

Combination of Qualitative Information with 2-Tuple Linguistic Representation in Dezert-Smarandache Theory

Xin-De Li^{1,*} (李新德), Member, IEEE, Florentin Smarandache², Jean Dezert³, and Xian-Zhong Dai¹ (戴先中)

¹MOE Key Laboratory of Measurement and Control of CSE, School of Automation, Southeast University
Nanjing 210096, China

²Mathematics and Sciences Department, University of New Mexico, 200 College Road, Gallup, NM 87301, U.S.A.

³French Aerospace Research Lab., 29 Avenue de la Division Leclerc 92320 Châtillon, France

E-mail: {xindeli; xzdai}@seu.edu.cn; smarand@unm.edu; jean.dezert@onera.fr

Received November 25, 2008; revised April 9, 2009.

Abstract Modern systems for information retrieval, fusion and management need to deal more and more with information coming from human experts usually expressed qualitatively in natural language with linguistic labels. In this paper, we propose and use two new 2-Tuple linguistic representation models (i.e., a distribution function model (DFM) and an improved Herrera-Martínez's model) jointly with the fusion rules developed in Dezert-Smarandache Theory (DSmT), in order to combine efficiently qualitative information expressed in term of qualitative belief functions. The two models both preserve the precision and improve the efficiency of the fusion of linguistic information expressing the global expert's opinion. However, DFM is more general and efficient than the latter, especially for unbalanced linguistic labels. Some simple examples are also provided to show how the 2-Tuple qualitative fusion rules are performed and their advantages.

Keywords Dezert-Smarandache Theory (DSmT), information fusion, qualitative reasoning, linguistic labels

1 Introduction

Qualitative methods for reasoning under uncertainty have gained more and more attention because traditional methods based only on quantitative representation and analysis are not able to adequately satisfy the need of the development of science and technology integrating at higher fusion level human beliefs and reports in complex systems. Therefore qualitative knowledge representation and analysis become important and necessary in next generation decision-making support systems. Most of the existing approaches use the 1-Tuple classical linguistic representation model consisting, in a given finite ordered set, of pure linguistic labels, say $L = \{L_0, \tilde{L}, L_{n+1}\}$, where $\tilde{L} = \{L_1, \dots, L_n\}$. Smarandache and Dezert^[1,2] give a detailed introduction of the major work of 1-Tuple qualitative reasoning under uncertainty. In 2007, Li *et al.*^[3] proposed in the DSmT framework the extension of 1-Tuple linguistic representation model to Qualitative Enriched labels, denoted as $L_i(\sigma_i^e)$, for taking into account a possible quantitative or qualitative *confidence factor* σ_i^e . However, some available richer information

content is lost in the classical/1-Tuple qualitative information processing. To overcome this limitation, Herrera and Martínez^[4,5] proposed a 2-Tuple fuzzy linguistic representation model for computing with words (CW). Their 2-Tuple labels could be used to solve the problem of non-equidistant labels according to Multi-Granular Hierarchical Linguistic Contexts^[6,7] at the cost of huge computation, but recently, Wang and Hao proposed a more interesting 2-Tuple linguistic representation model, which consists of two proportional linguistic terms, i.e., proportional 2-Tuples^[8,9]. Proportional 2-Tuples solve the problem of non-equidistant labels more efficiently than Herrera-Martínez's model, and this approach can be generalized as we propose. Because the previous 2-Tuples cannot be directly used for uncertain reasoning in DST or DSmT framework, we present in this paper some improvements of Herrera-Martínez's model^[10] and Wang-Hao's model and we define a general model of proportional 2-Tuples called the Distribution Function Model (DFM) to deal with either equidistant or non-equidistant labels for qualitative information fusion. The 2-Tuple linguistic representation model proposed is also extended directly

Regular Paper

This work is supported by the National Natural Science Foundation of China under Grant No. 60804063.

*Presently, he is taking charge of one project supported by the National Natural Science Foundation of China under Grant No.60804063 and one subproject of Jiangsu Province Science and Technology Transformation Project under Grant No. BA2007058.

to its *enriched* version which can be useful in practice in some particular situations. Some examples are presented with qualitative *DSm* fusion rules (denoted as q_2DSmC , q_2PCR5) for combining 2-Tuple qualitative beliefs based on the direct extension of quantitative *DSm* fusion rules. These qualitative fusion rules keep the precision in the information processing. This work extends the field of information fusion (usually too limited for quantitative information processing only) and opens new tracks for human-originated information retrieval, combination and management.

This paper is organized as follows. After a short presentation of Dezert-Smarandache Theory (DSMT) in Section 2, we recall the linguistic representation models, i.e., 1-Tuple classical model, 1-Tuple enriched model, Herrera-Martínez model and Wang-Hao model in Section 3. In Section 4, we present the extended/improved Herrera-Martínez model and also our new generalized model with the basic operators. To overcome the limitations of 1-Tuple enriched model, we directly and simply extend 2-Tuple linguistic enriched model for more complex fusion of 1-Tuple enriched model. In Section 5, we present the extension of the 2-Tuple qualitative DSm Classic (DSmC) fusion rule^[1] and the Proportional Conflict Redistribution rule No.5^[1] (PCR5) adapted for these new models. We provide some examples to show how to combine 2-Tuple qualitative beliefs with these fusion rules. We also compare the results with those obtained from other models. Concluding remarks are given in Section 6.

2 DSMT for the Fusion of Beliefs

In the following, we assume the reader is familiar with the theory of belief functions, which, also called Dempster-Shafer Theory (DST), was introduced in 1970s by Shafer^[11] and is well-known in the information fusion community.

2.1 Basic Belief Assignment

The main differences between Dempster-Shafer Theory^[11], and Dezert-Smarandache Theory (DSMT)^[1] are as follows.

1) The model with which one works. Typically, if one considers a finite frame of possible exhaustive solutions $\Theta = \{\theta_1, \dots, \theta_m\}$, Shafer assumes the exclusivity of θ_i and defines belief masses on the classical power set $2^\Theta \triangleq (\Theta, \cup)$ while we do not need such assumption in DSMT and the belief masses can be defined directly on Dedekind's lattice/hyper-power set $D^\Theta \triangleq (\Theta, \cup, \cap)$ and even on the super-power set $S^\Theta \triangleq (\Theta, \cup, \cap, c(\cdot))$ if one really needs/wants to work on the refined frame

Θ_{ref} of Θ . In the sequel, we use the generic notation G^Θ for denoting either 2^Θ , D^Θ or S^Θ . A quantitative basic belief assignment (bba) is a mapping $m(\cdot) : G^\Theta \rightarrow [0, 1]$ associated with a given body of evidence \mathcal{B} which satisfies $m(\emptyset) = 0$ and $\sum_{A \in G^\Theta} m(A) = 1$. The qualitative basic belief assignment (qbba) is defined in Section 3.

2) The choice of the combination and conditioning rules. Basically Dempster's rule in DST is versus PCR5 rule in DSMT (see [1] for detail).

3) Aside from working only with numerical/quantitative beliefs as within DST, DSMT also combines directly qualitative belief masses.

2.2 Fusion of Quantitative Belief Masses

In DSMT, we usually use the Proportional Conflict Redistribution rule No.5 (PCR5)^[1,12], which transfers conflicting masses (total or partial) proportionally to non-empty sets involved in the model according to all integrity constraints. PCR5 works for any degree of conflict in $[0, 1]$, for any models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSMT frameworks for static or dynamical fusion problems.

PCR5 for two sources is defined by

$$\begin{aligned}
 m_{PCR5}(\emptyset) &= 0 \text{ and } \forall X \in G^\Theta \setminus \{\emptyset\}, \\
 m_{PCR5}(X) &= m_{12}(X) \\
 &+ \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right],
 \end{aligned}
 \tag{1}$$

where each element X , and Y , is in the disjunctive normal form. $m_{12}(X) = \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)$ corresponds to the conjunctive consensus on X between the two sources. All denominators are different from zero. If a denominator is zero, that fraction is discarded.

No matter how big or small the conflicting mass is, PCR5 mathematically does a better redistribution of the conflicting mass than Dempster's rule and other rules since PCR5 goes backwards on the track of the conjunctive rule and redistributes the partial conflicting masses only to the sets involved in the conflict and proportionally to their masses put in the conflict, considering the conjunctive normal form of the partial conflict. PCR5 is quasi-associative and preserves the neutral impact of the vacuous belief assignment. General PCR5 fusion formula and improvement for the combination of $k \geq 2$ sources of evidence with many detailed examples can be found in [1].

3 Linguistic Representation Models

3.1 1-Tuple Linguistic Models

3.1.1 1-Tuple Classical Model

To deal with a 1-Tuple qualitative belief over G^θ , one defined in [1, 2, 13] a *qualitative basic belief assignment* $q_1m(\cdot)$ as a mapping function from G^θ into a set of linguistic labels $L = \{L_0, \tilde{L}, L_{n+1}\}$, where $\tilde{L} = \{L_1, \dots, L_n\}$ is a finite set of linguistic labels and $n \geq 2$ is an integer. For example, L_1 can take the linguistic value “poor”, L_2 the linguistic value “good”, etc. \tilde{L} is endowed with a total order relationship \prec , so that $L_1 \prec L_2 \prec \dots \prec L_n$. To work on a true closed linguistic set L under linguistic addition and multiplication operators, Smarandache and Dezert extended naturally \tilde{L} with two extreme values $L_0 = L_{\min}$ and $L_{n+1} = L_{\max}$, where L_0 corresponds to the minimal qualitative value and L_{n+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \dots \prec L_n \prec L_{n+1}$, where \prec means inferior to, or less (in quality) than, or smaller than, etc. In the sequel $L_i \in L$ are assumed linguistically equidistant labels as shown in Fig.1, where we can make an isomorphism between $L = \{L_0, L_1, L_2, \dots, L_n, L_{n+1}\}$ and $\{0, 1/(n+1), 2/(n+1), \dots, n/(n+1), 1\}$, defined as $L_i \mapsto i/(n+1)$ for all $i = 0, 1, 2, \dots, n, n+1$.

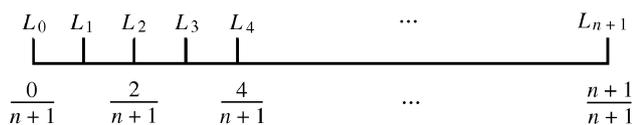


Fig.1. Isomorphic relationship between numbers and 1-Tuple labels.

From the extension of the isomorphism between the set of linguistic equidistant labels and a set of numbers in the interval $[0, 1]$, one can build exact operators on linguistic labels^[3]. For simplicity, here we use only the following approximate operators.

- q -addition:

$$L_i + L_j = \begin{cases} L_{i+j}, & \text{if } i + j < n + 1; \\ L_{n+1} = L_{\max}, & \text{if } i + j \geq n + 1. \end{cases} \quad (2)$$

- q -subtraction:

$$L_i - L_j = \begin{cases} L_{i-j}, & \text{if } i \geq j; \\ -L_{j-i}, & \text{if } i < j; \end{cases} \quad (3)$$

where $-L = \{-L_1, -L_2, \dots, -L_n, -L_{n+1}\}$.

- q -multiplication:

$$L_i \cdot L_j = L_{[(i \cdot j)/(n+1)]}, \quad (4)$$

where $[x]$ means the closest integer to x (with $[n+0.5] = n+1, \forall n \in \mathbb{N}$). This operator is justified by the approximation of the product of equidistant labels given by $L_i \cdot L_j = \frac{i}{n+1} \cdot \frac{j}{n+1} = \frac{(i \cdot j)/(n+1)}{n+1}$. The q -multiplication for $n > 2$ linguistic labels is possible, by example $L_i \cdot L_j \cdot L_k = L_{[(i \cdot j \cdot k)/(n+1)(n+1)]}$, etc. When working with the labels, no matter how many operations we have, the best (most accurate) result is obtained if we do only one approximation, and that one should be just at the very end.

- Scalar multiplication of a linguistic label: let a be a real number. The multiplication of a linguistic label by a scalar is defined by:

$$a \cdot L_i = \frac{a \cdot i}{n+1} \approx \begin{cases} L_{[a \cdot i]}, & \text{if } [a \cdot i] \geq 0; \\ L_{-[a \cdot i]}, & \text{otherwise.} \end{cases} \quad (5)$$

- Division of linguistic labels:

- a) q -division as an internal operator: let $j \neq 0$, then if $[(i/j) \cdot (n+1)] < n+1$ one defines

$$L_i / L_j = L_{[(i/j) \cdot (n+1)]}, \quad (6)$$

otherwise $L_i / L_j = L_{n+1}$.

- b) Division as an external operator: \odot . Let $j \neq 0$. We define

$$L_i \odot L_j = i/j. \quad (7)$$

From the q -operators we can directly extend all quantitative fusion rules into their qualitative counterparts by replacing classical operators on numbers with those on linguistic labels defined just above in the formulas. Many useful examples can be found in [1–3].

3.1.2 1-Tuple Enriched Model

To take into account the confidence in a linguistic assertions L_i , we proposed^[3] in 2007 a qualitative enriched 1-Tuple model, denoted as $L_i(\sigma_i^e)$, where the first component is a standard linguistic label L_i and the second component is a confidence factor σ_i^e (either a numerical supporting degree in $[0, 1]$ called Type 1, or a qualitative supporting degree taking its value in a given (ordered) set X of linguistic labels called Type 2). In [3], we used $\sigma_i^e \in [0, \infty)$ to allow over (quantitative) confidence but since the confidence factor usually comes from statistics it is more natural to take it in $[0, 1]$. σ_i^e represents the confidence one grants to the source when it assigns its qualitative belief L_i to a given proposition $A \in G^\theta$.

For example, the enriched Type 1 label $L_1 \triangleq L_1(1)$ represents the linguistic variable *Good* with 100% confidence, whereas $L_1(\sigma_1^e = 0.4)$ means that the linguistic value L_1 is discounted by 60%, i.e., we are under confident in L_1 given by the source. It is important to recall that σ_i^e is related with a confidence measure and does

not reflect a positive or negative refinement of the linguistic value itself, contrariwise to α_i^h in Herrera's *et al.* approach (see Subsection 3.2.1). That is why σ_i^e and L_i are considered as two independent components of the enriched label $L_i(\sigma_i^e)$ in the derivations done in [3]. We recall how to define new q^e -operators and how to combine qualitative beliefs based on this enriched linguistic 1-Tuple representation model.

First, we use the q -operators as presented in Subsection 3.1.1 for manipulating L_i, L_j labels, but for confidences we propose three possible versions. If the confidence in L_i is σ_i^e and the confidence in L_j is σ_j^e , then the confidence in combining L_i with L_j can be:

- (a) either the average, i.e., $(\sigma_i^e + \sigma_j^e)/2$;
- (b) or $\min\{\sigma_i^e, \sigma_j^e\}$;
- (c) or we may consider a confidence interval as

in statistics, so we get $[\sigma_{\min}^e, \sigma_{\max}^e]$, where $\sigma_{\min}^e \triangleq \min\{\sigma_i^e, \sigma_j^e\}$ and $\sigma_{\max}^e \triangleq \max\{\sigma_i^e, \sigma_j^e\}$; if $\sigma_i^e = \sigma_j^e$ then the confidence interval is reduced to a single point, σ_i^e .

In the sequel, we denote by “ c ” any of the above resulting confidence of combined enriched labels. All these versions coincide when $\epsilon_i = \epsilon_j = 1$ (for Type 1) or when $\epsilon_i = \epsilon_j = 0$ (for Type 2), i.e., there is no reinforcement or no discounting of the linguistic label. However the confidence degree average operator (case (a)) is not associative, so in many cases it is inconvenient to use it. The best among these three, which is the most prudent and easier to use, is the min operator. The confidence interval operator provides both a lower and an upper confidence level, so in an optimistic way, we may at the end take the midpoint of this confidence interval as a confidence level.

The qualitative enriched q^e operators working with enriched labels of Type 1 or Type 2 are then defined by:

- q^e -addition of enriched labels:

$$L_i(\sigma_i^e) + L_j(\sigma_j^e) = \begin{cases} L_{n+1}(c), & \text{if } i + j \geq n + 1; \\ L_{i+j}(c), & \text{otherwise.} \end{cases} \quad (8)$$

- q^e -multiplication of linguistic labels:

(a) As direct extension of (4), the multiplication of enriched labels is defined by

$$L_i(\sigma_i^e) \times L_j(\sigma_j^e) = L_{[(i \cdot j)/(n+1)]}(c); \quad (9)$$

(b) As another multiplication of labels, easier, but less exact:

$$L_i(\sigma_i^e) \times L_j(\sigma_j^e) = L_{\min\{i, j\}}(c). \quad (10)$$

- Scalar multiplication of an enriched label:

Let a be a real number. We define the multiplication

of an enriched linguistic label by a scalar as follows:

$$a \cdot L_i(\sigma_i^e) \approx \begin{cases} L_{[a \cdot i]}(\sigma_i^e), & \text{if } [a \cdot i] \geq 0; \\ L_{-[a \cdot i]}(\sigma_i^e), & \text{otherwise.} \end{cases} \quad (11)$$

- q^e -division of enriched labels:

(a) Division as an internal operator:

Let $j \neq 0$, then

$$\frac{L_i(\sigma_i^e)}{L_j(\sigma_j^e)} = \begin{cases} L_{n+1}(c), & \text{if } [(i/j) \cdot (n+1)] \geq n+1; \\ L_{[(i/j) \cdot (n+1)]}(c), & \text{otherwise.} \end{cases} \quad (12)$$

(b) Division as an external operator:

Let $j \neq 0$, then we can also consider the division of enriched labels as an external operator as follows.

$$L_i(\sigma_i^e) \oslash L_j(\sigma_j^e) = (i/j)_{\text{supp}(c)}. \quad (13)$$

The notation $(i/j)_{\text{supp}(c)}$ means that the numerical value (i/j) is supported with the degree c .

- q^e -subtraction of enriched labels:

$$L_i(\sigma_i^e) - L_j(\sigma_j^e) = \begin{cases} L_{i-j}(c), & \text{if } i \geq j; \\ -L_{j-i}(c), & \text{if } i < j. \end{cases} \quad (14)$$

These enriched q^e operators, although appealing with respect to classical operators of Subsection 3.1.1, suffer from the fact that a part of precision is lost because of the approximations done in derivation of the integer index, that is why we propose to use the enriched model together with the 2-Tuples for reasoning under uncertainty in DST or DSMT framework. Before going further on this, we firstly recall in the next subsection Herrera-Martínez^[5] linguistic model and Wang-Hao's linguistic model^[8,9], which was historically proposed for manipulating refined labels and kept the precision in the process of operation.

3.2 2-Tuple Linguistic Models

3.2.1 Herrera-Martínez's 2-Tuples

In order to keep working with a coarse/reduced set of linguistic labels for maintaining a low computational complexity but for working with a richer/refined information, Herrera and Martínez proposed in 2000 a 2-Tuple model in [4, 5] denoted as (L_i, α_i^h) different from our previous 1-Tuple (enriched) representation, where α_i^h expressed a kind of refinement of the linguistic label L_i . Clearly σ_i^e and α_i^h corresponded to two kinds of distinct notions. σ_i^e was related with the reliability/confidence of the qualitative information, whereas α_i^h was related with the refinement of the qualitative information, i.e., $\alpha_i^h \in [-0.5, 0.5)$, with $i \in \{0, \dots, n\}$. The value used to aggregate linguistic information is $\gamma \in [0, n]$. The 2-Tuple (L_i, α_i^h)

that expressed the equivalent information to γ was obtained through Herrera-Martínez's transformation function $\blacktriangle(\cdot) : [0, n] \rightarrow L \times [-0.5, 0.5]$ defined in [4, 5] by

$$\blacktriangle(\gamma) = (L_i, \alpha^h) \triangleq \begin{cases} L_i, & i = \text{round}(\gamma); \\ \alpha^h = \gamma - i, & \alpha^h \in [-0.5, 0.5]. \end{cases} \quad (15)$$

Herrera and Martínez also defined in [4, 5] the dual/inverse function of $\blacktriangle(\cdot)$ as

$$\blacktriangle^{-1}(L_i, \alpha_i^h) = i + \alpha_i^h = \gamma. \quad (16)$$

In addition, a 2-Tuple negation operator was also defined in [4, 5] as follows:

$$\text{Neg}((L_i, \alpha_i^h)) = \blacktriangle(n - (\blacktriangle^{-1}(L_i, \alpha_i^h))). \quad (17)$$

In order to solve unbalanced labels, Herrera and Martínez introduced a hierarchical linguistic structure to deal with multigranular linguistic contexts^[6,7].

3.2.2 Wang and Hao's 2-Tuples

Recently, Wang and Hao proposed a kind of 2-Tuple ordinal information representation model based on a symbolic proportionalization. In [8, 9], they considered L as the ordered set of $n + 1$ ordinal terms $L = \{l_0, l_1, \dots, l_n\}$, and L only makes no difference with Herrera-Martínez's 2-Tuples model. But actually they considered $L = l_0, l_1, \dots, l_n$ and the interval $I = [0, 1]$, and they proposed working with $IL \equiv I \times L = \{(\alpha, l_i) : \alpha \in [0, 1] \text{ for } i = 0, 1, \dots, n\}$. The authors then considered a pair (l_i, l_{i+1}) of two successive ordinal terms of L associated with two parameters $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$. $(\alpha, l_i), (\beta, l_{i+1})$ of IL is called symbolic proportion pair and could be equivalently denoted as $(\alpha l_i, (1 - \alpha)l_{i+1})$. Wang and Hao also defined the corresponding transformation function π and its dual/inverse one π^{-1} , i.e.,

$$\pi((\alpha l_i, (1 - \alpha)l_{i+1})) = i + (1 - \alpha) = \chi, \quad (18)$$

where $\chi \in [0, n]$.

$$\pi^{-1}(\chi) = ((1 - \beta)l_i, \beta l_{i+1}), \quad (19)$$

where $i = E(\chi)$ and E is the integral part function, $\beta = \chi - i$. The negation operator is then defined as

$$\text{Neg}((\alpha l_i, (1 - \alpha)l_{i+1})) = \pi^{-1}(n - (\pi((\alpha l_i, (1 - \alpha)l_{i+1})))) \quad (20)$$

This proportional 2-Tuple can deal with equidistant labels and also more efficiently (i.e., with a less computation amount) with unbalanced labels than Herrera-Martínez's model.

4 Extended 2-Tuples

Although previous 2-Tuples have many advantages, they cannot be directly used for combination reasoning in DST or DSmT framework. In this section, we improve the previous linguistic models in order to deal more precisely and more efficiently with the qualitative information through the combination process.

4.1 Extended Herrera-Martínez's Model

Here we extend Herrera-Martínez's 2-Tuple label model (L_i, α_i^h) to (L_i, σ_i^h) . σ_i^h is distinct from α_i^h , which is chosen in $\Sigma \triangleq [-0.5/(n + 1), 0.5/(n + 1)]$, not $[-0.5, 0.5]$ with $i \in \{1, \dots, \infty\}$. It is a numerical value of the symbolic translation of our quantitative two-order support.

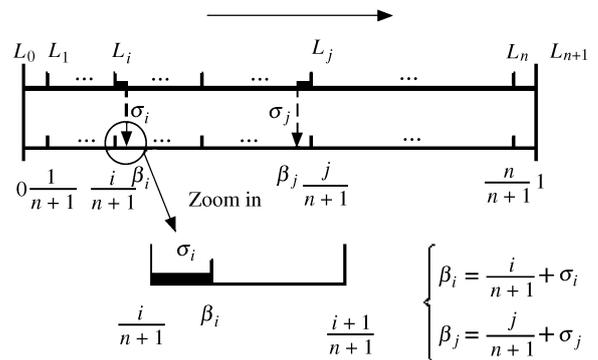


Fig.2. Isomorphism between numbers and 2-Tuples.

The 2-Tuple model is justified since each distance between two equidistant labels is $1/(n + 1)$ because of the isomorphism between L and $\{0, 1/(n + 1), \dots, n/(n + 1), 1\}$, so that $L_i = i/(n + 1)$ for all $i = 0, 1, 2, \dots, n, n + 1$. Therefore, we take half to the left and half to the right of each label, so $\sigma_i^h \in \Sigma$. This 2-Tuple equidistant linguistic representation model is used to represent the linguistic information by means of 2-Tuple item set $\Pi(L, \sigma^h)$ with $L = \{L_0, L_1, L_2, \dots, L_n, L_{n+1}\}$ isomorphic to $\{0, 1/(n + 1), 2/(n + 1), \dots, n/(n + 1), 1\}$ and the set of qualitative assessments isomorphic to Σ .

This 2-Tuple approach is an intricate/hybrid mechanism of derivation using jointly L_i and σ_i^h where σ_i^h is a positive or negative numerical remainder with respect to the labels as shown in Fig.2.

* *Symbolic Translation*: we define the normalized index $i = \text{round}((n + 1) \times \lambda) = [(n + 1) \times \lambda]$, with $i \in [0, (n + 1)]$ and $\lambda \in [0, 1]$, which is distinct from Herrera and Martínez's definition. And the *Symbolic Translation* $\sigma^h \triangleq \lambda - i/(n + 1) \in [-0.5/(n + 1), 0.5/(n + 1)]$; where $\text{round}(\cdot)$ is the rounding operation previously denoted $[\cdot]$ as in [3]. Roughly speaking, the symbolic translation of an assessment linguistic value

$(n + 1) \times \sigma_i^h$ is a numerical value that supports the difference of information between the (normalized) index obtained from the fusion rule and its closest value in $\{0, 1, \dots, n + 1\}$.

★ *Useful Transformations on 2-Tuples*

• $\Delta(\cdot)$: conversion of a numerical value into a 2-Tuple.

$\Delta(\cdot): [0, 1] \rightarrow L \times \Sigma$ is followed by Herrera and Martínez's definition^[4,5]

$$\Delta(\lambda) = (L_i, \sigma^h) \triangleq \begin{cases} L_i, & i = \text{round}((n + 1) \cdot \lambda); \\ \sigma^h = \lambda - i/(n + 1), & \sigma^h \in \Sigma. \end{cases} \quad (21)$$

Thus L_i has the closest index label to λ and σ^h is the value of its symbolic translation.

• $\nabla(\cdot)$: conversion of a 2-Tuple into a numerical value.

The inverse/dual function of $\Delta(\cdot)$ is denoted as $\nabla(\cdot)$ and $\nabla(\cdot) : L \times \Sigma \rightarrow [0, 1]$ is defined by

$$\nabla((L_i, \sigma_i^h)) = i/(n + 1) + \sigma_i^h = \lambda_i. \quad (22)$$

It has been proved in [4, 5] that any arithmetic operation commutes with $\Delta(\cdot)$ and/or with $\nabla(\cdot)$.

★ *Useful Operators on 2-Tuples*

Let us consider two 2-Tuples (L_i, σ_i^h) and (L_j, σ_j^h) , then the following operators are defined:

• q_2^h -Addition

$$\begin{aligned} (L_i, \sigma_i^h) + (L_j, \sigma_j^h) &\equiv \nabla((L_i, \sigma_i^h) + (L_j, \sigma_j^h)) \\ &= \nabla((L_i, \sigma_i^h)) + \nabla((L_j, \sigma_j^h)) = \lambda_i + \lambda_j = \lambda_z \\ &= \begin{cases} \Delta(\lambda_z), & \text{if } \lambda_z \in [0, 1]; \\ L_{n+1}, & \text{otherwise.} \end{cases} \end{aligned} \quad (23)$$

• q_2^h -Subtraction

Before giving the subtraction of 2-Tuples in extended Herrera and Martínez's model, it is necessary to improve Herrera and Martínez's negation operator, i.e., $Neg((L_i, \sigma_i^h)) = \Delta(0 - (\nabla(L_i, \sigma_i^h)))$ for some combination operations. According to it, we define the subtraction operator as follows:

$$\begin{aligned} (L_i, \sigma_i^h) - (L_j, \sigma_j^h) &\equiv \nabla((L_i, \sigma_i^h) - (L_j, \sigma_j^h)) \\ &= \nabla((L_i, \sigma_i^h)) - \nabla((L_j, \sigma_j^h)) = \lambda_i - \lambda_j = \lambda_z \\ &= \begin{cases} \Delta(\lambda_z), & \text{if } \lambda_z \in [0, 1]; \\ Neg(\Delta(-\lambda_z)), & \text{if } \lambda_z \in [-1, 0]; \\ \pm L_{n+1}, & \text{otherwise.} \end{cases} \end{aligned} \quad (24)$$

• q_2^h -product

$$\begin{aligned} (L_i, \sigma_i^h) \times (L_j, \sigma_j^h) &\equiv \nabla((L_i, \sigma_i^h) \times (L_j, \sigma_j^h)) \\ &= \nabla((L_i, \sigma_i^h)) \times \nabla((L_j, \sigma_j^h)) \\ &= \lambda_i \times \lambda_j = \lambda_p \equiv \Delta(\lambda_p) \end{aligned} \quad (25)$$

with $\lambda_p \in [0, 1]$. It can be proved that 2-Tuple addition and product operators are commutative and associative.

• q_2^h -scalar multiplication

$$\begin{aligned} \alpha \cdot (L_i, \sigma_i^h) &\equiv \nabla(\alpha \cdot (L_i, \sigma_i^h)) = \alpha \cdot \nabla((L_i, \sigma_i^h)) \\ &= \alpha \cdot \lambda_i = \lambda_\eta \equiv \begin{cases} \Delta(\lambda_\eta), & \lambda_\eta \in [0, 1]; \\ L_{n+1}, & \text{otherwise.} \end{cases} \end{aligned} \quad (26)$$

• q_2^h -division

Let us consider two 2-Tuples (L_i, σ_i^h) and (L_j, σ_j^h) , with $(L_i, \sigma_i^h) < (L_j, \sigma_j^h)$, where the comparison operator is defined in [4], then the division is defined as

$$\begin{aligned} \frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} &\equiv \nabla\left(\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}\right) = \frac{\nabla((L_i, \sigma_i^h))}{\nabla((L_j, \sigma_j^h))} \\ &= \frac{\lambda_i}{\lambda_j} = \lambda_d \equiv \Delta(\lambda_d) \quad \text{with } \lambda_d \in [0, 1]. \end{aligned} \quad (27)$$

If $(L_i, \sigma_i^h) \geq (L_j, \sigma_j^h)$, then

$$\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} \equiv \nabla\left(\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}\right) = \frac{\nabla((L_i, \sigma_i^h))}{\nabla((L_j, \sigma_j^h))} = \frac{\lambda_i}{\lambda_j} \geq 1.$$

In such a case, $\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}$ is set to the maximum label, i.e.,

$$\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} = (L_{n+1}, 0) \sim L_{n+1}.$$

In this extended Herrera and Martínez's model, all 1-Tuple classical label indexes together with their 2-order components generate the field of real numbers \mathbb{R} . All labels can be seen as continuous quantities, so that no information loss happens in the information processing. For unbalanced labels, Herrera and Martínez proposed a hierarchical representation model to deal with different granularity of uncertainty and/or semantics. Although there is no information loss in this process, this model is quite complex which makes its practical interest limited because of the huge amount of computation required. As already stated, Wang and Hao's proportional 2-Tuple offers advantages with respect to Herrera and Martínez's model in terms of complexity reduction. However, Wang and Hao's model reflects only a special case of unbalanced labels. So as an alternative, we propose in the next subsection a general linguistic model in DSMT framework called Distribution Function Model (DFM).

4.2 Distribution Function Model (DFM)

Dealing with equidistant labels is quite easy, but dealing with a given unbalanced label model (as shown in Fig.3) is more difficult. In the sequel, we propose a

new general representation model called the Distribution Function Model (DFM) for solving this problem.

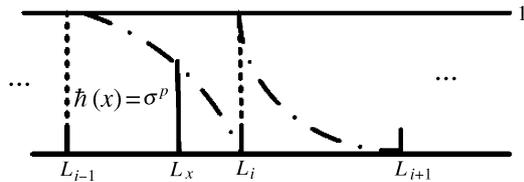


Fig.3. 2-Tuple label representation with unbalanced, or non-uniform distribution.

We assume that there exists a set of even distribution functions $\hbar(x) = -(|x| - i + 1)^k + 1$ ($k \in \mathbb{R}^+$) between any two labels L_{i-1} and L_i , $i \in [-n, n + 1]$. The inverse function of $\hbar(\cdot)$ always exists and is given by $\hbar^{-1}(\cdot) \in [i - 1, i]$. This 2-Tuple label model is then denoted as $\langle L_i, \hbar(\cdot) \rangle$ and for convenience we also denote it as q_2^p (standing for qualitative precise 2-Tuple representation model for short). $i - \hbar^{-1}(\cdot)$ represents the remainder done in the standard label L_i approximation. Let us consider a simple linear distribution function (when $k = 1$) $\hbar(x) = \sigma^p = -|x| + i$, $x \in [i - 1, i]$ as shown in Fig.4.

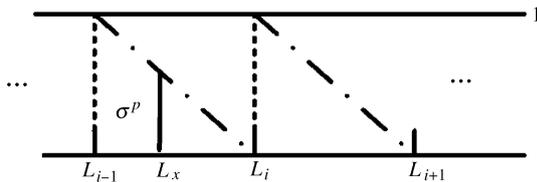


Fig.4. 2-Tuple label representation model within the proportional assessment.

$\hbar^{-1}(\cdot) = i - \sigma^p$ is continuous with the interval $[i - 1, i]$, $i \in [1, n + 1]$, where σ^p is a proportional factor used as the 2-order component modifier between two neighboring labels, i.e., $\frac{i-x}{i-(i-1)} = \frac{\sigma^p}{1}$, $x = i - \sigma^p$. We denote this kind of 2-Tuple label model as $\langle L_i, \sigma^p \rangle = L_x = L_{i-\sigma^p}$, which is a bit similar to the proportional 2-Tuple^[8,9], but distinct from it.

Example. Let us consider two labels L_{i-1}, L_i , $i \in [1, n + 1]$ and we assume that there is a 2-Tuple label $\langle L_i, 0.6 \rangle$, then, $\langle L_i, 0.6 \rangle = L_{(i-0.6)}$. Of course, if $\langle L_i, \sigma^h \rangle = \langle L_i, \sigma^p \rangle$, there is a relation between them: $i = j$, $\sigma^p = -(n + 1)\sigma^h$, when $\sigma^h \leq 0$, and $j = i + 1$, $\sigma^p = 1 - (n + 1)\sigma^h$, when $\sigma^h > 0$, where, if $\sigma^p = 1$, then $\langle L_i, 1 \rangle = L_{i-1}$. If $\sigma^p = 0$, then $\langle L_i, 0 \rangle = L_i$.

★ Some Useful q_2^p Operators

● Comparison Operator

The comparison operator for any two labels $\langle L_i, \hbar(i) \rangle, \langle L_j, \hbar(j) \rangle$ under DFM is defined as

1) $\langle L_i, \hbar(i) \rangle \geq \langle L_j, \hbar(j) \rangle$, if $i > j$, $i, j \in [-(n+1), n+1]$;

2) $\langle L_i, \hbar(i) \rangle \geq \langle L_j, \hbar(j) \rangle$ if $i = j$, $i, j \in [-(n+1), n+1]$ and if $\hbar(i) \leq \hbar(j)$. Otherwise, $\langle L_i, \hbar(i) \rangle < \langle L_j, \hbar(j) \rangle$;

3) $\langle L_i, \hbar(i) \rangle \leq \langle L_j, \hbar(j) \rangle$ if $i < j$, $i, j \in [-(n+1), n+1]$.

● Negation Operator

The negation operator is defined as

$$Neg(\langle L_i, \hbar(i) \rangle) = \langle L_{-i}, -\hbar(-i) \rangle \quad (28)$$

where, $\hbar(-i) = \hbar(i)$. For example, for $\langle L_i, \sigma^p \rangle$ we obtain $Neg\langle L_i, \sigma^p \rangle = L_{-i+\sigma^p}$.

● q_2^p -Addition: for any two labels $\langle L_i, \hbar(i) \rangle, \langle L_j, \hbar(j) \rangle$, we define

$$\langle L_i, \hbar(i) \rangle + \langle L_j, \hbar(j) \rangle = L_{\hbar^{-1}(i)+\hbar^{-1}(j)}. \quad (29)$$

Special case,

$$\langle L_i, \sigma_i^p \rangle + \langle L_j, \sigma_j^p \rangle = L_{i+j-\sigma_i^p-\sigma_j^p}. \quad (30)$$

● q_2^p -Subtraction: for any two labels $\langle L_i, \hbar(i) \rangle, \langle L_j, \hbar(j) \rangle$, we define

$$\langle L_i, \hbar(i) \rangle - \langle L_j, \hbar(j) \rangle = L_{\hbar^{-1}(i)-\hbar^{-1}(j)}. \quad (31)$$

Special case,

$$\langle L_i, \sigma_i^p \rangle - \langle L_j, \sigma_j^p \rangle = L_{i-j+\sigma_j^p-\sigma_i^p}. \quad (32)$$

● q_2^p -Product: for any two labels $\langle L_i, \hbar(i) \rangle, \langle L_j, \hbar(j) \rangle$, we define

$$\langle L_i, \hbar(i) \rangle \times \langle L_j, \hbar(j) \rangle = L_{\frac{\hbar^{-1}(i) \times \hbar^{-1}(j)}{n+1}}. \quad (33)$$

Special case,

$$\langle L_i, \sigma_i^p \rangle \times \langle L_j, \sigma_j^p \rangle = L_{\frac{(i-\sigma_i^p) \times (j-\sigma_j^p)}{n+1}} \quad (34)$$

where, the product operators in (33) and (34) can be easily justified according to the product operator in extended Herrera and Martínez's model because of their consistency.

● q_2^p -Scalar Multiplication: for any label $\langle L_i, \hbar(i) \rangle$, $i \in n + 1$, and a real number α , we define

$$\alpha \cdot \langle L_i, \hbar(i) \rangle = \langle L_i, \hbar(i) \rangle \times \alpha = L_{\alpha \cdot \hbar^{-1}(i)}. \quad (35)$$

Special case,

$$\alpha \cdot \langle L_i, \sigma_i^p \rangle = \langle L_i, \sigma_i^p \rangle \times \alpha = L_{\alpha(i-\sigma_i^p)}. \quad (36)$$

● q_2^p -Division: for any two labels $\langle L_i, \hbar(i) \rangle, \langle L_j, \hbar(j) \rangle$, if $\langle L_i, \hbar(i) \rangle < \langle L_j, \hbar(j) \rangle$, then we define

$$\langle L_i, \hbar(i) \rangle / \langle L_j, \hbar(j) \rangle = L_{\frac{\hbar^{-1}(i)}{(\hbar^{-1}(j)) \times (n+1)}}. \quad (37)$$

Special case,

$$\langle L_i, \sigma_i^p \rangle / \langle L_j, \sigma_j^p \rangle = L_{\frac{(i-\sigma_i^p)}{(j-\sigma_j^p)} \times (n+1)}. \quad (38)$$

All these operators can be also easily justified according to DSMT Field and Linear Algebra of Refined Labels (FLARL)^[14]. We can also easily transform all the operators in (29)~(38) to their standard forms according to 2-Tuple definition in DFM.

4.3 Enriched 2-Tuple Linguistic Model

Similarly to the extension/enrichment of the 1-Tuple model, it is possible to extend the 2-Tuple model into a 2-Tuple enriched model as well, i.e., $(L_i, \sigma^h)(\sigma^e)$ in extended Herrera and Martínez's model or $\langle L_i, \sigma^p \rangle(\sigma^e)$ in DFM. Actually, the operations on (L_i, σ^h) or $\langle L_i, \sigma^p \rangle$ like those in the 1-Tuple classical label L_i are independent of σ^e . That is why we do not reintroduce them here for the sake of space limitation. Let us just introduce a simple example to show how it works using extended Herrera and Martínez's enriched model. Let us consider $n = 9$ linguistic labels in L and the two specific enriched 2-Tuple labels $(L_3, 0.03)(0.4)$, $(L_4, 0.02)(0.5)$, then

$$\begin{aligned} & (L_3, 0.03)(0.4) + (L_4, 0.02)(0.5) \\ &= ((L_3, 0.03) + (L_4, 0.02))(\min(0.4, 0.5)) \\ &= (L_8, -0.05)(0.4). \end{aligned}$$

Similarly, for DFM with linear distribution function, one will get two labels $\langle L_3, 0.3 \rangle(0.4)$, $\langle L_4, 0.2 \rangle(0.5)$, then

$$\begin{aligned} & \langle L_3, 0.3 \rangle(0.4) + \langle L_4, 0.2 \rangle(0.5) \\ &= (\langle L_3, 0.3 \rangle + \langle L_4, 0.2 \rangle)(\min(0.4, 0.5)) \\ &= \langle L_7, 0.5 \rangle(0.4). \end{aligned}$$

The other operations can be done easily and similarly for dealing with enriched 2-Tuples.

4.4 Remarks on Linguistic Models

The use of 1-Tuple representation model involving approximate operators on the labels provides only approximate results because of the rounding approximation function $[x]$ required to round the indexes of labels to integers in $\{0, \dots, n\}$. Therefore, the number and order of operations do count in the final result. When working with the labels, no matter how many operations we have, the best (most accurate) result is obtained if we do only one approximation on the final label index at the very end. A better solution is then to use non-approximate operators and/or switch to 2-Tuple representation models.

Herrera-Martínez and Wang-Hao models both keep the precision in the representation of the qualitative information. However, these models cannot be used directly for fusing qualitative information in DST or in DSMT frameworks because the set of 2-Tuple labels is mapped into the interval $[0, n]$ according to the transformation function \blacktriangle . For working within DST or DSMT frameworks, the masses of belief must take their values in $[0, 1]$ and that is why we need to extend Herrera-Martínez and Wang-Hao models to DFM.

The DFM model follows Wang-Hao's idea since it uses also proportional 2-Tuples to represent qualitative information. However, DFM is more general than Wang-Hao's model since DFM allows any kind of distribution function, contrariwise to Wang-Hao's model which represents qualitative information only with a simple linear distribution functions. Moreover, the representation (and computation requirement) of DFM is simpler than that of Wang-Hao's model according to FLARL because DFM avoids the complex and repeated transformation operations between two 2-Tuple labels. When working with equidistant labels, there is not a big difference between the extended Herrera-Martínez model and DFM. The difference increases when one wants to deal with unbalanced labels because the extended Herrera-Martínez model must adopt Herrera-Martínez's hierarchal linguistic structure to deal with multigranular linguistic contexts which requires a great amount of computation with respect to DFM. That is why DFM is more interesting in this case.

5 Fusion of 2-Tuple Qualitative Beliefs

Since the 2-Tuple DFM representation denoted as q_2^p (or extended Herrera-Martínez denoted as q_2^h) is able to deal precisely with qualitative information for both equidistant labels or unbalanced labels and since it fits well with DST and DSMT frameworks, we present in the next subsections some combination rules for fusing qualitative information based on DFM (and, for comparison, also on extended Herrera-Martínez model).

5.1 Fusion Rules of 2-Tuples

From the previous 2-Tuple models of qualitative beliefs and the previous operators, we are able to extend the PCR5 and Dempster-Shafer's (DS) fusion rules in the qualitative domain in a more precise way than done before. The qualitative belief mass/assignment (qba) $q_2m(\cdot)$ based on 2-Tuple representation is defined as $q_2m(\cdot): G^\Theta \rightarrow L \times \sigma^h$ (for extended Herrera-Martínez based approach) or $G^\Theta \rightarrow L \times \sigma^p$ (for DFM based approach) such that $q_2^h m(\emptyset) = (L_0, 0) = L_0$ or $q_2^p m(\emptyset) = \langle L_0, 0 \rangle = L_0$ and $\sum_{A \in G^\Theta} q_2^h m(A) = (L_{n+1}, 0) = L_{n+1}$ or $\sum_{A \in G^\Theta} q_2^p m(A) = \langle L_{n+1}, 0 \rangle =$

L_{n+1} . The q_2 -extensions of DSmC, PCR5 (1) and Dempster-Shafer's fusion rules^[11] for two sources on a frame Θ based on the 2-Tuple operators are then given by (the direct extension for $N > 2$ sources is possible but will not be detailed in this paper):

- q_2 -extension of DSmC fusion rule: $q_2m_{DSmC}(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$,

$$q_2m_{DSmC}(X) = \sum_{\substack{x_1, x_2, \dots, x_k \in D^\Theta \\ x_1 \cap x_2 \cap \dots \cap x_k = X}} \prod_{i=1}^k q_2m_i(X_i). \quad (39)$$

- q_2 -extension of PCR5 fusion rule: $q_2m_{PCR5}(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$,

$$q_2m_{PCR5}(X) = q_2m_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} \left[\frac{q_2m_1(X)^2 q_2m_2(Y)}{q_2m_1(X) + q_2m_2(Y)} + \frac{q_2m_2(X)^2 q_2m_1(Y)}{q_2m_2(X) + q_2m_1(Y)} \right], \quad (40)$$

where $q_2m_{12}(X)$ corresponds to the qualitative q_2 -extension of the conjunctive consensus.

- q_2 -extension of Dempster-Shafer fusion rule: $q_2m_{DS}(\emptyset) = L_0$ and $\forall X \in 2^\Theta \setminus \{\emptyset\}$,

$$q_2m_{DS}(X) = \frac{\sum_{\substack{x_1, x_2 \in 2^\Theta \\ x_1 \cap x_2 = X}} q_2m_1(X_1)q_2m_2(X_2)}{L_{n+1} - K_{12}}, \quad (41)$$

where the total degree of qualitative conflict is given by $K_{12} \triangleq \sum_{\substack{x_1, x_2 \in 2^\Theta \\ x_1 \cap x_2 = \emptyset}} q_2m_1(X_1)q_2m_2(X_2)$.

It is important to note that the addition, the subtraction, the product and the division operators involved in the previous formulas are the 2-Tuple operators defined in the previous section. The extensions (39), (40) and (41) are well justified since every 2-Tuple $\langle L_i, \sigma_i^h \rangle$ or $\langle L_i, \sigma_i^p \rangle$ can be mapped into a unique numerical value, which makes q_2DSmC , q_2PCR5 and q_2DS equivalent to DSmC, PCR5 and DS.

5.2 Examples of Fusion

Let us consider an investment corporation which must choose one of three projects in $\Theta = \{\theta_1, \theta_2, \theta_3\}$ (assume here that Shafer's model holds, for simplicity) to invest through two consulting departments. A set of qualitative values are used to describe the opinions of two consulting companies, i.e., $I \mapsto$ Impossible, $EU \mapsto$ Extremely-Unlikely, $VLC \mapsto$ Very-Low-Chance, $LLC \mapsto$ Little-Low-Chance, $SC \mapsto$ Small-Chance, $IM \mapsto$ IT-May, $MC \mapsto$ Meaningful-Chance, $LBC \mapsto$ Little-Big-Chance, $BC \mapsto$ Big-Chance, $ML \mapsto$ Most-Likely, C

\mapsto Certain. So, we consider the set of ordered linguistic labels $L = \{L_0 \equiv I, L_1 \equiv EU, L_2 \equiv VLC, L_3 \equiv LLC, L_4 \equiv SC, L_5 \equiv IM, L_6 \equiv MC, L_7 \equiv LBC, L_8 \equiv BC, L_9 \equiv ML, L_{10} \equiv C\}$ and in this example $n = 9$.

Case 1. The opinions of the two consulting companies/sources are given in Table 1 according to the extended Herrera and Martínez's model.

Table 1. 2-Tuple Belief Masses in Extended Herrera and Martínez's Model

$m(\cdot)$	θ_1	θ_2	θ_3
Source No. 1	$(L_4, 0.03)$	$(L_3, -0.03)$	$(L_3, 0)$
Source No. 2	$(L_5, 0)$	$(L_2, 0.01)$	$(L_3, -0.01)$

Following PCR5, the masses of the partial conflicts $\theta_1 \cap \theta_2$, $\theta_1 \cap \theta_3$ and $\theta_2 \cap \theta_3$ are redistributed to those belief masses involved in these conflicts according to (40). We obtain

$$q_2^h m_{xA1}(\theta_1) = \frac{(L_4, 0.03) \times (L_1, -0.0097)}{(L_6, 0.04)} \approx (L_1, -0.0393),$$

$$q_2^h m_{yA1}(\theta_2) = \frac{(L_2, 0.01) \times (L_1, -0.0097)}{(L_6, 0.04)} \approx (L_0, 0.0296),$$

$$q_2^h m_{xB1}(\theta_1) = \frac{(L_5, 0) \times (L_1, 0.035)}{(L_8, -0.03)} \approx (L_1, -0.0123),$$

$$q_2^h m_{yB1}(\theta_2) = \frac{(L_3, -0.03) \times (L_1, 0.035)}{(L_8, -0.03)} \approx (L_0, 0.0473),$$

$$q_2^h m_{xA2}(\theta_1) = \frac{(L_4, 0.03) \times (L_1, 0.0247)}{(L_7, 0.02)} \approx (L_1, -0.0255),$$

$$q_2 m_{xA2}(\theta_1) = \frac{(L_4, 0.03) \times (L_1, 0.0247)}{(L_7, 0.02)} \approx (L_1, -0.0255),$$

$$q_2^h m_{yA2}(\theta_3) = \frac{(L_3, -0.01) \times (L_1, 0.0247)}{(L_7, 0.02)} \approx (L_1, -0.0498),$$

and similarly, we have $q_2^h m_{xB2}(\theta_1) \approx (L_1, -0.0062)$, $q_2^h m_{yB2}(\theta_3) \approx (L_1, -0.0437)$, $q_2^h m_{xA3}(\theta_2) \approx (L_0, 0.0377)$, $q_2^h m_{yA3}(\theta_3) \approx (L_0, 0.0405)$, $q_2^h m_{xB3}(\theta_2) \approx (L_0, 0.0259)$ and $q_2^h m_{yB3}(\theta_3) \approx (L_0, 0.0370)$. Thus, we finally obtain

$$q_2^h m_{PCR5}(\theta_1) = q_2^h m_{12}(\theta_1) + q_2^h m_{xA1}(\theta_1) + q_2^h m_{xB1}(\theta_1) + q_2^h m_{xA2}(\theta_1) + q_2^h m_{xB2}(\theta_1) \approx (L_5, 0.0315),$$

$$\begin{aligned}
 q_2^h m_{\text{PCR5}}(\theta_2) &= q_2^h m_{12}(\theta_2) + q_2^h m_{yA1}(\theta_2) \\
 &\quad + q_2^h m_{yB1}(\theta_2) + q_2^h m_{xA3}(\theta_2) + q_2^h m_{xB3}(\theta_2) \\
 &\approx (L_2, -0.0026), \\
 q_2^h m_{\text{PCR5}}(\theta_3) &= q_2^h m_{\text{DSmC}}(\theta_3) + q_2^h m_{yA2}(\theta_3) \\
 &\quad + q_2^h m_{yB2}(\theta_3) + q_2^h m_{yA3}(\theta_3) + q_2^h m_{yB3}(\theta_3) \\
 &\approx (L_3, -0.0289).
 \end{aligned}$$

Since $q_2^h m_{\text{PCR5}}(\theta_1) > q_2^h m_{\text{PCR5}}(\theta_2)$ and $q_2^h m_{\text{PCR5}}(\theta_1) > q_2^h m_{\text{PCR5}}(\theta_3)$, the investment corporation must invest in the project θ_1 . Using DS fusion (41), the total conflict is $qK_{12} = q_2^h m_{12}(\theta_1 \cap \theta_2) + q_2^h m_{12}(\theta_1 \cap \theta_3) + q_2^h m_{12}(\theta_3 \cap \theta_2) = (L_6, 0.0413)$. Thus $q_2^h m_{\text{DS}}(\emptyset) \triangleq (L_0, 0)$ and

$$\begin{aligned}
 q_2^h m_{\text{DS}}(\theta_1) &= \frac{q_2^h m_{12}(\theta_1)}{L_{10} - qK_{12}} = \frac{(L_2, 0.015)}{L_{10} - (L_6, 0.0413)} \\
 &\approx (L_6, -0.0006), \\
 q_2^h m_{\text{DS}}(\theta_2) &= \frac{q_2^h m_{12}(\theta_2)}{L_{10} - qK_{12}} = \frac{(L_1, -0.0413)}{L_{10} - (L_6, 0.0413)} \\
 &\approx (L_2, -0.0419), \\
 q_2^h m_{\text{DS}}(\theta_3) &= \frac{q_2^h m_{12}(\theta_3)}{L_{10} - qK_{12}} = \frac{(L_1, -0.013)}{L_{10} - (L_6, 0.0413)} \\
 &\approx (L_2, 0.0425).
 \end{aligned}$$

$q_2^h m_{\text{DS}}(\theta_1)$ is still larger than $q_2^h m_{\text{DS}}(\theta_2)$, and $q_2^h m_{\text{DS}}(\theta_1) > q_2^h m_{\text{DS}}(\theta_3)$, and the first project is also chosen to invest based on q_2 DS rule. The final decision is the same as the previous one based on $q_2^h m_{\text{PCR5}}$. However, when the total conflict increases up to L_{10} , then $q_2^h m_{\text{DS}}$ results for decision-making can become counter-intuitive and it can yield to wrong decision (see [1] for counter examples of DS rule).

Case 2. The opinions of the two consulting companies/sources are given in Table 2 according to DFM.

Table 2. 2-Tuple Belief Masses in DFM

$m(\cdot)$	θ_1	θ_2	θ_3
Source No. 1	$\langle L_5, 0.7 \rangle$	$\langle L_3, 0.3 \rangle$	$\langle L_3, 0 \rangle$
Source No. 2	$\langle L_5, 0 \rangle$	$\langle L_3, 0.9 \rangle$	$\langle L_3, 0.1 \rangle$

Thus, we finally obtain

$$\begin{aligned}
 q_2^p m_{\text{PCR5}}(\theta_1) &\approx \langle L_6, 0.685 \rangle = (L_5, 0.0315), \\
 q_2^p m_{\text{PCR5}}(\theta_2) &\approx \langle L_2, 0.026 \rangle = (L_2, -0.0026), \\
 q_2^p m_{\text{PCR5}}(\theta_3) &\approx \langle L_3, 0.289 \rangle = (L_3, -0.0289).
 \end{aligned}$$

From the previous two examples, we see that the final results in DS_mT framework are the same. The same conclusion holds in DST framework.

Case 3. The opinions of the two consulting companies/sources are given in Table 3 according to 1-Tuples, which are crude approximations of Tables 1 and 2.

Table 3. 1-Tuple Belief Masses

$m(\cdot)$	θ_1	θ_2	θ_3
Source No. 1	L_4	L_3	L_3
Source No. 2	L_5	L_2	L_3

After applying qualitative fusion rules, one finally gets

$$\begin{aligned}
 q_1^p m_{\text{PCR5}}(\theta_1) &= L_6, \\
 q_1^p m_{\text{PCR5}}(\theta_2) &= L_1, \\
 q_1^p m_{\text{PCR5}}(\theta_3) &= L_3.
 \end{aligned}$$

Case 4. From the point of view of quantitative fusion, let us consider two (quantitative) sources providing the numerical masses in Table 4, which are equivalent in some sense with 2-Tuples given to Tables 1 and 2.

Table 4. Quantitative Belief Masses

$m(\cdot)$	θ_1	θ_2	θ_3
Source No. 1	0.43	0.27	0.30
Source No. 2	0.50	0.21	0.29

According to DS_mC and PCR5 combination rules, we get as final result $qm_{\text{PCR5}}(\theta_1) = 0.5315 \approx 0.5$, $qm_{\text{PCR5}}(\theta_2) = 0.1974 \approx 0.2$, $qm_{\text{PCR5}}(\theta_3) = 0.2711 \approx 0.3$. Recall that from our previous results, we had $q_2^p m_{\text{PCR5}}(\theta_1) = (L_5, 0.0315) \approx L_5$, $q_2^p m_{\text{PCR5}}(\theta_2) = (L_2, -0.0026) \approx L_2$, $q_2^p m_{\text{PCR5}}(\theta_3) = (L_3, -0.0289) \approx L_3$, however, $q_1^p m_{\text{PCR5}}(\theta_1) = L_6$, $q_1^p m_{\text{PCR5}}(\theta_2) = L_1$, $q_1^p m_{\text{PCR5}}(\theta_3) = L_3$. Therefore, the 2-Tuples result is more consistent with the quantitative fusion when using 2-Tuple than when using 1-Tuple.

The advantages of the two kinds of 2-Tuples qualitative representation models in DS_mT framework are listed as follows.

(a) High precision: based on 2-Tuple qualitative enriched representation, q_2 operators on 2-Tuples provide a higher precision than q_1 operators on 1-Tuples because for every q_2 label, i.e., (L, σ^h) , or $\langle L, \sigma^p \rangle$, we always find a unique real number corresponding to it and thus one does not lose precision in the computations. This has been shown in our previous examples.

(b) Wide adaptive capacity: since q_2 labels express a continuous qualitative belief through a standard label and the remainder/proportional factor, which is equivalent to real number. Consequently, all quantitative fusion rules and belief conditioning rules^[11] can be used directly in this framework. Since it is already

proved that the quantitative PCR5 fusion rule proposed in DSmT outperforms Dempster-Shafer's rule, specially in all high conflicting situations^[1], the qualitative q_2 DSmT will naturally also outperform q_2 DST. Moreover, since DFM is more general and simpler than Wang-Hao's model for dealing with unbalanced labels, q_2^p DSmT is theoretically expected to work better (although we have not yet conducted experiments to prove this conjecture).

(c) Low complexity: since the q_2 operators are commutative and associative, while classical q_1 models depend on the order/approximation of the operations carried out, the fusion based on q_2 labels works better than the one based on classical q_1 models. In addition, q_2^p DSmT will be more efficient than q_2^h DSmT in dealing with unbalanced labels, because there is no need of a hierarchal linguistic structure to deal with the multigranular linguistic contexts. There is also no need for repeated transformation computations according to FLARL^[14].

6 Conclusion

In this paper, we have presented two kinds of 2-Tuple linguistic representation models, i.e., the extended Herrera and Martínez's model and the DFM. Their corresponding operators have been also presented in order to efficiently combine qualitative information in DSmT framework. DFM is more general and simpler than Wang-Hao's model, and DFM deals directly and easily with unbalanced labels contrariwise to the extended Herrera-Martínez model. We have also proposed a direct/natural extension of these 2-Tuple models to 2-Tuple enriched models for dealing with a possible confidence factor related to each label (when available and if necessary). The q_2 DSmC and q_2 PCR5 fusion rules have been introduced as direct extensions of their quantitative counterparts already available in DSmT framework. Some simple examples on how to apply these fusion rules have been provided for equidistant labels. For unbalanced labels, although currently we cannot give an appropriate application example to testify our models. With our recognition of uncertain information, especially for a more complex case^[15], their importance will be found gradually. This work enlarges the scope of the linguistic representation and preserves all the precision in the qualitative information processing. Our approach can be useful for the development of future information retrieval, fusion and management systems. Since DSmT with its fusion rules has been applied successfully in robotics^[16], the new q_2^p DSmT and q_2^h DSmT fusion rules are now under evaluation in our current research work related with environment perception in hybrid systems (robot with human feedback

interaction^[17]). We hope that this work will be also useful for other applications in cognitive sciences and for human-computer interface development.

References

- [1] Smarandache F, Dezert J. (eds.). Advances and Applications of DSmT for Information Fusion, Vol.1 & Vol.2, Rehoboth: American Research Press, 2004 & 2006, <http://fs.gallup.unm.edu//DSmT.htm>.
- [2] Dezert J, Smarandache F. A new probabilistic transformation of belief mass assignment. In *Proc. Fusion 2008*, Cologne, Germany, June 30–July 3, 2008.
- [3] Li X, Huang X, Dezert J, Smarandache F. Enrichment of qualitative beliefs for reasoning under uncertainty. In *Proc. Fusion 2007*, Quebec, Canada, July 9–12, 2007.
- [4] Herrera F, Martínez L. A 2-Tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Systems*, 2000, 8(6): 746–752.
- [5] Herrera F, Martínez L. The 2-Tuple linguistic computational model. Advantages of its linguistic description, accuracy and consistency. *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.*, 2001, 9(Suppl.): 33–49.
- [6] Herrera F, Martínez L. A model based on linguistic 2-Tuples for dealing with multi-granular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Trans. Systems, Man, and Cybernetics, Part B: Cybernetics*, 2001, 31(2): 227–234.
- [7] Herrera F, Herrera-Viedma E, Martínez L. A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Trans. Fuzzy Systems*, 2008, 16(2): 354–370.
- [8] Wang J H, Hao J. A new version of 2-Tuple fuzzy linguistic representation model for computing with words. *IEEE Trans. Fuzzy Systems.*, 2006, 14(3): 435–445.
- [9] Wang J H, Hao J. An approach to computing with words based on canonical characteristic values of linguistic labels. *IEEE Trans. Fuzzy Systems.*, 2007, 15(4): 593–604.
- [10] Li X, Dai X, Dezert J, Smarandache F. DSmT qualitative reasoning based on 2-Tuple linguistic representation model. In *Proc. the 9th Int. Conf. Young Computer Scientists.*, Zhangjiajie, China, Nov. 18–21, 2008, pp.1671–1676.
- [11] Shafer G. A Mathematical Theory of Evidence. Princeton, NJ: Princeton University Press, 1976.
- [12] Smarandache F, Dezert J. Information fusion based on new proportional conflict redistribution rules. In *Proc. Fusion 2005*, Philadelphia, PA, USA, July 25–28, 2005.
- [13] Smarandache F, Dezert J. Qualitative belief conditioning rules. In *Proc. Fusion 2007 Int. Conf.*, Quebec, Canada, July 9–12, 2007.
- [14] Smarandache F, Dezert J, Li X. DSm Linear Algebra of Refined Labels. Advances and Applications of DSmT for Information Fusion, Rehoboth: American Research Press, Vol.3, 2009, <http://fs.gallup.unm.edu//DSmT.htm>.
- [15] Li X, Dai X, Dezert J, Smarandache F. Fusion of imprecise qualitative information. *Applied Intelligence*, 2009, 31(2): DOI: 10.1007/s10489-009-0170-2.
- [16] Li X, Huang X, Dezert J *et al.* A successful application of DSmT in sonar grid map building and comparison with DST-based approach. *Int. J. Innovative Comp., Inf. and Control*, 2007, 3(3): 539–551.
- [17] Li S, Zhang J, Huang X *et al.* Semantic computation in a Chinese question-answering system. *Journal of Computer Science and Technology*, 2002, 17(6): 933–939.



Xin-De Li is a faculty member in School of Automation, Southeast University and is also a member of the Speciality Committee of Intelligent Robotics in Chinese Association for Artificial Intelligence and a member of IEEE (RAS, CS, CI). He received his Master's degree from Shandong University in 2003, and Ph.D. degree from Huazhong University of Science and Technology in 2007, respectively. His main research interests include information fusion, uncertainty reasoning, robot perception, computer vision, pattern recognition, robot's map building and localization and multi-robot system.

His main research interests include information fusion, uncertainty reasoning, robot perception, computer vision, pattern recognition, robot's map building and localization and multi-robot system.



Florentin Smarandache was born in Balcesti, Romania, in 1954. He obtain an M.Sc. degree in both mathematics and computer science from the University of Craiova in 1979, received a Ph.D. degree in mathematics from the State University of Kishinev in 1997, and continued postdoctoral studies at various American Universities (New Mexico State Univ. in Las Cruces, Los Alamos National Lab.) after emigration. Dr. Smarandache worked as a professor of mathematics for many years at the University of New Mexico, Gallup Campus, USA. He is the author, co-author, and editor of 75 books, over 100 scientific notes and articles, In mathematics there are several entries named Smarandache Functions, Sequences, Constants, and especially Paradoxes in international journals and encyclopedias.

Dr. Smarandache worked as a professor of mathematics for many years at the University of New Mexico, Gallup Campus, USA. He is the author, co-author, and editor of 75 books, over 100 scientific notes and articles, In mathematics there are several entries named Smarandache Functions, Sequences, Constants, and especially Paradoxes in international journals and encyclopedias.



Jean Dezert obtained his Ph.D. degree from Paris XI Orsay University, France in 1990 in automatic control and signal processing. He is a senior research scientist at the French Aerospace Research Lab (ONERA). His current research interests include autonomous navigation, estimation theory and its applications to multisensor-multitarget tracking, information fusion and plausible reasoning. Dr. Dezert is developing since 2001 with Professor Smarandache a new theory of plausible and paradoxical reasoning for information fusion (DS_mT) and has edited two textbooks (collected works) devoted to it in 2004, 2006 respectively and a third volume is in preparation in 2009. He serves as reviewer for different international journals and has been involved in the Technical Program Committees of Fusion International Conferences. Since 2001, he is a member of the board of the International Society of Information Fusion (www.isif.org). He served also as executive vice-president of ISIF in 2004 and he is associate editor of the Journal of Advances in Information Fusion.



Xian-Zhong Dai obtained his Ph.D. degree from Tsinghua University in 1986. From 1986 to 1988, he worked in Tsinghua University. Then, he has been working at School of Automation, Southeast University. From 1991 to 1993, he joined in Erlangen University for postdoc research supported by Humboldt Foundation of Germany. In 1999, he carried out a cooperation research in Cornell University. He owns 5 patents, published 50 papers and 2 books. His main interests include measurement, intelligent robot and motion control, intelligent information processing, etc.