \{(a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6])\}$ and $N^+(w) = \{(a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$. Therefore, $N^+(s) \cap N^+(w) = \{(a, [0.2, 0.5], [0.2, 0.3], (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3])\}$. Now, $i = |\text{supp}(N^+(s) \cap N^+(w))| = 3$. For $p = 3$, we have, $t_B^i(s, w) = 0.02$, $t_B^{p-1}(s, w) = 0.08$, $i_B^p(s, w) = 0.04$, $i_B^q(s, w) = 0.1$, $f_B^i(s, w) = 0.01$ and $f_B^{p-1}(s, w) = 0.03$. This graph is depicted in Fig. 8.

We state the following theorem without its proof.

**Theorem 2.27.** Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. If

- $h_1^1(N^+(s) \cap N^+(w)) = 1$, $h_2^2(N^+(s) \cap N^+(w)) = 1$, $h_3^3(N^+(s) \cap N^+(w)) = 0$, $h_4^4(N^+(s) \cap N^+(w)) = 1$, $h_5^5(N^+(s) \cap N^+(w)) = 1$, $h_6^6(N^+(s) \cap N^+(w)) = 0$,
in $C[2](\overrightarrow{G})$, then the edge $(s, w)$ is strong, where $i = |\text{supp}(N^+(s) \cap N^+(w))|$. (Note that for any real number $s$, $[s]$ is the greatest integer not exceeding $s$.)

C. $m$-step interval-valued neutrosophic competition graphs

We now define another extension of IVNC-graph known as $m$-step IVNC-graph. We will use the following notations:

- $P^m_{s, w}$: An interval-valued neutrosophic path of length $m$ from $s$ to $w$.
- $\overrightarrow{P}^m_{s, w}$: A directed interval-valued neutrosophic path of length $m$ from $s$ to $w$.
- $N^+_m(s)$: $m$-step interval-valued neutrosophic out-neighbourhood of vertex $s$.
- $N^-_m(s)$: $m$-step interval-valued neutrosophic in-neighbourhood of vertex $s$.
- $N_m(G)$: $m$-step interval-valued neutrosophic neighbourhood graph of the IVN-graph $G$.
- $\overrightarrow{G}_m$: $m$-step IVNC-graph of the IVN-digraph $\overrightarrow{G}$.

**Definition 2.28.** Suppose $\overrightarrow{G} = (A, \overrightarrow{B})$ is an IVN-digraph. The $m$-step IVN-digraph of $\overrightarrow{G}$ is denoted by $\overrightarrow{G}^m = (A, B)$, where IVN-set of vertices of $\overrightarrow{G}^m$ is same with IVN-set of vertices of $\overrightarrow{G}$ and has an edge between $s$ and $w$ in $\overrightarrow{G}^m$ if and only if there exists an interval-valued neutrosophic directed path $\overrightarrow{P}^m_{s, w}$ in $\overrightarrow{G}$.

**Definition 2.29.** The $m$-step interval-valued neutrosophic out-neighbourhood (IVN-out-neighbourhood) of vertex $s$ in an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is IVN-set

$$N^+_m(s) = (X^+_s, [t^+_s, i^+_s])$$

where $X^+_s = \{w\}$ there exists a directed interval-valued neutrosophic path of length $m$ from $s$ to $w$, $\overrightarrow{P}^m_{s, w}$, $t^+_s : X^+_s \rightarrow [0, 1]$, $i^+_s : X^+_s \rightarrow [0, 1]$, $\overrightarrow{G}$ is a digraph of $G$.

**Example 2.30.** Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), (d, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}$, and $B = \{(a, d), (0.1, 0.4), (0.3, 0.6), (0.2, 0.6), (a, c), (0.2, 0.6), (0.3, 0.5), (0.2, 0.6), (w, b), (0.2, 0.6), (0.1, 0.6), (0.2, 0.3), (b, c), (0.2, 0.4), (0.1, 0.2), (0.1, 0.3), (d, (0.1, 0.3), (0.1, 0.2), (0.2, 0.4)), as shown in Fig. 9.
We calculate 2-step IVN-out-neighbourhoods as, \(\mathbb{N}^+(s) = \{(c, [0.1, 0.4], [0.3, 0.5], [0.2, 0.6])\}, \{(d, [0.1, 0.4], [0.3, 0.5], [0.2, 0.4])\}\) and \(\mathbb{N}^+(w) = \{(c, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3])\}, \{(d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.3])\}\).  

**Definition 2.31.** The \(m\)-step interval-valued neutrosophic in-neighbourhood (IVN-in-neighbourhood) of vertex \(s\) of an IVN-digraph \(\overline{G} = (A, \overline{B})\) is IVN-set  
\[
\mathbb{N}_m^-(s) = \{X_s^{-}, [t_s^{(l)}], t_s^{(u)}], [l_s^{-}, u_s^{(l)}], [f_s^{(l)}], f_s^{(u)}]\},
\]  

where \(X_s^{-} = \{w\} \) there exists a directed interval-valued neutrosophic path of length \(m\) from \(w\) to \(s\), \(\overline{P}_{w,s}^m\), \(t_s^{(l)} : X_s^{-} \rightarrow [0, 1]\), \(t_s^{(u)} : X_s^{-} \rightarrow [0, 1]\), \(t_s^{(l)} : X_s^{-} \rightarrow [0, 1]\), \(f_s^{(l)} : X_s^{-} \rightarrow [0, 1]\) \(f_s^{(u)} : X_s^{-} \rightarrow [0, 1]\) are defined by \(t_s^{(l)} = \min\{t^l(s_1, s_2)\}, (s_1, s_2)\) is an edge of \(\overline{P}_{w,s}^m\), \(t_s^{(u)} = \min\{t^u(s_1, s_2)\}, (s_1, s_2)\) is an edge of \(\overline{P}_{w,s}^m\), \(f_s^{(l)} = \min\{f^l(s_1, s_2)\}, (s_1, s_2)\) is an edge of \(\overline{P}_{w,s}^m\), \(f_s^{(u)} = \min\{f^u(s_1, s_2)\}, (s_1, s_2)\) is an edge of \(\overline{P}_{w,s}^m\), respectively.

**Example 2.32.** Consider an IVN-digraph \(\overline{G} = (A, \overline{B})\) on \(X = \{s, w, a, b, c, d\}\), such that \(A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), (d, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6])\}\) and \(B = \{(s, a), [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (a, c), [0.2, 0.6], [0.3, 0.5], [0.2, 0.6]), ([w, b], [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), ([b, c], [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), ([b, d], [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}, as shown in Fig. 10.
We calculate 2-step IVN-in-neighbourhoods as, $N_\pm^+(s) = \{ (c, [0.1,0.4], [0.3,0.5], [0.2,0.6]), (d, [0.1,0.4], [0.3,0.5], [0.2,0.4]) \}$ and $N_\pm^-(w) = \{ (c, [0.2,0.4], [0.1,0.2], [0.1,0.3]), (d, [0.1,0.3], [0.1,0.2], [0.2,0.3]) \}$.

**Definition 2.33.** Suppose $\vec{G} = (A, \overrightarrow{B})$ is an IVN-digraph. The $m$-step IVNC-graph of IVN-digraph $\vec{G}$ is denoted by $C_m(\vec{G}) = (A, B)$ which has same IVN-set of vertices as in $\vec{G}$ and has an edge between two vertices $s, w \in X$ in $C_m(\vec{G})$ if and only if $(N_m^+(s) \cap N_m^+(w))$ is a non-empty IVN-set in $\vec{G}$. The interval-valued truth-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_1^m(N_m^+(s) \cap N_m^+(w))$, and $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_1^m(N_m^+(s) \cap N_m^+(w))$, the interval-valued indeterminacy-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_2^m(N_m^+(s) \cap N_m^+(w))$, and $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_2^m(N_m^+(s) \cap N_m^+(w))$, the interval-valued falsity-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_3^m(N_m^+(s) \cap N_m^+(w))$, and $\ell_B^m(s, w) = [\ell_A(s) \wedge \ell_A(w)]h_3^m(N_m^+(s) \cap N_m^+(w))$.

The 2-step IVNC-graph is illustrated by the following example.

**Example 2.34.** Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{ s, w, a, b, c, d \}$, such that $A = \{ (s, [0.4,0.5], [0.5,0.7], [0.8,0.9]), (w, [0.6,0.7], [0.4,0.6], [0.2,0.3]), (a, [0.2,0.6], [0.3,0.6], [0.2,0.6]), (b, [0.2,0.6], [0.1,0.6], [0.2,0.6]), (c, [0.2,0.7], [0.3,0.5], [0.2,0.6]), d, [0.2,0.6], [0.3,0.6], [0.2,0.6]) \}$, and $B = \{ (s, a), [0.1,0.4], [0.3,0.6], [0.2,0.6], (a, c), [0.2,0.6], [0.3,0.5], [0.2,0.6], ((a, d), [0.2,0.6], [0.3,0.5], [0.2,0.4], ([w, b], [0.2,0.6], [0.1,0.6], [0.2,0.3]), ((b, c), [0.2,0.4], [0.1,0.2], [0.1,0.3]), (b, d), [0.1,0.3], [0.1,0.2], [0.2,0.4]) \}$, as shown in Fig. 11.
We calculate $N^{+}_{2}(s) = \{(c, [0.1, 0.4], [0.3, 0.5], [0.2, 0.6]), (d, [0.1, 0.4], [0.3, 0.5], [0.2, 0.4])\}$ and $N^{+}_{2}(w) = \{(c, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (d, [0.1, 0.2], [0.2, 0.6]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$. Therefore, $N^{+}_{2}(s) \cap N^{+}_{2}(w) = \{(c, [0.1, 0.4], [0.1, 0.2], [0.2, 0.6]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$. Thus, $t^{+}_{B}(s, w) = 0.04, t^{+}_{B}(s, w) = 0.20, i^{+}_{B}(s, w) = 0.04, i^{+}_{B}(s, w) = 0.12, f^{+}_{B}(s, w) = 0.04$ and $f^{+}_{B}(s, w) = 0.12$. This graph is depicted in Fig. 12.

If a predator $s$ attacks one prey $w$, then the linkage is shown by an edge $(s, w)$ in an IVN-digraph. But, if predator needs help of many other mediators $s_1$, $s_2$, \ldots, $s_{m-1}$, then linkage among them is shown by interval-valued neutrosophic directed path $\vec{P}_{s,w}^{m}$ in an IVN-digraph. So, $m$-step prey in an IVN-digraph is represented by a vertex which is the $m$-step out-neighbourhood of some vertices. Now, the strength of an IVNC-graphs is defined below.

**Definition 2.35.** Let $\vec{G} = (A, \vec{B})$ be an IVN-digraph. Let $w$ be a common vertex of $m$-step out-neighbourhoods of vertices $s_1$, $s_2$, \ldots, $s_l$. Also, let $\vec{B}^1_{s_1}(u_1, v_1)$, $\vec{B}^1_{s_2}(u_2, v_2)$, \ldots, $\vec{B}^1_{s_l}(u_r, v_r)$ and $\vec{B}^2_{s_1}(u_1, v_1)$, $\vec{B}^2_{s_2}(u_2, v_2)$, \ldots, $\vec{B}^2_{s_l}(u_r, v_r)$ be the minimum interval-valued truth-membership values, $\vec{B}^3_{s_1}(u_1, v_1)$, $\vec{B}^3_{s_2}(u_2, v_2)$, \ldots, $\vec{B}^3_{s_l}(u_r, v_r)$ and $\vec{B}^4_{s_1}(u_1, v_1)$, $\vec{B}^4_{s_2}(u_2, v_2)$, \ldots, $\vec{B}^4_{s_l}(u_r, v_r)$ be the minimum indeterminacy-membership values.
values, $\overrightarrow{B}_1^5(u_1, v_1), \overrightarrow{B}_2^5(u_2, v_2), \ldots, \overrightarrow{B}_r^5(u_r, v_r)$ and $\overrightarrow{B}_3^5(u_1, v_1), \overrightarrow{B}_3^5(u_2, v_2), \ldots, \overrightarrow{B}_3^5(u_r, v_r)$ be the maximum false-membership values, of edges of the paths $\overrightarrow{P}_s^{m_1}, \overrightarrow{P}_s^{m_2}, \ldots, \overrightarrow{P}_s^{m_r}$, respectively. The $m$-step prey $w \in X$ is strong prey if

$$\overrightarrow{B}_1(u_i, v_i) > 0.5, \quad \overrightarrow{B}_2(u_i, v_i) > 0.5, \quad \overrightarrow{B}_3(u_i, v_i) < 0.5,$$

$$\overrightarrow{B}_1^5(u_i, v_i) > 0.5, \quad \overrightarrow{B}_2^5(u_i, v_i) > 0.5, \quad \overrightarrow{B}_3^5(u_i, v_i) < 0.5, \quad \text{for all } i = 1, 2, \ldots, r.$$

The strength of the prey $w$ can be measured by the mapping $S : X \rightarrow [0, 1]$, such that:

$$S(w) = \frac{1}{r} \left\{ \sum_{i=1}^{r} \overrightarrow{B}_1^5(u_i, v_i) + \sum_{i=1}^{r} \overrightarrow{B}_2^5(u_i, v_i) + \sum_{i=1}^{r} \overrightarrow{B}_3^5(u_i, v_i) \right\}$$

$$+ \sum_{i=1}^{r} \overrightarrow{B}_1^5(u_i, v_i) - \sum_{i=1}^{r} \overrightarrow{B}_2^5(u_i, v_i) - \sum_{i=1}^{r} \overrightarrow{B}_3^5(u_i, v_i) \right\}. $$

**Example 2.36.** Consider an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ as shown in Fig. 11, the strength of the prey $c$ is equal to

$$\frac{(0.2 + 0.2) + (0.6 + 0.4) + (0.1 + 0.1) + (0.6 + 0.2) - (0.2 + 0.1) - (0.3 + 0.3)}{2} = 1.5$$

$$> 0.5.$$

Hence, $c$ is strong 2-step prey.

We state the following theorem without its proof.

**Theorem 2.37.** If a prey $w$ of $\overrightarrow{G} = (A, \overrightarrow{B})$ is strong, then the strength of $w$, $S(w) > 0.5$.

**Remark 2.38.** The converse of the above theorem is not true, i.e. if $S(w) > 0.5$, then all preys may not be strong. This can be explained as:

Let $S(w) > 0.5$ for a prey $w$ in $\overrightarrow{G}$. So,

$$S(w) = \frac{1}{r} \left\{ \sum_{i=1}^{r} \overrightarrow{B}_1^5(u_i, v_i) + \sum_{i=1}^{r} \overrightarrow{B}_2^5(u_i, v_i) + \sum_{i=1}^{r} \overrightarrow{B}_3^5(u_i, v_i) \right\}$$

$$+ \sum_{i=1}^{r} \overrightarrow{B}_1^5(u_i, v_i) - \sum_{i=1}^{r} \overrightarrow{B}_2^5(u_i, v_i) - \sum_{i=1}^{r} \overrightarrow{B}_3^5(u_i, v_i) \right\}.$$
This result does not necessarily imply that
\[ \overrightarrow{B}_j^1(u_i, v_i) > 0.5, \quad \overrightarrow{B}_j^2(u_i, v_i) > 0.5, \quad \overrightarrow{B}_j^3(u_i, v_i) < 0.5, \]
\[ \overrightarrow{B}_j^4(u_i, v_i) > 0.5, \quad \overrightarrow{B}_j^5(u_i, v_i) > 0.5, \quad \overrightarrow{B}_j^6(u_i, v_i) < 0.5, \]
for all \( i = 1, 2, \ldots, r \).

Since, all edges of the directed paths \( \overrightarrow{P}^m_{s_1,w}, \overrightarrow{P}^m_{s_2,w}, \ldots, \overrightarrow{P}^m_{s_r,w} \), are not strong. So, the converse of the above statement is not true i.e., if \( S(w) > 0.5 \), the prey \( w \) of \( \overrightarrow{G} \) may not be strong. Now, \( m \)-step interval-valued neutrosophic neighbourhood graphs are defined below.

**Definition 2.39.** The \( m \)-step IVN-out-neighbourhood of vertex \( s \) of an IVN-digraph \( \overrightarrow{G} = (A, B) \) is IVN-set
\[ N_m(s) = (X_s, [t_s^1, t_s^u], [i_s^1, i_s^u], [f_s^1, f_s^u]), \]
where \( X_s = \{ w \} \) there exists a directed interval-valued neutrosophic path of length \( m \) from \( s \) to \( w, \mathbb{P}^m_{s,w} \), \( t_s^1 : X_s \rightarrow [0, 1], i_s^1 : X_s \rightarrow [0, 1], i_s^u : X_s \rightarrow [0, 1], \)
\[ f_s^1 : X_s \rightarrow [0, 1], f_s^u : X_s \rightarrow [0, 1], \]
are defined by \( t_s^1 = \min \{t^1(s_1, s_2)\}, (s_1, s_2) \) is an edge of \( \mathbb{P}^m_{s,w} \), \( i_s^1 = \min \{i^1(s_1, s_2)\}, (s_1, s_2) \) is an edge of \( \mathbb{P}^m_{s,w} \), \( i_s^u = \min \{i^u(s_1, s_2)\}, (s_1, s_2) \) is an edge of \( \mathbb{P}^m_{s,w} \), \( f_s^1 = \min \{f^1(s_1, s_2)\}, (s_1, s_2) \) is an edge of \( \mathbb{P}^m_{s,w} \), \( f_s^u = \min \{f^u(s_1, s_2)\}, (s_1, s_2) \) is an edge of \( \mathbb{P}^m_{s,w} \), respectively.

**Definition 2.40.** Suppose \( G = (A, B) \) is an IVN-graph. Then \( m \)-step interval-valued neutrosophic neighbourhood graph \( N_m(G) \) is defined by \( N_m(G) = (A, B) \) where \( A = ([A_1^1, A_1^u], [A_2^1, A_2^u], [A_3^1, A_3^u]), B = ([\overrightarrow{B}_1^1, \overrightarrow{B}_1^u], [\overrightarrow{B}_2^1, \overrightarrow{B}_2^u], [\overrightarrow{B}_3^1, \overrightarrow{B}_3^u]), \)
\[ \overrightarrow{B}_1^1 : X \times X \rightarrow [0, 1], \overrightarrow{B}_1^u : X \times X \rightarrow [0, 1], \overrightarrow{B}_2^1 : X \times X \rightarrow [0, 1], \overrightarrow{B}_2^u : X \times X \rightarrow [0, 1], \]
\[ \overrightarrow{B}_3^1 : X \times X \rightarrow [0, 1], \] and \( \overrightarrow{B}_3^u : X \times X \rightarrow [0, 1] \) are such that:
\[ \overrightarrow{B}_1^1(s, w) = A_1^1(s) \land A_1^u(w)h_1^1(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}_2^1(s, w) = A_2^1(s) \land A_2^u(w)h_2^1(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}_3^1(s, w) = A_3^1(s) \land A_3^u(w)h_3^1(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}_1^u(s, w) = A_1^u(s) \land A_1^1(w)h_1^u(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}_2^u(s, w) = A_2^u(s) \land A_2^1(w)h_2^u(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}_3^u(s, w) = A_3^u(s) \land A_3^1(w)h_3^u(N_m(s) \cap N_m(w)), \]
respectively.

We state the following theorems without their proofs.

**Theorem 2.41.** If all preys of \( \overrightarrow{G} = (A, B) \) are strong, then all edges of \( C_m(\overrightarrow{G}) = (A, B) \) are strong.

A relation is established between \( m \)-step IVVC-graph of an IVN-digraph and IVVC-graph of \( m \)-step IVN-digraph.

**Theorem 2.42.** If \( \overrightarrow{G} \) is an IVN-digraph and \( \overrightarrow{G}_m \) is the \( m \)-step IVN-digraph of \( \overrightarrow{G} \), then \( C(\overrightarrow{G}_m) = C_m(\overrightarrow{G}) \).

**Theorem 2.43.** Let \( \overrightarrow{G} = (A, B) \) be an IVN-digraph. If \( m > |X| \) then \( C_m(\overrightarrow{G}) = (A, B) \) has no edge.
Theorem 2.44. If all the edges of IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ are independent strong, then all the edges of $C_m(\overrightarrow{G})$ are independent strong.

3. Conclusions

Graph theory is an enjoyable playground for the research of proof techniques in discrete mathematics. There are many applications of graph theory in different fields. We have introduced IVNC-graphs and $k$-competition IVN-graphs, $p$-competition IVN-graphs and $m$-step IVNC-graphs as the generalized structures of IVNC-graphs. We have described interval-valued neutrosophic open and closed-neighbourhood. Also we have established some results related to them. We aim to extend our research work to (1) Interval-valued fuzzy rough graphs; (2) Interval-valued fuzzy rough hypergraphs, (3) Interval-valued fuzzy rough neutrosophic graphs, and (4) Decision support systems based on IVN-graphs.

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