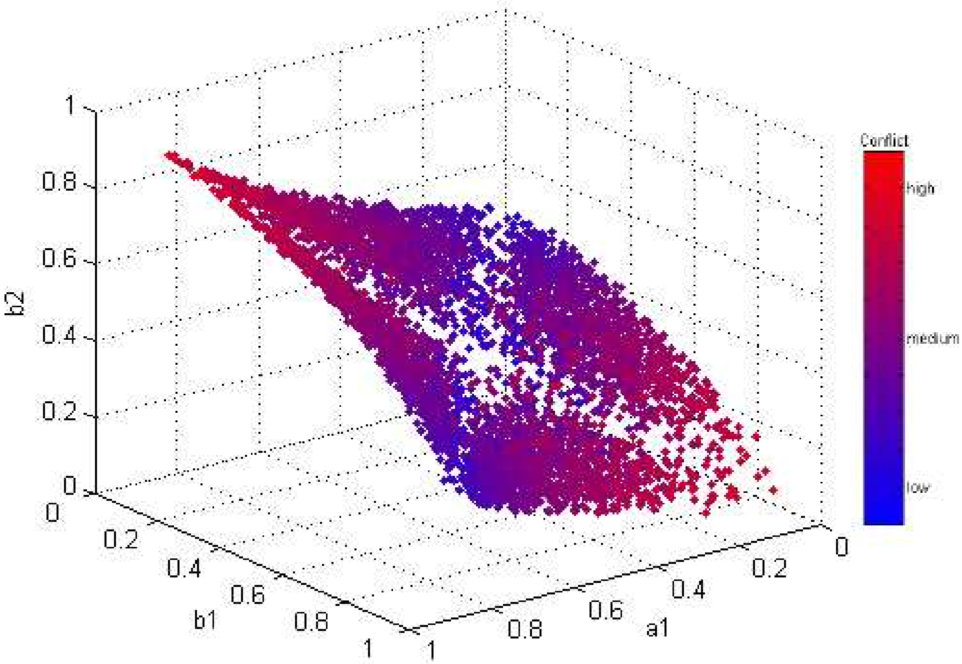
Florentin Smarandache & Jean Dezert

Editors

Advances and Applications of DSmT for Information Fusion

Collected Works

Volume 2



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About the cover: Decision space
Each point represents a pair of mass functions for which there is a difference of decision between the conjunctive rule and the PCR rule for the two sources and two classes case according. Color is the level of conflict between the two sources. This is the Monte-Carlo resolution of the system (2.26) from Chapter 2 entitled *A new generalization of the proportional conflict redistribution rule stable in terms of decision.* This figure was provided by Arnaud Martin and Christophe Osswald, ENSIETA, Brest, France, and generated with Python and Matlab$^{\text{TM}}$.

This book is dedicated to the memory
of Professor Philippe Smets,
our missing friend and colleague.

Preamble

This second book devoted on advances and applications of Dezert-Smarandache Theory (DSmT) for information fusion collects recent papers from different researchers working in engineering and mathematics. Part 1 of this book presents the current state-of-the-art on theoretical investigations while, Part 2 presents several applications of this new theory. Some ideas in this book are still under current development or improvements, but we think it is important to propose them in order to share ideas and motivate new debates with people interested in new reasoning methods and information fusion. So, we hope that this second volume on DSmT will continue to stir up some interests to researchers and engineers working in data fusion and in artificial intelligence.

This second volume brings several theoretical advances and applications which some of them have not been published until now, or only partially published and presented since summer 2004 in some past international conferences, journals or in some workshops and seminars. Through this volume, the readers will discover a new family of Proportional Conflict Redistribution (PCR) rules for efficient combination of uncertain, imprecise and highly conflicting sources of information; new investigations on continuous belief functions; investigations on new fusion rules based on T-norms/T-conorms or N-norms/N-conorms (hence using fuzzy/neutrosophy logic in information fusion); an extension of DSmT for dealing with qualitative information expressed directly with linguistic labels; some proposals for new belief conditioning rules (BCR), and more. Also, applications of DSmT are showing up to multitarget tracking in clutter based on generalized data association, or target type tracking, to robot's map reconstruction, sonar imagery and radar target classification.

We want to thank all people who have invited us, or our colleagues, to give lectures on DSmT in workshops and seminars during the last two years at NIA/NASA Hampton, VA, USA (Nov. 2004), Czech Society for Cybernetics and Informatics, Praha (Dec. 2004), University Kolkata, India (Dec. 2004), NATO Advanced Study Institute, Albena, Bulgaria (May 2005), NATO Advanced Research Workshop, Tallinn, Estonia (June 2005), Marcus Evans Workshop, Barcelona, Spain (Nov. 2005), ENSIETA, Brest, France (Dec. 2005), Information Days on Advanced Computing, Velingrad, Bulgaria (May 2006), University Sekolah Tinggi Informatika & Komputer Indonesia, Malang, Indonesia, (May 2006), University Kristen Satya Wacana, Salatiga, Indonesia (May 2006) and at the Round panel Discussion on Prevision Methods, 38ièmes Journées de Statistique, EDF Recherche et Développement (ICAME/SOAD), Clamart, France (Mai 2006).

We want to thank Dr. Frédéric Dambreville, Dr. Milan Daniel, Mr. Pascal Djiknavorian, Prof. Dominic Grenier, Prof. Xinhan Huang, Dr. Pavlina Konstantinova, Mr. Xinde Li, Dr. Arnaud Martin, Dr. Christophe Osswald, Dr. Andrew Schumann, Prof. Tzvetan Semerdjiev, Dr. Albena Tchamova and Prof. Min Wang for their contributions to this second volume, and for their interests and support of these new ideas. We are grateful to our colleagues for encouraging us to edit this second book and for sharing with us many ideas and questions on DSmT since the publication of the first volume in June 2004. We specially thank Albena Tchamova for her devotion in helping us in the preparation of this book and Arnaud Martin and Christophe Osswald for kindly providing us an interesting image for the front cover of the volume. We also thank all colleagues and reviewers of our papers who have manifested their interests in our works and have brought either positive or negative comments and, in all cases, interesting, passionate and exciting discussions. Without feedbacks from them, new ideas would have probably emerged more slowly. So, more than ever, we encourage you, if you are interested in Information Fusion and by DSmT to share your comments, criticisms, notes and articles with us for maybe a next volume . . .

We are very grateful to Doctor Éloi Bossé and Professor Bassel Solaiman for accepting to peer-review this second volume and writing a preface for it. We want also to thank Professor Pierre Valin for his deep review of this book and all his valuable comments which were very helpful for improvement of this volume.

Jean Dezert is grateful to Department of Information Modelling and Processing (DTIM) at the Office National d'Études et de Recherches Aérospatiales (ONERA), Châtillon, France for encouraging him to carry on this research and for its financial support. Florentin Smarandache is grateful to The University of New Mexico that many times partially sponsored him to attend international conferences, workshops and seminars on Information Fusion and to the University Sekolah Tinggi Informatika & Komputer Indonesia - Malang, and the University Kristen Satya Wacana - Salatiga, both from Indonesia, that invited him to present the DSmT in May 2006.

We want to thank everyone.

The Editors

Prefaces

Data and information fusion clearly is a key enabler in the provision of decision quality information to the decision maker. The essence of decision-making in civilian, military and public security operations is people making timely decisions in the face of uncertainty, and acting on them. This process has been immeasurably complicated by the overwhelming and increasing volume of raw data and information available in the current age. Knowledge, belief and uncertainty are three key notions of the data/information fusion process. Belief and knowledge representation is a crucial step needed to transform data into knowledge that I believe is the ultimate goal of information fusion. The data/information coming from the different sources must be converted into a certain language or with other means (e.g. visualization) so as they can be processed and used by the human to build his mental model in order to decide and act. To this end, formalization is necessary to be able to deal with knowledge or uncertainty: a formal framework in which knowledge, information and uncertainty can be represented, combined and managed. An ideal framework would be one mixing quantified evaluations of uncertainty and high reasoning capabilities.

It is a great pleasure to welcome this second volume on 'Advances and Applications of DSmT for Information Fusion'. As already mentioned in Volume 1, The Dezert-Smarandache Theory (DSmT) is considered as an extension of the Dempster-Shafer (DS) as well as the Bayesian theories to formalize the fusion process for efficient combination of uncertain, imprecise and highly conflicting sources of information. This second volume brings in depth presentation of several theoretical advances and applications of that theory. In particular, the combination rules have been treated in a way that we can consider to be almost exhaustive. The book also presents very interesting applications of DSmT to multitarget tracking and classification, robotics and sonar imagery. The quantitative approaches have been addressed quite extensively in this volume and we must congratulate the authors to have brought contributions addressing the qualitative information sources. Even though the book did not provide that ideal framework mixing quantified evaluations of uncertainty and high reasoning capabilities, the contributions are significant and will certainly motivate researchers and engineers working in data/information fusion to be more innovative and creative.

I specifically thank Florentin Smarandache and Jean Dezert for having taken the responsibility to edit that book and the authors for their original contribution in bringing more light on this promising approach.

Éloi Bossé, Ph.D.
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With the continuous technologies development, we assist to an explosion of information sources. It is not one or two sensors, which are available but sometimes more than a hundred. The sensors multiplicity makes the decision-making process more complex. Thus, it is very difficult to find the "credible" information in such information mass.

In 2004, F. Smarandache and J. Dezert have published volume 1 of "Advances and Applications of DSmT for information Fusion". The so active DSmT community pursues its own development and few years after, it is so great to produce the second volume with two complementary and interesting issues that readers will certainly have pleasure to read.

In the first part, the authors present the current state of the art related to the Dezert-Smarandache Theory (DSmT) for information fusion. In this "theoretical" part, we discover a set of new topics and new extensions. This certainly gives several good tools for engineering applications.

The second part is perhaps the most exciting from a practical point of view. First, four concrete applications show that DSmT in association with proportional conflict redistribution rules are very efficient. In real application, the real time response is necessary, a solution of this problem is presented in the chapter untitled "Reducing DSmT hybrid rule complexity through optimization of the calculation algorithm" optimization and complexity reducing.

In the first application, the uncertainty plays a major role. The classification of underwater sediment using a sonar image and human experts decision or in the case of a target recognition using virtual experts, is detailed. The main problem is to make a decision when two or more experts give contradictory information? In this case the association of the DSmT with combination rules is clearly shown to be efficient.

The second and third applications illustrate the problem of targets tracking or recognition in real situations. These two applications pursue a previous work and the efficiency of the DSmT in association with PCR in a complex system is detailed.

Robot exploration in an unknown environment is a difficult task. This application uses several sensors (16 simulated sonar detectors, location of robot, velocity...) and a redistribution of the conflict mass to build the grid map. Several methods are tested in order to show the advantages of the association between DSmT and PRC5.

The problem of optimizing and algorithmic complexity reducing is very useful when a real time decision is concerned. This illustrates the constant growing of the DSmT community. I would like to thank the authors for their original contributions and to encourage the development of this fascinating approach.

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Contents

Part I

Advances on DSmT

Chapter 1

Proportional Conflict Redistribution Rules for Information Fusion

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Abstract: *In this chapter we propose five versions of a Proportional Conflict Redistribution rule (PCR) for information fusion together with several examples. From PCR1 to PCR2, PCR3, PCR4, PCR5 one increases the complexity of the rules and also the exactitude of the redistribution of conflicting masses. PCR1 restricted from the hyper-power set to the power set and without degenerate cases gives the same result as the Weighted Average Operator (WAO) proposed recently by Jøsang, Daniel and Vannoorenberghe but does not satisfy the neutrality property of vacuous belief assignment (VBA). That's why improved PCR rules are proposed in this chapter. PCR4 is an improvement of minC and Dempster's rules. The PCR rules redistribute the conflicting mass, after the conjunctive rule has been applied, proportionally with some functions depending on the masses assigned to their corresponding columns in the mass matrix. There are infinitely many ways these functions (weighting factors) can be chosen depending on the complexity one wants to deal with in specific applications and fusion systems. Any fusion combination rule is at some degree ad-hoc.*

1.1 Introduction

This chapter presents a new set of alternative combination rules based on different proportional conflict redistributions (PCR) which can be applied in the framework of the two principal theories dealing the combination of belief functions. We remind briefly the basic ideas of these two theories:

- The first and the oldest one is the Dempster-Shafer Theory (DST) developed by Shafer in 1976 in [17]. In DST framework, Glenn Shafer starts with a so-called *frame of discernment* $\Theta = \{\theta_1, \dots, \theta_n\}$ consisting in a finite set of exclusive and exhaustive hypotheses. This is Shafer's model. Then, a basic belief assignment (bba) $m(.)$ is defined as the mapping $m : 2^\Theta \to [0, 1]$ with:

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in 2^\Theta} m(X) = 1 \tag{1.1}$$

The combination of belief assignments provided by several sources of evidence is done with Dempster's rule of combination.

- The second and the most recent theory is the Dezert-Smarandache Theory (DSmT) developed by the authors since 2001 [18]. In the DSmT framework, one starts with a frame $\Theta = \{\theta_1, \ldots, \theta_n\}$ consisting only in a finite set of exhaustive[1] hypotheses. This is the so-called *free DSm model.* The exclusivity assumption between elements (i.e. requirement for a refinement) of Θ is not necessary within DSmT. However, in DSmT any integrity constraints between elements of Θ can also be introduced, if necessary, depending on the fusion problem under consideration. A free DSm model including some integrity constraints is called a *hybrid DSm model.* DSmT can deal also with Shafer's model as well which appears actually only as a specific hybrid DSm model. The DSmT framework is much larger that the DST one since it offers the possibility to deal with any model and any intrinsic nature of elements of Θ including continuous/vague concepts having subjective/relative interpretation which cannot be refined precisely into finer exclusive subsets. In DSmT, a generalized basic belief assignment (gbba) $m(.)$ is defined as the mapping $m : D^\Theta \to [0, 1]$ with

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in D^\Theta} m(X) = 1 \tag{1.2}$$

D^Θ represents the hyper-power set of Θ (i.e. Dedekind's lattice). Since the power set 2^Θ is closed under $\cup$ operator, while the hyper-power set D^Θ is closed under both $\cup$ and $\cap$ operators, $| D^\Theta | > | 2^\Theta |$. A detailed presentation of DSmT with many examples and comparisons between rules of combination can be found in [18].

Among all possible bba's or gbba's, the belief vacuous belief assignment (VBA), denoted $m_v(.)$ and defined by $m_v(\Theta) = 1$ which characterizes a full ignorant source, plays a particular and important role for the construction of a satisfying combination rule. Indeed, the major properties that a good rule of combination must satisfy, upon to authors' opinion, are :

1. the coherence of the combination result in all possible cases (i.e. for any number of sources, any values of bba's or gbba's and for any types of frames and models which can change or stay invariant over time).

2. the commutativity of the rule of combination

3. the neutral impact of the VBA into the fusion.

The requirement for conditions 1 and 2 is legitimate since we are obviously looking for best performances (we don't want a rule yielding to counter-intuitive or wrong solutions) and we don't want that the result depends on the arbitrary order the sources are combined. The neutral impact of VBA to be satisfied by a fusion rule (condition 3), denoted by the generic $\oplus$ operator is very important too. This condition states that the combination of a full ignorant source

[1] The exhaustivity assumption is not restrictive since one always can close any non-exhaustive set by introducing a closure element, say θ_0, representing all missing unknown hypotheses.

with a set of $s \geq 1$ non-totally ignorant sources doesn't change the result of the combination of the s sources because the full ignorant source doesn't bring any new specific evidence on any problems under consideration. This condition is thus perfectly reasonable and legitimate. The condition 3 is mathematically represented as follows: for all possible $s \geq 1$ non-totally ignorant sources and for any $X \in 2^{\Theta}$ (or for any $X \in D^{\Theta}$ when working in the DSmT framework), the fusion operator $\oplus$ must satisfy

$$[m_1 \oplus \ldots \oplus m_s \oplus m_v](X) = [m_1 \oplus \ldots \oplus m_s](X) \tag{1.3}$$

The associativity property, while very attractive and generally useful for sequential implementation is not actually a crucial property that a combination rule must satisfy if one looks for the best coherence of the result. The search for an optimal solution requires to process all bba's or gbba's altogether. Naturally, if several different rules of combination satisfy conditions 1-3 and provide similar performances, the simplest rule endowing associativity will be preferentially chosen (from engineering point of view). Up to now and unfortunately, no combination rule available in literature satisfy incontrovertibly the three first primordial conditions. Only three fusion rules based on the conjunctive operator are known associative: Dempster's rule in DST, Smets' rule (conjunctive consensus based on the open-world assumption), and the DSm classic rule on free DSm model. The disjunctive rule is associative and satisfy properties 1 and 2 only. All alternative rules developed in literature until now don't endow properties 1-3 and the associativity property. Although, some rules such as Yager's, Dubois & Prade's, DSm hybrid, WAO, minC, PCR rules, which are not associative become quasi-associative if one stores the result of the conjunctive rule at each time when a new bba arises in the combination process (see section 1.14 for details).

This chapter extends a previous paper on Proportional Conflict Redistribution Rule no 1 (PCR1) detailed in [20, 21] in order to overcome its inherent limitation (i.e. the neutral impact of VBA - condition 3 - is not fulfilled by PCR1). In the DSm hybrid rule of combination [18], the transfer of partial conflicts (taking into account all integrity constraints of the model) is done directly onto the most specific sets including the partial conflicts but without proportional redistribution. In this chapter, we propose to improve this rule by introducing a more effective proportional conflict redistribution to get a more efficient and precise rule of combination PCR5.

The main steps in applying all the PCR rules of combination (i.e. fusion) are as follows:

- Step 1: use the conjunctive rule,
- Step 2: compute the conflicting masses (partial and/or total),
- Step 3: redistribute the conflicting masses to non-empty sets.

The way the redistribution is done makes the distinction between all existing rules available in literature in the DST and DSmT frameworks (to the knowledge of the authors) and the PCR rules, and also the distinction among the different PCR versions themselves. One also studies the impact of the vacuous belief assignment (VBA) on PCR rules and one makes a short discussion on the degree of the fusion rules' ad-hoc-ity.

Before presenting the PCR rules, and after a brief reminder on the notion of total and partial conflicts, we browse the main rules of combination proposed in the literature in the frameworks of DST and DSmT in the next section. More rules of combination are presented in Chapter 8. Then we present the general Weighted Operator (WO), the Weighted Average Operator (WAO) and the minC operator. MinC is historically the first sophisticated rule using the idea of proportional conflict redistribution. The last part of this chapter is devoted to the development of a new family of PCR rules. Several examples and comparisons with other rules are also provided.

1.2 The principal rules of combination

In the sequel, we assume non degenerate void[2] problems and thus we always consider the frame Θ as a truly non empty finite set (i.e. $\Theta \neq \{\emptyset\}$), unless specified expressly.

1.2.1 Notion of total and partial conflicting masses

The *total conflicting mass* drawn from two sources, denoted k_{12}, is defined as follows:

$$k_{12} = \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2) \tag{1.4}$$

The total conflicting mass is nothing but the sum of *partial conflicting masses*, i.e.

$$k_{12} = \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset}} m(X_1 \cap X_2) \tag{1.5}$$

Here, $m(X_1 \cap X_2)$, where $X_1 \cap X_2 = \emptyset$, represents a partial conflict, i.e. the conflict between the sets X_1 and X_2. Formulas (1.4) and (1.5) can be directly generalized for $s \geq 2$ sources as follows:

$$k_{12\ldots s} = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta \\ X_1 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_i(X_i) \tag{1.6}$$

$$k_{12\ldots s} = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta \\ X_1 \cap \ldots \cap X_s = \emptyset}} m(X_1 \cap X_2 \cap \ldots \cap X_s) \tag{1.7}$$

1.2.2 The conjunctive rule

1.2.2.1 Definition

For $n \geq 2$, let's $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of the fusion problem under consideration. In the case when these n elementary hypotheses $\theta_1, \theta_2, \ldots, \theta_n$ are known to be truly *exhaustive and exclusive* (i.e. Shafer's model holds), one can use the DST [17] framework with Dempster's rule,

[2]The degenerate void problem considers $\Theta = \{\emptyset\}$ which is actually a meaningless fusion problem in static fusion applications since the frame contains no hypothesis on which we can work with. In dynamic fusion application, a non degenerate void problem can sometimes turn into a degenerate void problem at a given time depending of the evolution of integrity constraints and thus the dynamic fusion problem can vanish with time. To overcome such possibility (if required by the fusion system designer), it is more cautious to always introduce at least one closure - possibly unknown - element $\theta_0 \neq \emptyset$ in Θ.

Yager's rule [29, 30], the TBM [25, 26] approach, Dubois-Prade approach [6–8] or the DSmT framework as well using the general DSm hybrid rule of combination [18] adapted to deal with any DSm model (including Shafer's model). When the hypotheses (or some of them) are not exclusive and have potentially vague boundaries, the DSmT [18] is adopted. If hypotheses are known to be *non-exhaustive*, one can either use Smets' open-world approach [25, 26] or apply the hedging closure procedure [28] and work back with DST or DSmT.

The conjunctive rule (known also as conjunctive consensus) for $s \geq 2$ sources can be applied both in DST and in DSmT frameworks. In the DST framework, it is defined $\forall X \in 2^\Theta$ by

$$m_\cap(X) = \sum_{\substack{X_1,\ldots,X_s \in 2^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \tag{1.8}$$

$m_\cap(.)$ is not a proper belief assignment satisfying Shafer's definition (1.1), since in most of cases the sources do not totally agree (there exists partial and/or total conflicts between sources of evidence), so that $m_\cap(\emptyset) > 0$. In Smets' open-world approach and TBM, one allows $m_\cap(\emptyset) \geq 0$ and the empty set is then interpreted not uniquely as the classical empty set (i.e. the set having no element) but also as the set containing all missing hypotheses of the original frame Θ to which all the conflicting mass is committed.

In the DSmT framework, the formula is similar, but instead of the power set 2^Θ, one uses the hyper-power set D^Θ and the generalized basic belief assignments, i.e. $\forall X \in D^\Theta$

$$m_\cap(X) = \sum_{\substack{X_1,\ldots,X_s \in D^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \tag{1.9}$$

$m_\cap(.)$ remains, in the DSmT framework based on the free DSm model, a proper generalized belief assignment as defined in (1.2). Formula (1.9) allowing the use of intersection of sets (for the non-exclusive hypotheses) is called the *DSm classic rule*.

1.2.2.2 Example

Let's consider $\Theta = \{\theta_1, \theta_2\}$ and two sources with belief assignments

$$m_1(\theta_1) = 0.1 \quad m_1(\theta_2) = 0.2 \quad m_1(\theta_1 \cup \theta_2) = 0.7$$

$$m_2(\theta_1) = 0.4 \quad m_2(\theta_2) = 0.3 \quad m_2(\theta_1 \cup \theta_2) = 0.3$$

In the DST framework based on Shafer's model, one gets

$$m_\cap(\emptyset) = 0.11 \quad m_\cap(\theta_1) = 0.35$$

$$m_\cap(\theta_2) = 0.33 \quad m_\cap(\theta_1 \cup \theta_2) = 0.21$$

In the DSmT framework based on the free DSm model, one gets

$$m_\cap(\emptyset) = 0 \quad m_\cap(\theta_1 \cap \theta_2) = 0.11$$

$$m_\cap(\theta_1) = 0.35 \quad m_\cap(\theta_2) = 0.33 \quad m_\cap(\theta_1 \cup \theta_2) = 0.21$$

We can easily verify that the condition 3 (neutral impact of VBA) is satisfied with the conjunctive operator in both cases and that the commutativity and associativity are also preserved. The main drawback of this operator is that it doesn't generate a proper belief assignment in both DST and DSmT frameworks when integrity constraints are introduced in the model as in dynamic fusion problems where the frame and/or the model itself can change with time.

1.2.3 The disjunctive rule

The disjunctive rule of combination [6, 7, 24] is a commutative and associative rule proposed by Dubois & Prade in 1986 and denoted here by the index $\cup$. $m_\cup(.)$ is defined $\forall X \in 2^\Theta$ by $m_\cup(\emptyset) = 0$ and $\forall (X \neq \emptyset) \in 2^\Theta$ by

$$m_\cup(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cup X_2 = X}} m_1(X_1)m_2(X_2)$$

The core of the belief function (i.e. the set of focal elements having a positive mass) given by $m_\cup$ equals the union of the cores of m_1 and m_2. This rule reflects the disjunctive consensus and is usually preferred when one knows that one of the sources (some of the sources in the case of s sources) could be mistaken but without knowing which one. The disjunctive rule can also be defined similarly in DSmT framework by replacing 2^Θ by D^Θ in the previous definition.

1.2.4 Dempster's rule of combination

Dempster's rule of combination is the most widely used rule of combination so far in many expert systems based on belief functions since historically it was proposed in the seminal book of Shafer in [17]. This rule, although presenting interesting advantages (mainly the commutativity, associativity and the neutral impact of VBA) fails however to provide coherent results due to the normalization procedure it involves. Some proponents of Dempster's rule claim that this rule provides correct and coherent result, but actually under strictly satisfied probabilistic conditions, which are rarely satisfied in common real applications. Discussions on the justification of Dempster's rule and its well-known limitations can be found by example in [18, 27, 31–33]. Let's a frame of discernment Θ based on Shafer's model and two independent and equally reliable belief assignments $m_1(.)$ and $m_2(.)$. Dempster's rule of combination of $m_1(.)$ and $m_2(.)$ is obtained as follows: $m_{DS}(\emptyset) = 0$ and $\forall (X \neq \emptyset) \in 2^\Theta$ by

$$m_{DS}(X) = \frac{\displaystyle\sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)}{1 - \displaystyle\sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2)} = \frac{1}{1-k_{12}} \cdot \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) \tag{1.10}$$

where the *degree of conflict* k_{12} is defined by $k_{12} \triangleq \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2)$.

$m_{DS}(.)$ is a proper basic belief assignment if and only if the denominator in equation (1.10) is non-zero, i.e. the *degree of conflict* k_{12} is less than one.

1.2.5 Smets' rule of combination

Smets' rule of combination [25, 26] is nothing but the non-normalized version of the conjunctive consensus (equivalent to the non-normalized version of Dempster's rule). It is commutative and associative and allows positive mass on the null/empty set $\emptyset$ (i.e. open-world assumption). Smets' rule of combination of two independent (equally reliable) sources of evidence (denoted here by index S) is given by:

$$m_S(\emptyset) \equiv k_{12} = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2)$$

and $\forall (X \neq \emptyset) \in 2^\Theta$, by

$$m_S(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)$$

1.2.6 Yager's rule of combination

Yager's rule of combination [28–30] admits that in case of conflict the result is not reliable, so that k_{12} plays the role of an absolute discounting term added to the weight of ignorance. This commutative but not associative rule, denoted here by index Y is given[3] by $m_Y(\emptyset) = 0$ and $\forall X \in 2^\Theta, X \neq \emptyset, X \neq \Theta$ by

$$m_Y(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)$$

and when $X = \Theta$ by

$$m_Y(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2)$$

1.2.7 Dubois & Prade's rule of combination

Dubois & Prade's rule of combination [7] admits that the two sources are reliable when they are not in conflict, but one of them is right when a conflict occurs. Then if one observes a value in set X_1 while the other observes this value in a set X_2, the truth lies in $X_1 \cap X_2$ as long $X_1 \cap X_2 \neq \emptyset$. If $X_1 \cap X_2 = \emptyset$, then the truth lies in $X_1 \cup X_2$ [7]. According to this principle, the commutative (but not associative) Dubois & Prade hybrid rule of combination, denoted here by index DP, which is a reasonable trade-off between precision and reliability, is defined by $m_{DP}(\emptyset) = 0$ and $\forall X \in 2^\Theta, X \neq \emptyset$ by

$$m_{DP}(X) = \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X \\ X_1 \cap X_2 \neq \emptyset}} m_1(X_1)m_2(X_2) + \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cup X_2 = X \\ X_1 \cap X_2 = \emptyset}} m_1(X_1)m_2(X_2) \tag{1.11}$$

[3] Θ represents here the full ignorance $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ on the frame of discernment according the notation used in [17].

1.2.8 The hybrid DSm rule

The hybrid DSm rule of combination is the first general rule of combination developed in the DSmT framework [18] which can work on any DSm models (including Shafer's model) and for any level of conflicting information. The hybrid DSm rule can deal with the potential dynamicity of the frame and its model as well. The DSmT deals properly with the granularity of information and intrinsic vague/fuzzy nature of elements of the frame Θ to manipulate. The basic idea of DSmT is to define belief assignments on hyper-power set D^Θ (i.e. free Dedekind's lattice) and to integrate all integrity constraints (exclusivity and/or non-existential constraints) of the model, say $\mathcal{M}(\Theta)$, fitting with the problem into the rule of combination. Mathematically, the hybrid DSm rule of combination of $s \geq 2$ independent sources of evidence is defined as follows (see chap. 4 in [18]) for all $X \in D^\Theta$,

$$m_{\mathcal{M}(\Theta)}(X) \triangleq \phi(X)\Big[S_1(X) + S_2(X) + S_3(X)\Big] \tag{1.12}$$

where <u>all sets involved in formulas are in canonical form</u>[4], and where $\phi(X)$ is the *characteristic non-emptiness function* of a set X, i.e. $\phi(X) = 1$ if $X \notin \boldsymbol{\emptyset}$ and $\phi(X) = 0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\boldsymbol{\emptyset}_{\mathcal{M}}, \emptyset\}$. $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(X)$, $S_2(X)$ and $S_3(X)$ are defined by

$$S_1(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s \in D^\Theta \\ X_1\cap X_2\cap\ldots\cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \tag{1.13}$$

$$S_2(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s \in \boldsymbol{\emptyset} \\ [\mathcal{U}=X]\vee[(\mathcal{U}\in\boldsymbol{\emptyset})\wedge(X=I_t)]}} \prod_{i=1}^{s} m_i(X_i) \tag{1.14}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s \in D^\Theta \\ X_1\cup X_2\cup\ldots\cup X_s = A \\ X_1\cap X_2\cap\ldots\cap X_s \in \boldsymbol{\emptyset}}} \prod_{i=1}^{s} m_i(X_i) \tag{1.15}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_s)$ where $u(X)$ is the union of all θ_i that compose X and $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(A)$ corresponds to the classic DSm rule for k independent sources based on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the (canonical) disjunctive form of non-empty sets[5]. The hybrid DSm rule generalizes the classic DSm rule of combination and is not equivalent to Dempster's rule. It works for any DSm models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic

[4]The canonical form of a set is its easiest (or standard) form. We herein use the disjunctive normal form (which is a disjunction of conjunctions). In Boolean logic (and equivalently in the classical set theory) every statement of sentential calculus can be reduced to its disjunctive normal form. Of course the canonical form depends on the model.

[5]We have voluntarily removed the *canonicity function* $c(.)$ in expression of $S_3(.)$ with respect to some formulas in earlier publications because such notation appears actually totally useless since all sets involved in formulas must be expressed in canonical form.

belief functions. Extension of this hybrid DSm rule for the fusion of imprecise belief can be found in [18].

In the case of a dynamic fusion problem, when all elements become empty because one gets new evidence on integrity constraints (which corresponds to a specific hybrid model $\mathcal{M}$), then the conflicting mass is transferred to the total ignorance, which also turns to be empty, therefore the empty set gets now mass equals one which shows that the problem has no solution at all (actually the problem is a degenerate void problem since all elements became empty at a given time). If we prefer to adopt an optimistic vision, we can consider that one (or more missing hypotheses), say θ_0, has entered in the frame but we did pay attention to it in the dynamicity and thus, one must expressly consider $m(\theta_0) = 1$ instead of $m(\emptyset) = 1$. For example, Let's consider the frame $\Theta = \{A, B\}$ with the 2 following bba's $m_1(A) = 0.5$, $m_1(B) = 0.3$, $m_1(A \cup B) = 0.2$ and $m_2(A) = 0.4$, $m_2(B) = 0.5$, $m_2(A \cup B) = 0.1$, but one finds out with new evidence that A and B are truly empty, then $A \cup B \equiv \Theta \overset{\mathcal{M}}{\equiv} \emptyset$. Then $m(\emptyset) = 1$ which means that this is a totally impossible problem because this degenerate problem turns out to be void. The only escape is to include a third or more missing hypotheses C, D, etc into the frame to warranty its true closure.

The hybrid DSm rule of combination is not equivalent to Dempster's rule even working on Shafer's model. DSmT is an extension of DST in the way that the hyper-power set is an extension of the power set; hyper-power set includes, besides, unions, also intersections of elements; and when all intersections are empty, the hyper-power set coincides with the power set. Consequently, the DSm hybrid models include Shafer's model. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is possible and is presented in [18]. The hybrid DSm rule can be seen as an improved version of Dubois & Prade's rule which mix the conjunctive and disjunctive consensus applied in the DSmT framework to take into account the possibility for any dynamical integrity constraint in the model.

1.3 The general weighted operator (WO)

In the framework of Dempster-Shafer Theory (DST), a unified formula has been proposed recently by Lefèvre, Colot and Vanoorenberghe in [12] to embed all the existing (and potentially forthcoming) combination rules involving conjunctive consensus in the same general mechanism of construction. It turns out that such unification formula had been already proposed by Inagaki [10] in 1991 as reported in [16]. This formulation is known as *the Weighted Operator* (WO) in literature [11]. The WO for 2 sources is based on two steps.

- **Step 1**: Computation of the total conflicting mass based on the conjunctive consensus

$$k_{12} \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{1.16}$$

- **Step 2**: This second step consists in the reallocation (convex combination) of the conflicting masses on $(X \neq \emptyset) \subseteq \Theta$ with some given coefficients $w_m(X) \in [0, 1]$ such that $\sum_{X \subseteq \Theta} w_m(X) = 1$ according to

$$m(\emptyset) = w_m(\emptyset) \cdot k_{12}$$

and $\forall(X \neq \emptyset) \in 2^\Theta$

$$m(X) = [\sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)] + w_m(X)k_{12} \tag{1.17}$$

The WO can be easily generalized for the combination of $s \geq 2$ independent and equally reliable sources of information as well by substituting k_{12} in step 1 by

$$k_{12\ldots s} \triangleq \sum_{\substack{X_1,\ldots,X_s \in 2^\Theta \\ X_1 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_i(X_i)$$

and for step 2 by deriving for all $(X \neq \emptyset) \in 2^\Theta$ the mass $m(X)$ by

$$m(X) = [\sum_{\substack{X_1,\ldots,X_s \in 2^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i)] + w_m(X)k_{12\ldots s}$$

The particular choice of coefficients $w_m(.)$ provides a particular rule of combination (Dempster's, Yager's, Smets', Dubois & Prade's rules, by example, are particular cases of WO [12]). Actually this nice and important general formulation shows there exists an infinite number of possible rules of combination. Some rules are more justified or criticized with respect to the other ones mainly on their ability to, or not to, preserve the commutativity, associativity of the combination, to maintain the neutral impact of VBA and to provide what we feel coherent/acceptable solutions in high conflicting situations. It can be easily shown in [12] that such general procedure provides all existing rules involving conjunctive consensus developed in the literature based on Shafer's model.

1.4 The weighted average operator (WAO)

1.4.1 Definition

This operator has been recently proposed (only in the framework of Dempster-Shafer theory) by Jøsang, Daniel and Vannoorenberghe in [11] only for static fusion case. It is a new particular case of WO where the weighting coefficients $w_m(A)$ are chosen as follows: $w_m(\emptyset) = 0$ and $\forall X \in 2^\Theta \setminus \{\emptyset\}$,

$$w_m(X) = \frac{1}{s}\sum_{i=1}^{s} m_i(X) \tag{1.18}$$

where s is the number of independent sources to combine.

From the general expression of WO and this particular choice of weighting coefficients $w_m(X)$, one gets, for the combination of $s \geq 2$ independent sources and $\forall(X \neq \emptyset) \in 2^\Theta$

$$m_{WAO}(X) = [\sum_{\substack{X_1,\ldots,X_s \in 2^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i)] + [\frac{1}{s}\sum_{i=1}^{s} m_i(X)] \cdot [\sum_{\substack{X_1,\ldots,X_s \in 2^\Theta \\ X_1 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_i(X_i)] \tag{1.19}$$

1.4.2 Example for WAO

Let's consider Shafer's model (exhaustivity and exclusivity of hypotheses) on $\Theta = \{A, B\}$ and the two following bba's

$$m_1(A) = 0.3 \quad m_1(B) = 0.4 \quad m_1(A \cup B) = 0.3$$

$$m_2(A) = 0.5 \quad m_2(B) = 0.1 \quad m_2(A \cup B) = 0.4$$

The conjunctive consensus yields[6]

$$m_{12}(A) = 0.42 \quad m_{12}(B) = 0.23 \quad m_{12}(A \cup B) = 0.12$$

with the conflicting mass $k_{12} = 0.23$. The weighting average coefficients are given by

$$w_m(A) = 0.40 \quad w_m(B) = 0.25 \quad w_m(A \cup B) = 0.35$$

The result of the WAO is therefore given by

$$\begin{aligned}
m_{WAO|12}(A) &= m_{12}(A) + w_m(A) \cdot k_{12} = 0.42 + 0.40 \cdot 0.23 = 0.5120 \\
m_{WAO|12}(B) &= m_{12}(B) + w_m(B) \cdot k_{12} = 0.23 + 0.25 \cdot 0.23 = 0.2875 \\
m_{WAO|12}(A \cup B) &= m_{12}(A \cup B) + w_m(A \cup B) \cdot k_{12} = 0.12 + 0.35 \cdot 0.23 = 0.2005
\end{aligned}$$

1.4.3 Limitations of WAO

From the previous simple example, one can easily verify that the WAO doesn't preserve the neutral impact of VBA (condition expressed in (1.3)). Indeed, if one combines the two first sources with a third (but totally ignorant) source represented by the vacuous belief assignment (i.e. $m_3(.) = m_v(.)$), $m_3(A \cup B) = 1$ altogether, one gets same values from conjunctive consensus and conflicting mass, i.e. $k_{123} = 0.23$ and

$$m_{123}(A) = 0.42 \quad m_{123}(B) = 0.23 \quad m_{123}(A \cup B) = 0.12$$

but the weighting average coefficients are now given by

$$w_m(A) = 0.8/3 \quad w_m(B) = 0.5/3 \quad w_m(A \cup B) = 1.7/3$$

so that

$$m_{WAO|123}(A) = 0.42 + (0.8/3) \cdot 0.23 \approx 0.481333$$

$$m_{WAO|123}(B) = 0.23 + (0.5/3) \cdot 0.23 \approx 0.268333$$

$$m_{WAO|123}(A \cup B) = 0.12 + (1.7/3) \cdot 0.23 \approx 0.250334$$

Consequently, WAO doesn't preserve the neutral impact of VBA since one has found at least one example in which condition (1.3) is not satisfied because

$$m_{WAO|123}(A) \neq m_{WAO|12}(A)$$

[6] We use m_{12} instead of $m_\cap$ to indicate explicitly that only 2 sources enter in the conjunctive operator. The notation $m_{WAO|12}$ denotes the result of the WAO combination for sources 1 and 2. When $s \geq 2$ sources are combined, we use similarly the notations $m_{12...s}$ and $m_{WAO|12...s}$.

$$m_{WAO|123}(B) \neq m_{WAO|12}(B)$$

$$m_{WAO|123}(A \cup B) \neq m_{WAO|12}(A \cup B)$$

Another limitation of WAO concerns its impossibility to deal with dynamical evolution of the frame (i.e. when some evidence arises after a while on the true vacuity of elements of power set). As example, let's consider three different suspects A, B and C in a criminal investigation (i.e. $\Theta = \{A, B, C\}$) and the two following simple Bayesian witnesses reports

$$m_1(A) = 0.3 \quad m_1(B) = 0.4 \quad m_1(C) = 0.3$$

$$m_2(A) = 0.5 \quad m_2(B) = 0.1 \quad m_2(C) = 0.4$$

The conjunctive consensus is

$$m_{12}(A) = 0.15 \quad m_{12}(B) = 0.04 \quad m_{12}(C) = 0.12$$

with the conflicting mass $k_{12} = 0.69$. Now let's assume that a little bit later, one learns that $B = \emptyset$ because the second suspect brings a perfect alibi, then the initial consensus on B (i.e. $m_{12}(B) = 0.04$) must enter now in the new conflicting mass $k'_{12} = 0.69 + 0.04 = 0.73$ since $B = \emptyset$. k'_{12} is then re-distributed to A and C according to the WAO formula:

$$m_{WAO|12}(B) = 0$$

$$m_{WAO|12}(A) = 0.15 + (1/2)(0.3 + 0.5)(0.73) = 0.4420$$

$$m_{WAO|12}(C) = 0.12 + (1/2)(0.3 + 0.4)(0.73) = 0.3755$$

From this WAO result, one sees clearly that the sum of the combined belief assignments $m_{WAO|12}(.)$ is $0.8175 < 1$. Therefore, the WAO proposed in [12] doesn't manage properly the combination with VBA neither the possible dynamicity of the fusion problematic. This limitation is not very surprising since the WAO was proposed actually only for the static fusion[7] based on Shafer's model. The improvement of WAO for dynamic fusion is an open problem, but Milan Daniel in a private communication to the authors, proposed to use the following normalized coefficients for WAO in dynamic fusion:

$$w_m(X) = \frac{1}{s} \frac{\sum_X \sum_{i=1}^{s} m_i(X)}{\sum_{X \neq \emptyset} \sum_{i=1}^{s} m_i(X)} \sum_{i=1}^{s} m_i(X) \tag{1.20}$$

1.5 Daniel's minC rule of combination

1.5.1 Principle of the minC rule

MinC fusion rule is a recent interesting rule based on proportional redistribution of partial conflicts. Actually it was the first rule, to the knowledge of authors, that uses the idea for sophisticated proportional conflict redistribution. This rule was developed in the DST framework only. MinC rule is commutative and preserves the neutral impact of VBA but, as the majority of rules, MinC is not fully associative. MinC has been developed and proposed by Milan Daniel

[7]Note that the static fusion aspect was not explicitly stated and emphasized in [12] but only implicitly assumed.

in [1–4]. A detailed presentation of MinC can also be found in [18] (Chap. 10).

The basic idea of minC is to identify all different types of partial conflicts and then transfer them with some proportional redistribution. Two versions of proportional redistributions have been proposed by Milan Daniel:

- The minC (version a)): the mass coming from a partial conflict (called contradiction by M. Daniel) involving several sets X_1,X_2,...,X_k is proportionalized among all unions $\bigcup_{i=1}^{j}$, of $j \leq k$ sets X_i of $\{X_1, \dots, X_k\}$ (after a proper reallocation of all equivalent propositions containing partial conflit onto elements of power set).
- The minC (version b)): the mass coming from a partial conflict involving several sets X_1,X_2,...,X_k is proportionalized among all non empty subsets of $X_1 \cup, \dots \cup X_k$.

The preservation of the neutral impact of the VBA by minC rule can been drawn from the following demonstration: Let's consider two basic belief assignments $m_1(.)$ and $m_2(.)$. The first stage of minC consists in deriving the conjunctive consensus $m_{12}(.)$ from $m_1(.)$ and $m_2(.)$ and then transfer the mass of conflicting propositions to its components and unions of its components proportionally to their masses $m_{12}(.)$. Since the vacuous belief assignment $m_v(.)$ is the neutral element of the conjunctive operator, one always has $m_{12v}(.) = m_{12}(.)$ and thus the result of the minC at the first stage and after the first stage not affected by the introduction of the vacuous belief assignment in the fusion process. That's why minC preserves the neutral impact of VBA.

Unfortunately no analytic expression for the minC rules (version a) and b)) has been provided so far by the author. As simply stated, minC transfers $m(A \cap B)$ when $A \cap B = \emptyset$ with specific proportionalization factors to A, B, and $A \cup B$; More generally, minC transfers the conflicting mass $m(X)$, when $X = \emptyset$, to all subsets of $u(X)$ (the disjunctive form of X), which is not the most exact issue. As it will be shown in the sequel of this chapter, the PCR5 rule allows a more judicious proportional conflict redistribution. For a better understanding of the minC rule, here is a simple illustrative example drawn from [18] (p. 237).

1.5.2 Example for minC

Let's consider Shafer's model with $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the two following bba's to combine (here we denotes $\theta_1 \cup \theta_2 \cup \theta_3$ by Θ for notation convenience).

$$\begin{aligned}
m_1(\theta_1) &= 0.3 & m_2(\theta_1) &= 0.1\\
m_1(\theta_2) &= 0.2 & m_2(\theta_2) &= 0.1\\
m_1(\theta_3) &= 0.1 & m_2(\theta_3) &= 0.2\\
m_1(\theta_1 \cup \theta_2) &= 0.1 & m_2(\theta_1 \cup \theta_2) &= 0.0\\
m_1(\theta_1 \cup \theta_3) &= 0.1 & m_2(\theta_1 \cup \theta_3) &= 0.1\\
m_1(\theta_2 \cup \theta_3) &= 0.0 & m_2(\theta_2 \cup \theta_3) &= 0.2\\
m_1(\Theta) &= 0.2 & m_2(\Theta) &= 0.3
\end{aligned}$$

The results of the three steps of the minC rules are given in Table 1.1. For notation convenience, the square symbol $\square$ represents $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$.

	m_{12}	$m^{\star}_{12}$	$m^{a)}_{\text{minC}}$	$m^{b)}_{\text{minC}}$
θ_1	0.19	0.20	0.2983	0.2999
θ_2	0.15	0.17	0.2318	0.2402
θ_3	0.14	0.16	0.2311	0.2327
$\theta_1 \cup \theta_2$	0.03	0.03	0.0362	0.0383
$\theta_1 \cup \theta_3$	0.06	0.06	0.0762	0.0792
$\theta_2 \cup \theta_3$	0.04	0.04	0.0534	0.0515
$\theta_1 \cup \theta_2 \cup \theta_3$	0.06	0.06	0.0830	0.0692
$\theta_1 \cap \theta_2$	0.05	0.05		
$\theta_1 \cap \theta_3$	0.07	0.07		
$\theta_2 \cap \theta_3$	0.05	0.05		
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.06	0.06		
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0.03	0.03		
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0.02	0.02		
$\theta_1 \cup (\theta_2 \cap \theta_3)$	0.01			
$\theta_2 \cup (\theta_1 \cap \theta_3)$	0.02			
$\theta_3 \cup (\theta_1 \cap \theta_2)$	0.02			
$\theta_1 \cap \theta_2 \cap \theta_3$	0			
$\square$	0			

Table 1.1: minC result (versions a) and b))

- **Step 1 of minC** : the conjunctive consensus

  The first column of Table 1.1 lists all the elements involved in the combination. The second column gives the result of the first step of the minC rule which consists in applying the conjunctive consensus operator $m_{12}(.)$ defined on the hyper-power set D^{Θ} of the free-DSm model.

- **Step 2 of minC** : the reallocation

  The second step of minC consists in the reallocation of the masses of all partial conflicts which are equivalent to some non empty elements of the power set. This is what we call *the equivalence-based reallocation principle* (EBR principle). The third column $m^{\star}_{12}$ of Table 1.1 gives the basic belief assignment *after reallocation of partial conflicts* based on EBR principle before proportional conflict redistribution (i.e. the third and final step of minC).

  Let's explain a bit what EBR is from this simple example. Because we are working with Shafer's model all elements θ_1, θ_2 and θ_3 of Θ are exclusive and therefore $\theta_1 \cap \theta_2 = \emptyset$, $\theta_1 \cap \theta_3 = \emptyset$, $\theta_3 \cap \theta_3 = \emptyset$ and $\theta_1 \cap \theta_2 \cap \theta_3 = \emptyset$. Consequently, the propositions $\theta_1 \cup (\theta_2 \cap \theta_3)$, $\theta_2 \cup (\theta_1 \cap \theta_3)$, and $\theta_3 \cup (\theta_1 \cap \theta_2)$ corresponding to the 14th, 15th and 16th rows of the Table 1.1 are respectively equivalent to θ_1, θ_2 and θ_3 so that their committed masses can be directly reallocated (added) onto $m_{12}(\theta_1)$, $m_{12}(\theta_2)$ and $m_{12}(\theta_3)$. No other mass containing partial conflict can be directly reallocated onto the first seven elements of the table based on the EBR principle in this example. Thus finally, one gets $m^\star_{12}(.) = m_{12}(.)$ for all non-equivalent elements and for elements θ_1, θ_2 and θ_3 for which a reallocation has been done

$$\begin{aligned}
m^\star_{12}(\theta_1) &= m_{12}(\theta_1) + m_{12}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 0.19 + 0.01 = 0.20\\
m^\star_{12}(\theta_2) &= m_{12}(\theta_2) + m_{12}(\theta_2 \cup (\theta_1 \cap \theta_3)) = 0.15 + 0.02 = 0.17\\
m^\star_{12}(\theta_3) &= m_{12}(\theta_3) + m_{12}(\theta_3 \cup (\theta_1 \cap \theta_2)) = 0.14 + 0.02 = 0.16
\end{aligned}$$

- **Step 3 of minC** : proportional conflict redistribution

  The fourth and fifth columns of the Table 1.1 ($m^{a)}_{\text{minC}}$ and $m^{b)}_{\text{minC}}$) provide the minC results with the two versions of minC proposed by Milan Daniel and explained below. The column 4 of the Table 1.1 corresponds to the version a) of minC while the column 5 corresponds to the version b). Let's explain now in details how the values of columns 4 and 5 have be obtained.

  Version a) of minC: The result for the minC (version a) corresponding to the fourth column of the Table 1.1 is obtained from $m^\star_{12}(.)$ by the proportional redistribution of the partial conflict onto the elements entering in the partial conflict and their union. By example, the mass $m^\star_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) = 0.06$ will be proportionalized from the mass of θ_1, $\theta_2 \cup \theta_3$ and $\theta_1 \cup \theta_2 \cup \theta_3$ only. The parts of the mass of $\theta_1 \cap (\theta_2 \cup \theta_3)$ added to θ_1, $\theta_2 \cup \theta_3$ and $\theta_1 \cup \theta_2 \cup \theta_3$ will be given by

$$\begin{aligned}
k(\theta_1) &= m^\star_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^\star_{12}(\theta_1)}{K} = 0.06 \cdot \frac{0.20}{0.30} = 0.040\\
k(\theta_2 \cup \theta_3) &= m^\star_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^\star_{12}(\theta_2 \cup \theta_3)}{K} = 0.06 \cdot \frac{0.04}{0.30} = 0.008\\
k(\theta_1 \cup \theta_2 \cup \theta_3) &= m^\star_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^\star_{12}(\Theta)}{K} = 0.06 \cdot \frac{0.06}{0.30} = 0.012
\end{aligned}$$

  where the normalization constant is $K = m^\star_{12}(\theta_1) + m^\star_{12}(\theta_2 \cup \theta_3) + m^\star_{12}(\theta_1 \cup \theta_2 \cup \theta_3) = 0.20 + 0.04 + 0.06 = 0.30$.

  The proportional redistribution is done similarly for all other partial conflicting masses. We summarize in Tables 1.2-1.4 all the proportions (rounded at the fifth decimal) of conflicting masses to transfer onto elements of the power set. The sum of each column of the Tables 1.2-1.4 is transferred onto the mass of the element of power set it corresponds to get the final result of minC (version a)). By example, $m^{a)}_{\text{minC}}(\theta_1)$ is obtained by

$$m^{a)}_{\text{minC}}(\theta_1) = m^\star_{12}(\theta_1) + (0.025 + 0.03333 + 0.04) = 0.20 + 0.09833 = 0.29833$$

  which corresponds to the first value (rounded at the 4th decimal) of the 4th column of Table 1.1. All other values of the minC (version a)) result of Table 1.1 can be easily verified similarly.

  Version b) of minC: In this second version of minC, the proportional redistribution of any partial conflict X remaining after step 2 uses all subsets of $u(X)$ (i.e. the disjunctive form of X). As example, let's consider the partial conflict $X = \theta_1 \cap (\theta_2 \cup \theta_3)$ in the Table 1.1 having the belief mass $m^\star_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) = 0.06$. Since $u(X) = \theta_1 \cup \theta_2 \cup \theta_3$, all elements

	θ_1	θ_2	θ_3
$\theta_1 \cap \theta_2$	0.025	0.02125	
$\theta_1 \cap \theta_3$	0.03333		0.02667
$\theta_2 \cap \theta_3$		0.02297	0.02162
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.04		
$\theta_2 \cap (\theta_1 \cup \theta_3)$		0.01758	
$\theta_3 \cap (\theta_1 \cup \theta_2)$			0.0128

Table 1.2: Version a) of minC Proportional conflict redistribution factors

	$\theta_1 \cup \theta_2$	$\theta_1 \cup \theta_3$
$\theta_1 \cap \theta_2$	0.00375	
$\theta_1 \cap \theta_3$		0.01
$\theta_2 \cap \theta_3$		
$\theta_1 \cap (\theta_2 \cup \theta_3)$		
$\theta_2 \cap (\theta_1 \cup \theta_3)$		0.00621
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0.0024	

Table 1.3: Version a) of minC Proportional conflict redistribution factors (continued)

	$\theta_2 \cup \theta_3$	$\theta_1 \cup \theta_2 \cup \theta_3$
$\theta_1 \cap \theta_2$		
$\theta_1 \cap \theta_3$		
$\theta_2 \cap \theta_3$	0.00541	
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.008	0.012
$\theta_2 \cap (\theta_1 \cup \theta_3)$		0.00621
$\theta_3 \cap (\theta_1 \cup \theta_2)$		0.0048

Table 1.4: Version a) of minC Proportional conflict redistribution factors (continued)

of the power set 2^Θ will enter in the proportional redistribution and we will get for this X

$$k(\theta_1) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_1)}{K} \approx 0.01666$$

$$k(\theta_2) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_2)}{K} \approx 0.01417$$

$$k(\theta_3) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_3)}{K} \approx 0.01333$$

$$k(\theta_1 \cup \theta_2) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_1 \cup \theta_2)}{K} = 0.06 \cdot \frac{0.03}{0.72} = 0.0025$$

$$k(\theta_1 \cup \theta_3) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_1 \cup \theta_3)}{K} = 0.06 \cdot \frac{0.06}{0.72} = 0.005$$

$$k(\theta_2 \cup \theta_3) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\theta_2 \cup \theta_3)}{K} = 0.06 \cdot \frac{0.04}{0.72} \approx 0.00333$$

$$k(\Theta) = m^{\star}_{12}(\theta_1 \cap (\theta_2 \cup \theta_3)) \cdot \frac{m^{\star}_{12}(\Theta)}{K} = 0.005$$

where the normalization constant $K = 0.72$ corresponds here to $K = \sum_{Y \in 2^\Theta} m^\star_{12}(Y)$.

If one considers now $X = \theta_1 \cap \theta_2$ with its belief mass $m^\star_{12}(\theta_1 \cap \theta_2) = 0.05$, then only θ_1, θ_2 and $\theta_1 \cup \theta_2$ enter in the proportional redistribution (version b) because $u(X) = \theta_1 \cup \theta_2$ doesn't not carry element θ_3. One then gets for this element X the new set of proportional redistribution factors:

$$\begin{aligned}
k(\theta_1) &= m^\star_{12}(\theta_1 \cap \theta_2) \cdot \frac{m^\star_{12}(\theta_1)}{K} = 0.05 \cdot \frac{0.20}{0.40} = 0.025 \\
k(\theta_2) &= m^\star_{12}(\theta_1 \cap \theta_2) \cdot \frac{m^\star_{12}(\theta_2)}{K} = 0.05 \cdot \frac{0.17}{0.40} = 0.02125 \\
k(\theta_1 \cup \theta_2) &= m^\star_{12}(\theta_1 \cap \theta_2) \cdot \frac{m^\star_{12}(\theta_1 \cup \theta_2)}{K} = 0.05 \cdot \frac{0.03}{0.40} = 0.00375
\end{aligned}$$

where the normalization constant $K = 0.40$ corresponds now to the sum $K = m^\star_{12}(\theta_1) + m^\star_{12}(\theta_2) + m^\star_{12}(\theta_1 \cup \theta_2)$.

The proportional redistribution is done similarly for all other partial conflicting masses. We summarize in the Tables 1.5-1.7 all the proportions (rounded at the fifth decimal) of conflicting masses to transfer onto elements of the power set based on this second version of proportional redistribution of minC.

The sum of each column of the Tables 1.5-1.7 is transferred onto the mass of the element of power set it corresponds to get the final result of minC (version b)). By example, $m^{b)}_{\text{minC}}(\theta_1)$ will be obtained by

$$\begin{aligned}
m^{b)}_{\text{minC}}(\theta_1) &= m^\star_{12}(\theta_1) + (0.02500 + 0.03333 + 0.01666 + 0.00834 + 0.00555) \\
&= 0.20 + 0.08888 = 0.28888
\end{aligned}$$

which corresponds to the first value (rounded at the 4th decimal) of the 5th column of Table 1.1. All other values of the minC (version b)) result of Table 1.1 can be easily verified similarly.

	θ_1	θ_2	θ_3
$\theta_1 \cap \theta_2$	0.02500	0.02125	
$\theta_1 \cap \theta_3$	0.03333		0.02667
$\theta_2 \cap \theta_3$		0.02298	0.02162
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.01666	0.01417	0.01333
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0.00834	0.00708	0.00667
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0.00555	0.00472	0.00444

Table 1.5: Version b) of minC Proportional conflict redistribution factors

	$\theta_1 \cup \theta_2$	$\theta_1 \cup \theta_3$
$\theta_1 \cap \theta_2$	0.00375	
$\theta_1 \cap \theta_3$		0.01000
$\theta_2 \cap \theta_3$		
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.00250	0.00500
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0.00125	0.00250
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0.00084	0.00167

Table 1.6: Version b) of minC Proportional conflict redistribution factors (continued)

	$\theta_2 \cup \theta_3$	$\theta_1 \cup \theta_2 \cup \theta_3$
$\theta_1 \cap \theta_2$		
$\theta_1 \cap \theta_3$		
$\theta_2 \cap \theta_3$	0.00540	
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0.00333	0.00500
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0.00166	0.00250
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0.00111	0.00167

Table 1.7: Version b) of minC Proportional conflict redistribution factors (continued)

1.6 Principle of the PCR rules

Let's $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of the fusion problem under consideration and two belief assignments $m_1, m_2 : G^\Theta \to [0, 1]$ such that $\sum_{X \in G^\Theta} m_i(X) = 1$, $i = 1, 2$. The general principle of the Proportional Conflict Redistribution Rules (PCR for short) is:

- apply the conjunctive rule (1.8) or (1.9) depending on theory, i.e. G^Θ can be either 2^Θ or D^Θ,
- calculate the total or partial conflicting masses,
- then redistribute the conflicting mass (total or partial) proportionally on non-empty sets involved in the model according to all integrity constraints.

The way the conflicting mass is redistributed yields to five versions of PCR, denoted PCR1, PCR2, ..., PCR5 as it will be shown in the sequel. The PCR combination rules work for any degree of conflict $k_{12} \in [0, 1]$ or $k_{12\ldots s} \in [0, 1]$, for any DSm models (Shafer's model, free DSm model or any hybrid DSm model). PCR rules work both in DST and DSmT frameworks and for static or dynamical fusion problematic. The sophistication/complexity (but correctness) of proportional conflict redistribution increases from the first PCR1 rule up to the last rule PCR5. The development of different PCR rules presented here comes from the fact that the first initial PCR rule developed (PCR1) does not preserve the neutral impact of VBA. All other improved rules PCR2-PCR5 preserve the commutativity, the neutral impact of VBA and propose, upon to our opinion, a more and more exact solution for the conflict management to satisfy as best as possible the condition 1 (in section 1) that any satisfactory combination rule must tend to. The general proof for the neutrality of VBA within PCR2, PCR3, PCR4 and PCR5 rules is given in section 1.11.1 and some numerical examples are given in the section related with the presentation of each rule.

1.7 The PCR1 rule

1.7.1 The PCR1 formula

PCR1 is the simplest and the easiest version of proportional conflict redistribution for combination. PCR1 is described in details in [20]. The basic idea for PCR1 is only to compute the total conflicting mass k_{12} (not worrying about the partial conflicting masses). The total conflicting mass is then distributed to *all non-empty sets* proportionally with respect to their corresponding non-empty column sum of the associated mass matrix. The PCR1 is defined $\forall(X \neq \emptyset) \in G^\Theta$ by:

- For the combination of $s = 2$ sources

$$m_{PCR1}(X) = [\sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)] + \frac{c_{12}(X)}{d_{12}} \cdot k_{12} \tag{1.21}$$

  where $c_{12}(X)$ is the non-zero sum of the column of X in the mass matrix $\mathbf{M} = \begin{bmatrix}\mathbf{m}_1 \\ \mathbf{m}_2\end{bmatrix}$ (where $\mathbf{m}_i$ for $i = 1, 2$ is the row vector of belief assignments committed by the source i to elements of G^Θ), i.e. $c_{12}(X) = m_1(X) + m_2(X) \neq 0$, k_{12} is the total conflicting mass, and d_{12} is the sum of all non-zero column sums of all non-empty sets (in many cases $d_{12} = 2$, but in some degenerate cases it can be less) (see [20]).

- For the combination of $s \geq 2$ sources

$$m_{PCR1}(X) = [\sum_{\substack{X_1,X_2,\ldots,X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i)] + \frac{c_{12\ldots s}(X)}{d_{12\ldots s}} \cdot k_{12\ldots s} \tag{1.22}$$

  where $c_{12\ldots s}(X)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12\ldots s}(X) = m_1(X) + m_2(X) + \ldots + m_s(X) \neq 0$, $k_{12\ldots s}$ is the total conflicting mass, and $d_{12\ldots s}$ is the sum of all non-zero column sums of all non-empty sets (in many cases $d_{12\ldots s} = s$, but in some degenerate cases it can be less).

PCR1 is an alternative combination rule to WAO (Weighted Average Operator) proposed by Jøsang, Daniel and Vannoorenberghe in [11]. Both are particular cases of WO (The Weighted Operator) because the conflicting mass is redistributed with respect to some weighting factors. In the PCR1, the proportionalization is done for each non-empty set with respect to the non-zero sum of its corresponding mass matrix - instead of its mass column average as in WAO. But, PCR1 extends WAO, since PCR1 works also for the degenerate cases when all column sums of all non-empty sets are zero because in such cases, the conflicting mass is transferred to the non-empty disjunctive form of all non-empty sets together; when this disjunctive form happens to be empty, then either the problem degenerates truly to a void problem and thus all conflicting mass is transferred onto the empty set, or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0.

A nice feature of PCR1 rule, is that it works in all cases (degenerate and non degenerate). PCR1 corresponds to a specific choice of proportionality coefficients in the infinite continuum family[8] of possible rules of combination involving conjunctive consensus operator. The PCR1 on the power set and for non-degenerate cases gives the same results as WAO (as Philippe Smets pointed out); yet, for the storage requirement in a dynamic fusion when the associativity is requested, one needs to store for PCR1 only the last sum of masses, besides the previous conjunctive rule's result, while in WAO one needs also to store the number of the steps (see [20] for details) – and both rules become quasi-associative. In addition to WAO, we propose a general formula for PCR1 (WAO for non-degenerate cases).

Unfortunately, a severe limitation of PCR1 (as for WAO) is the non-preservation of the neutral impact of the VBA as shown in [20]. In other words, for $s \geq 1$, one gets for $m_1(.) \neq m_v(.)$, $\ldots$, $m_s(.) \neq m_v(.)$:

$$m_{PCR1}(.) = [m_1 \oplus \ldots m_s \oplus m_v](.) \neq [m_1 \oplus \ldots m_s](.)$$

For the cases of the combination of only one non-vacuous belief assignment $m_1(.)$ with the vacuous belief assignment $m_v(.)$ where $m_1(.)$ has mass assigned to an empty element, say $m_1(\emptyset) > 0$ as in Smets' TBM, or as in DSmT dynamic fusion where one finds out that a previous non-empty element A, whose mass $m_1(A) > 0$, becomes empty after a certain time, then this mass of an empty set has to be transferred to other elements using PCR1, but for such case $[m_1 \oplus m_v](.)$ is different from $m_1(.)$. This severe drawback of WAO and PCR1 forces us to develop the next PCR rules satisfying the neutrality property of VBA with better redistributions of the conflicting information.

1.7.2 Example for PCR1 (degenerate case)

For non degenerate cases with Shafer's model, PCR1 and WAO provide the same results. So it is interesting to focus the reader's attention on the difference between PCR1 and WAO in a simple degenerate case corresponding to a dynamic fusion problem. Let's take the following example showing the restriction of applicability of *static*-WAO[9]. As example, let's consider three different suspects A, B and C in a criminal investigation (i.e. $\Theta = \{A, B, C\}$) and the two following simple Bayesian witnesses reports

$$m_1(A) = 0.3 \quad m_1(B) = 0.4 \quad m_1(C) = 0.3$$

$$m_2(A) = 0.5 \quad m_2(B) = 0.1 \quad m_2(C) = 0.4$$

The conjunctive consensus is

$$m_{12}(A) = 0.15 \quad m_{12}(B) = 0.04 \quad m_{12}(C) = 0.12$$

[8] pointed out independently by Inagaki in 1991 and Lefèvre, Colot and Vannoorenberghe in 2002.

[9] *static*-WAO stands for the WAO rule proposed in [11, 12] based on Shafer's model for the implicit static fusion case (i.e. Θ remains invariant with time), while *dynamic*-WAO corresponds to Daniel's improved version of WAO using (1.20).

with the conflicting mass $k_{12} = 0.69$. Now let's assume that a little bit later, one learns that $B = \emptyset$ because the second suspect brings a strong alibi, then the initial consensus on B (i.e. $m_{12}(B) = 0.04$) must enter now in the new conflicting mass $k'_{12} = 0.69 + 0.04 = 0.73$ since $B = \emptyset$. Applying the PCR1 formula, one gets now:

$$\begin{aligned}
m_{PCR1|12}(B) &= 0 \\
m_{PCR1|12}(A) &= 0.15 + \frac{0.8}{0.8+0.7} \cdot 0.73 = 0.5393 \\
m_{PCR1|12}(C) &= 0.12 + \frac{0.7}{0.8+0.7} \cdot 0.73 = 0.4607
\end{aligned}$$

Let's remind (see section 4.3) that in this case, the *static*-WAO provides

$$m_{WAO|12}(B) = 0 \qquad m_{WAO|12}(A) = 0.4420 \qquad m_{WAO|12}(C) = 0.3755$$

We can verify easily that $m_{PCR1|12}(A) + m_{PCR1|12}(B) + m_{PCR1|12}(C) = 1$ while $m_{WAO|12}(A) + m_{WAO|12}(B) + m_{WAO|12}(C) = 0.8175 < 1$. This example shows clearly the difference between PCR1 and *static*-WAO originally proposed in [11, 12] and the ability of PCR1 to deal with degenerate/dynamic cases contrariwise to original WAO. The improved *dynamic*-WAO version suggested by Daniel coincides with PCR1.

1.8 The PCR2 rule

1.8.1 The PCR2 formula

In PCR2, the total conflicting mass k_{12} is distributed only to *the non-empty sets involved in the conflict* (not to all non-empty sets) and taken the canonical form of the conflict proportionally with respect to their corresponding non-empty column sum. The redistribution is then more exact (accurate) than in PCR1 and WAO. A nice feature of PCR2 is the preservation of the neutral impact of the VBA and of course its ability to deal with all cases/models.

A non-empty set $X_1 \in G^\Theta$ is considered *involved in the conflict* if there exists another set $X_2 \in G^\Theta$ which is neither included in X_1 nor includes X_1 such that $X_1 \cap X_2 = \emptyset$ and $m_{12}(X_1 \cap X_2) > 0$. This definition can be generalized for $s \geq 2$ sources.

- The PCR2 formula for two sources ($s = 2$) is $\forall (X \neq \emptyset) \in G^\Theta$,

$$m_{PCR2}(X) = [\sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)] + \mathcal{C}(X) \frac{c_{12}(X)}{e_{12}} \cdot k_{12} \tag{1.23}$$

  where

$$\mathcal{C}(X) = \begin{cases} 1, & \text{if } X \text{ involved in the conflict,} \\ 0, & \text{otherwise;} \end{cases}$$

  and where $c_{12}(X)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12}(X) = m_1(X) + m_2(X) \neq 0$, k_{12} is the total conflicting mass, and e_{12} is the sum of all non-zero column sums of all non-empty sets *only* involved in the conflict (resulting from the conjunctive normal form of their intersection after using the conjunctive rule). In many cases $e_{12} = 2$, but in some degenerate cases it can be less.

- For the the combination of $s \geq 2$ sources, the previous PCR2 formula can be easily generalized as follows $\forall (X \neq \emptyset) \in G^\Theta$:

$$m_{PCR2}(X) = [\sum_{\substack{X_1,X_2,\ldots,X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i)] + \mathcal{C}(X) \frac{c_{12\ldots s}(X)}{e_{12\ldots s}} \cdot k_{12\ldots s} \tag{1.24}$$

where

$$\mathcal{C}(X) = \begin{cases} 1, & \text{if } X \text{ involved in the conflict,} \\ 0, & \text{otherwise;} \end{cases}$$

and $c_{12\ldots s}(X)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12\ldots s}(X) = m_1(X) + m_2(X) + \ldots + m_s(X) \neq 0$, $k_{12\ldots s}$ is the total conflicting mass, and $e_{12\ldots s}$ is the sum of all non-zero column sums of all non-empty sets involved in the conflict (in many cases $e_{12\ldots s} = s$, but in some degenerate cases it can be less).

In the degenerate case when all column sums of all non-empty sets involved in the conflict are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets together which were involved in the conflict together. But if this disjunctive form happens to be empty, then the problem reduces to a degenerate void problem and thus all conflicting mass is transferred to the empty set or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0.

1.8.2 Example for PCR2 versus PCR1

Let's have the frame of discernment $\Theta = \{A, B\}$, Shafer's model (i.e. all intersections empty), and the following two bba's:

$$m_1(A) = 0.7 \quad m_1(B) = 0.1 \quad m_1(A \cup B) = 0.2$$

$$m_2(A) = 0.5 \quad m_2(B) = 0.4 \quad m_2(A \cup B) = 0.1$$

The sums of columns of the mass matrix are

$$c_{12}(A) = 1.2 \quad c_{12}(B) = 0.5 \quad c_{12}(A \cup B) = 0.3$$

Then the conjunctive consensus yields

$$m_{12}(A) = 0.52 \quad m_{12}(B) = 0.13 \quad m_{12}(A \cup B) = 0.02$$

with the total conflict $k_{12} = m_{12}(A \cap B) = 0.33$.

- Applying the PCR1 rule yields ($d_{12} = 1.2 + 0.5 + 0.3 = 2$):

$$m_{PCR1|12}(A) = m_{12}(A) + \frac{c_{12}(A)}{d_{12}} \cdot k_{12} = 0.52 + \frac{1.2}{2} \cdot 0.33 = 0.7180$$

$$m_{PCR1|12}(B) = m_{12}(B) + \frac{c_{12}(B)}{d_{12}} \cdot k_{12} = 0.13 + \frac{0.5}{2} \cdot 0.33 = 0.2125$$

$$m_{PCR1|12}(A \cup B) = m_{12}(A \cup B) + \frac{c_{12}(A \cup B)}{d_{12}} \cdot k_{12} = 0.02 + \frac{0.3}{2} \cdot 0.33 = 0.0695$$

- While applying the PCR2 rule yields ($e_{12} = 1.2 + 0.5 = 1.7$):

$$
\begin{aligned}
m_{PCR2}(A) &= m_{12}(A) + \frac{c_{12}(A)}{e_{12}} \cdot k_{12} = 0.52 + \frac{1.2}{1.7} \cdot 0.33 = 0.752941 \\
m_{PCR2}(B) &= m_{12}(B) + \frac{c_{12}(B)}{e_{12}} \cdot k_{12} = 0.12 + \frac{0.5}{1.7} \cdot 0.33 = 0.227059 \\
m_{PCR2}(A \cup B) &= m_{12}(A \cup B) = 0.02
\end{aligned}
$$

1.8.3 Example of neutral impact of VBA for PCR2

Let's keep the previous example and introduce now a third but totally ignorant source $m_v(.)$ and examine the result of the combination of the 3 sources with PCR2. So, let's start with

$$m_1(A) = 0.7 \quad m_1(B) = 0.1 \quad m_1(A \cup B) = 0.2$$

$$m_2(A) = 0.5 \quad m_2(B) = 0.4 \quad m_2(A \cup B) = 0.1$$

$$m_v(A) = 0.0 \quad m_v(B) = 0.0 \quad m_v(A \cup B) = 1.0$$

The sums of columns of the mass matrix are

$$c_{12v}(A) = 1.2 \quad c_{12v}(B) = 0.5 \quad c_{12v}(A \cup B) = 1.3$$

Then the conjunctive consensus yields

$$m_{12v}(A) = 0.52 \quad m_{12v}(B) = 0.13 \quad m_{12v}(A \cup B) = 0.02$$

with the total conflict $k_{12v} = m_{12v}(A \cap B) = 0.33$. We get naturally $m_{12v}(.) = m_{12}(.)$ because the vacuous belief assignment $m_v(.)$ has no impact in the conjunctive consensus.

Applying the PCR2 rule yields:

$$
\begin{aligned}
m_{PCR2|12v}(A) &= m_{12v}(A) + \frac{c_{12v}(A)}{e_{12v}} \cdot k_{12v} = 0.52 + \frac{1.2}{1.2 + 0.5} \cdot 0.33 = 0.752941 \\
m_{PCR2|12v}(B) &= m_{12v}(B) + \frac{c_{12v}(B)}{e_{12v}} \cdot k_{12v} = 0.52 + \frac{0.5}{1.2 + 0.5} \cdot 0.33 = 0.227059 \\
m_{PCR2|12v}(A \cup B) &= m_{12v}(A \cup B) = 0.02
\end{aligned}
$$

In this example one sees that the neutrality property of VBA is effectively well satisfied since

$$m_{PCR2|12v}(.) = m_{PCR2|12}(.)$$

A general proof for neutrality of VBA within PCR2 is given in section 1.11.1.

1.9 The PCR3 rule

1.9.1 Principle of PCR3

In PCR3, one transfers *partial conflicting masses*, instead of the total conflicting mass, to non-empty sets involved in partial conflict (taken the canonical form of each partial conflict). If an intersection is empty, say $A \cap B = \emptyset$, then the mass $m(A \cap B)$ of the partial conflict is transferred to the non-empty sets A and B proportionally with respect to the non-zero sum of masses assigned to A and respectively to B by the bba's $m_1(.)$ and $m_2(.)$. The PCR3 rule works if at least one set between A and B is non-empty and its column sum is non-zero.

When both sets A and B are empty, or both corresponding column sums of the mass matrix are zero, or only one set is non-empty and its column sum is zero, then the mass $m(A \cap B)$ is transferred to the non-empty disjunctive form $u(A) \cup u(B)$ defined in (1.25); if this disjunctive form is empty then $m(A \cap B)$ is transferred to the non-empty total ignorance; but if even the total ignorance is empty then either the problem degenerates truly to a void problem and thus all conflicting mass is transferred onto the empty set, or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0.

If another intersection, say $A \cap C \cap D = \emptyset$, then again the mass $m(A \cap C \cap D) > 0$ is transferred to the non-empty sets A, C, and D proportionally with respect to the non-zero sum of masses assigned to A, C, and respectively D by the sources; if all three sets A, C, D are empty or the sets which are non-empty have their corresponding column sums equal to zero, then the mass $m(A \cap C \cap D)$ is transferred to the non-empty disjunctive form $u(A) \cup u(C) \cup u(D)$; if this disjunctive form is empty then the mass $m(A \cap C \cap D)$ is transferred to the non-empty total ignorance; but if even the total ignorance is empty (a completely degenerate void case) all conflicting mass is transferred onto the empty set (which means that the problem is truly void), or (if we prefer to adopt an optimistic point of view) all conflicting mass is transferred onto a new unknown extra and closure element θ_0 representing all missing hypotheses of the frame Θ.

The *disjunctive form* is defined[10] as [18]:

$$\begin{cases} u(X) = X \quad \text{if } X \text{ is a singleton} \\ u(X \cup Y) = u(X) \cup u(Y) \\ u(X \cap Y) = u(X) \cup u(Y) \end{cases} \tag{1.25}$$

1.9.2 The PCR3 formula

- For the combination of two bba's, the PCR3 formula is given by: $\forall (X \neq \emptyset) \in G^\Theta$,

[10] These relationships can be generalized for any number of sets.

$$
\begin{aligned}
m_{PCR3}(X) = [& \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)] \\
& + [c_{12}(X) \cdot \sum_{\substack{Y \in G^\Theta \\ Y \cap X = \emptyset}} \frac{m_1(Y)m_2(X) + m_1(X)m_2(Y)}{c_{12}(X) + c_{12}(Y)}] \\
& + [\sum_{\substack{X_1,X_2 \in (G^\Theta \setminus \{X\}) \cap \boldsymbol{\emptyset} \\ X_1 \cap X_2 = \emptyset \\ u(X_1) \cup u(X_2) = X}} [m_1(X_1)m_2(X_2) + m_1(X_2)m_2(X_1)]] \\
& + [\phi_\Theta(X) \sum_{\substack{X_1,X_2 \in (G^\Theta \setminus \{X\}) \cap \boldsymbol{\emptyset} \\ X_1 \cap X_2 = \emptyset \\ u(X_1) = u(X_2) = \emptyset}} [m_1(X_1)m_2(X_2) + m_1(X_2)m_2(X_1)]]
\end{aligned} \tag{1.26}
$$

where all sets are in canonical form, $c_{12}(X_i)$ ($X_i \in G^\Theta$) is the non-zero sum of the mass matrix column corresponding to the set X_i, i.e. $c_{12}(X_i) = m_1(X_i) + m_2(X_i) \neq 0$, and where $\phi_\Theta(.)$ is the characteristic function of the total ignorance (assuming $\mid \Theta \mid = n$) defined by

$$
\begin{cases}
\phi_\Theta(X) = 1 \,\text{if}\, X = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n \,\text{(total ignorance)} \\
\phi_\Theta(X) = 0 \quad \text{otherwise}
\end{cases} \tag{1.27}
$$

- For the fusion of $s \geq 2$ bba's, one extends the above procedure to formulas (1.25) and (1.26) to more general ones. One then gets the following PCR3 general formula. Let $G^\Theta = \{X_1, \ldots, X_n\} \neq \emptyset$ (G^Θ being either the power-set or hyper-power set depending on the model we want to deal with), $n \geq 2$, $\forall X \neq \emptyset$, $X \in G^\Theta$, one has:

$$
\begin{aligned}
m_{PCR3}(X) = & \, m_{12\ldots s}(X) \\
& + c_{12\ldots s}(X) \cdot \sum_{k=1}^{s-1} S_1^{PCR3}(X,k) + \sum_{k=1}^{s} S_2^{PCR3}(X,k) \\
& + \phi_\Theta(X) \sum_{k=1}^{s} S_3^{PCR3}(X,k)
\end{aligned} \tag{1.28}
$$

For convenience, the following notation is used

$$
m_{12\ldots s}(X) = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{k=1}^{s} m_k(X_k)
$$

$$
m_{12\ldots s}(\bigcap_{j=1}^{k} X_{i_j}) = m_{12\ldots s}(X_{i_1} \cap \ldots \cap X_{i_k})
$$

$$
S_1^{PCR3}(X,k) \triangleq \sum_{\substack{X_{i_1},\ldots,X_{i_k} \in G^\Theta \setminus \{X\} \\ \{i_1,\ldots,i_k\} \in \mathcal{P}^k(\{1,2,\ldots,n\}) \\ X \cap X_{i_1} \cap \ldots \cap X_{i_k} = \emptyset}} R_k^{i_1,\ldots,i_k}(X)
$$

with

$$R_k^{i_1,\ldots,i_k}(X) \triangleq \frac{m_{12\ldots s}(X \cap X_{i_1} \cap \ldots \cap X_{i_k})}{c_{12\ldots s}(X) + \sum_{j=1}^{k} c_{12\ldots s}(X_{i_j})}$$

and

$$S_2^{PCR3}(X,k) \triangleq \sum_{\substack{X_{i_1},\ldots,X_{i_k} \in (G^\Theta \setminus \{X\}) \cap \boldsymbol{\emptyset} \\ \{i_1,\ldots,i_k\} \in \mathcal{P}^k(\{1,2,\ldots,n\}) \\ X_{i_1} \cap \ldots \cap X_{i_k} = \emptyset \\ u(X_{i_1}) \cup \ldots \cup u(X_{i_k}) = X}} m_{12\ldots s}(\bigcap_{j=1}^{k} X_{i_j})$$

$$S_3^{PCR3}(X,k) \triangleq \sum_{\substack{X_{i_1},\ldots,X_{i_k} \in (G^\Theta \setminus \{X\}) \cap \boldsymbol{\emptyset} \\ \{i_1,\ldots,i_k\} \in \mathcal{P}^k(\{1,2,\ldots,n\}) \\ X_{i_1} \cap \ldots \cap X_{i_k} = \emptyset \\ u(X_{i_1}) = \ldots = u(X_{i_k}) = \boldsymbol{\emptyset}}} m_{12\ldots s}(\bigcap_{j=1}^{k} X_{i_j})$$

where $\boldsymbol{\emptyset}$ is the set of elements (if any) which have been forced to be empty by the integrity constraints of the model of the problem (in case of dynamic fusion) and ($\mathcal{P}^k(\{1,2,\ldots,n\})$ is the set of all subsets ok k elements from $\{1,2,\ldots,n\}$ (permutations of n elements taken by k), the order of elements doesn't count.

The sum $\sum_{k=1}^{s} S_2^{PCR3}(X,k)$ in (1.28) is for cases when $X_{i_1},\ldots,\ X_{i_k}$ become empty in dynamic fusion; their intersection mass is transferred to their disjunctive form: $u(X_{i_1}) \cup \ldots \cup u(X_{i_k}) \neq \emptyset$.

The sum $\sum_{k=1}^{s} S_3^{PCR3}(X,k)$ in (1.28) is for degenerate cases, i.e. when $X_{i_1},\ldots,\ X_{i_k}$ and their disjunctive form become empty in dynamic fusion; their intersection mass is transferred to the total ignorance.

PCR3 preserves the neutral impact of the VBA and works for any cases/models.

1.9.3 Example for PCR3

Let's have the frame of discernment $\Theta = \{A, B, C\}$, Shafer's model (i.e. all intersections empty), and the 2 following Bayesian bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(C) = 0.1$$

$$m_2(A) = 0.4 \quad m_2(B) = 0.4 \quad m_2(C) = 0.2$$

The sums of columns of the mass matrix are

$$c_{12}(A) = 1.0 \quad c_{12}(B) = 0.7 \quad c_{12}(C) = 0.3$$

Then the conjunctive consensus yields

$$m_{12}(A) = 0.24 \quad m_{12}(B) = 0.12 \quad m_{12}(C) = 0.02$$

with the total conflict $k_{12} = m_{12}(A\cap B)+m_{12}(A\cap C)+m_{12}(B\cap C) = 0.36+0.16+0.10 = 0.62$, which is a sum of factors.

Applying the PCR3 rule yields for this very simple (Bayesian) case:

$$\begin{aligned}
m_{PCR3|12}(A) &= m_{12}(A) + c_{12}(A)\cdot \frac{m_1(B)m_2(A)+m_1(A)m_2(B)}{c_{12}(A)+c_{12}(B)} \\
&\quad + c_{12}(A)\cdot \frac{m_1(C)m_2(A)+m_1(A)m_2(C)}{c_{12}(A)+c_{12}(C)} \\
&= 0.24 + 1\cdot \frac{0.3\cdot 0.4+0.6\cdot 0.4}{1+0.7} + 1\cdot \frac{0.1\cdot 0.4+0.6\cdot 0.2}{1+0.3} = 0.574842
\end{aligned}$$

$$\begin{aligned}
m_{PCR3|12}(B) &= m_{12}(B) + c_{12}(B)\cdot \frac{m_1(A)m_2(B)+m_1(B)m_2(A)}{c_{12}(B)+c_{12}(A)} \\
&\quad + c_{12}(B)\cdot \frac{m_1(C)m_2(B)+m_1(B)m_2(C)}{c_{12}(B)+c_{12}(C)} \\
&= 0.12 + 0.7\cdot \frac{0.6\cdot 0.4+0.3\cdot 0.4}{0.7+1} + 0.7\cdot \frac{0.1\cdot 0.4+0.3\cdot 0.2}{0.7+0.3} = 0.338235
\end{aligned}$$

$$\begin{aligned}
m_{PCR3|12}(C) &= m_{12}(C) + c_{12}(C)\cdot \frac{m_1(C)m_2(A)+m_1(A)m_2(C)}{c_{12}(C)+c_{12}(A)} \\
&\quad + c_{12}(C)\cdot \frac{m_1(C)m_2(B)+m_1(B)m_2(C)}{c_{12}(C)+c_{12}(B)} \\
&= 0.02 + 0.3\cdot \frac{0.1\cdot 0.4+0.6\cdot 0.2}{0.3+1} + 0.3\cdot \frac{0.1\cdot 0.4+0.2\cdot 0.3}{0.3+0.7} = 0.086923
\end{aligned}$$

Note that in this simple case, the two last sums involved in formula (1.26) are equal to zero because here there doesn't exist positive mass products $m_1(X_1)m_2(X_2)$ to compute for any $X \in 2^\Theta$, $X_1, X_2 \in 2^\Theta \setminus \{X\}$ such that $X_1 \cap X_2 = \emptyset$ and $u(X_1) \cup u(X_2) = X$, neither for $X_1 \cap X_2 = \emptyset$ and $u(X_1) = u(X_2) = \emptyset$.

In this example, PCR3 provides a result different from PCR1 and PCR2 (PCR2 provides same result as PCR1) since

$$\begin{aligned}
m_{PCR1}(A) &= 0.24 + \frac{1}{1+0.7+0.3}\cdot 0.62 = 0.550 \\
m_{PCR1}(B) &= 0.12 + \frac{0.7}{1+0.7+0.3}\cdot 0.62 = 0.337 \\
m_{PCR1}(C) &= 0.02 + \frac{0.3}{1+0.7+0.3}\cdot 0.62 = 0.113
\end{aligned}$$

1.9.4 Example of neutral impact of VBA for PCR3

Let's keep the previous example and introduce now a third but totally ignorant source $m_v(.)$ and examine the result of the combination of the 3 sources with PCR3. Θ denotes here for notation convenience $A \cup B \cup C$. So, Let's start with

$$\begin{array}{llll} m_1(A) = 0.6 & m_1(B) = 0.3 & m_1(C) = 0.1 & \\ m_2(A) = 0.4 & m_2(B) = 0.4 & m_2(C) = 0.2 & \\ m_v(A) = 0.0 & m_v(B) = 0.0 & m_v(C) = 0.0 & m_v(\Theta) = 1 \end{array}$$

The sums of columns of the mass matrix are

$$c_{12v}(A) = 1,\ c_{12v}(B) = 0.7,\ c_{12v}(C) = 0.3,\ c_{12v}(\Theta) = 1$$

The conjunctive consensus yields

$$m_{12v}(A) = 0.24 \quad m_{12v}(B) = 0.12 \quad m_{12v}(C) = 0.02$$

with the total conflict $k_{12v} = m_{12v}(A \cap B) + m_{12v}(A \cap C) + m_{12v}(B \cap C) = 0.36 + 0.16 + 0.10 = 0.62$, which is a sum of factors. We get naturally $m_{12v}(.) = m_{12}(.)$ because the vacuous belief assignment $m_v(.)$ has no impact on the conjunctive consensus.

Applying the PCR3 rule yields for this case

$$\begin{aligned} m_{PCR3|12v}(A) =& m_{12v}(A) \\ &+ c_{12v}(A) \cdot [\frac{m_1(B)m_2(A)m_v(\Theta)}{c_{12v}(A) + c_{12v}(B)} + \frac{m_1(A)m_2(B)m_v(\Theta)}{c_{12v}(A) + c_{12v}(B)}] \\ &+ c_{12v}(A) \cdot [\frac{m_1(C)m_2(A)m_v(\Theta)}{c_{12v}(A) + c_{12v}(C)} + \frac{m_1(A)m_2(C)m_v(\Theta)}{c_{12v}(A) + c_{12v}(C)}] \\ =& 0.24 + 1 \cdot \frac{0.3 \cdot 0.4 \cdot 1 + 0.6 \cdot 0.4 \cdot 1}{1 + 0.7} + 1 \cdot \frac{0.1 \cdot 0.4 \cdot 1 + 0.6 \cdot 0.2 \cdot 1}{1 + 0.3} \\ =& 0.574842 = m_{PCR3|12}(A) \end{aligned}$$

Similarly, one obtains

$$\begin{aligned} m_{PCR3|12v}(B) =& 0.12 + 0.7 \cdot \frac{0.6 \cdot 0.4 \cdot 1 + 0.3 \cdot 0.4 \cdot 1}{0.7 + 1} + 0.7 \cdot \frac{0.1 \cdot 0.4 \cdot 1 + 0.3 \cdot 0.2 \cdot 1}{0.7 + 0.3} \\ =& 0.338235 = m_{PCR3|12}(B) \end{aligned}$$

$$\begin{aligned} m_{PCR3|12v}(C) =& 0.02 + 0.3 \cdot \frac{0.1 \cdot 0.4 \cdot 1 + 0.6 \cdot 0.2 \cdot 1}{0.3 + 1} + 0.3 \cdot \frac{0.1 \cdot 0.4 \cdot 1 + 0.2 \cdot 0.3 \cdot 1}{0.3 + 0.7} \\ =& 0.086923 = m_{PCR3|12}(C) \end{aligned}$$

In this example one sees that the neutrality property of VBA is effectively well satisfied by PCR3 rule since

$$m_{PCR3|12v}(.) = m_{PCR3|12}(.)$$

A general proof for neutrality of VBA within PCR3 is given in section 1.11.1.

1.10 The PCR4 rule

1.10.1 Principle of PCR4

PCR4 redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the canonical form of the partial conflict. PCR4 is an improvement of previous PCR rules but also of Milan Daniel's minC operator [18]. Daniel uses the proportionalization with respect to the results of the conjunctive rule, but not with respect to the masses assigned to each set by the sources of information as done in PCR1-3 and also as in the most effective PCR5 rule explained in the next section. Actually, PCR4 also uses the proportionalization with respect to the results of the conjunctive rule, but with PCR4 the conflicting mass $m_{12}(A \cap B)$ when $A \cap B = \emptyset$ is distributed to A and B only because only A and B were involved in the conflict ($A \cup B$ was not involved in the conflict since $m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B)$), while minC redistributes $m_{12}(A \cap B)$ to A, B, *and* $A \cup B$ in both of its versions a) and b) (see section 5 and [18] for details). Also, for the mixed elements such as $C \cap (A \cup B) = \emptyset$, the mass $m(C \cap (A \cup B))$ is redistributed to C, $A \cup B$, $A \cup B \cup C$ in minC version a), and worse in minC version b) to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$ and $A \cup B \cup C$ (see example in section 5). PCR4 rule improves this and redistributes the mass $m(C \cap (A \cup B))$ to C and $A \cup B$ *only*, since only them were involved in the conflict: i.e. $m_{12}(C \cap (A \cup B)) = m_1(C)m_2(A \cup B) + m_2(C)m_1(A \cup B)$, clearly the other elements A, B, $A \cup B \cup C$ that get some mass in minC were not involved in the conflict $C \cap (A \cup B)$. If at least one conjunctive rule result is null, then the partial conflicting mass which involved this set is redistributed proportionally to the column sums corresponding to each set. Thus PCR4 does a more exact redistribution than both minC versions (versions a) and b)) explained in section 5. The PCR4 rule partially extends Dempster's rule in the sense that instead of redistributing the total conflicting mass as within Dempster's rule, PCR4 redistributes partial conflicting masses, hence PCR4 does a better refined redistribution than Dempster's rule; PCR4 and Dempster's rule coincide for $\Theta = \{A, B\}$, in Shafer's model, with $s \geq 2$ sources, and such that $m_{12\ldots s}(A) > 0$, $m_{12\ldots s}(B) > 0$, and $m_{12\ldots s}(A \cup B) = 0$. Thus according to authors opinion, PCR4 rule redistributes better than Dempster's rule since in PCR one goes on partial conflicting, while Dempster's rule redistributes the conflicting mass to all non-empty sets whose conjunctive mass is nonzero, even those not involved in the conflict.

1.10.2 The PCR4 formula

The PCR4 formula for $s = 2$ sources: $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$m_{PCR4}(X) = m_{12}(X) \cdot [1 + \sum_{\substack{Y \in G^\Theta \\ Y \cap X = \emptyset}} \frac{m_{12}(X \cap Y)}{m_{12}(X) + m_{12}(Y)}] \tag{1.29}$$

with $m_{12}(X)$ and $m_{12}(Y)$ nonzero. $m_{12}(.)$ corresponds to the conjunctive consensus, i.e.

$$m_{12}(X) \triangleq \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2)\,.$$

If at least one of $m_{12}(X)$ or $m_{12}(Y)$ is zero, the fraction is discarded and the mass $m_{12}(X \cap Y)$ is transferred to X and Y proportionally with respect to their non-zero column sum of masses; if both their column sums of masses are zero, then one transfers to the partial ignorance $X \cup Y$;

if even this partial ignorance is empty then one transfers to the total ignorance.

Let $G = \{X_1, \ldots, X_n\} \neq \emptyset$ (G^Θ being either the power-set or hyper-power set depending on the model we want to deal with), $n \geq 2$, $\forall X \neq \emptyset$, $X \in G^\Theta$, the general PCR4 formula for $s \geq 2$ sources is given by $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$m_{PCR4}(X) = m_{12\ldots s}(X) \cdot [1 + \sum_{k=1}^{s-1} S^{PCR4}(X,k)] \tag{1.30}$$

with

$$S^{PCR4}(X,k) \triangleq \sum_{\substack{X_{i_1},\ldots,X_{i_k} \in G^\Theta \setminus \{X\} \\ \{i_1,\ldots,i_k\} \in \mathcal{P}^k(\{1,2,\ldots,n\}) \\ X \cap X_{i_1} \cap \ldots \cap X_{i_k} = \emptyset}} \frac{m_{12\ldots s}(X \cap X_{i_1} \cap \ldots \cap X_{i_k})}{m_{12\ldots s}(X) + \sum_{j=1}^{k} m_{12\ldots s}(X_{i_j})} \tag{1.31}$$

with all $m_{12\ldots s}(X)$, $m_{12\ldots s}(X_1)$, $\ldots$, $m_{12\ldots s}(X_n)$ nonzero and where the first term of the right side of (1.30) corresponds to the conjunctive consensus between s sources (i.e. $m_{12\ldots s}(.)$). If at least one of $m_{12\ldots s}(X)$, $m_{12\ldots s}(X_1)$, $\ldots$, $m_{12\ldots s}(X_n)$ is zero, the fraction is discarded and the mass $m_{12\ldots s}(X \cap X_1 \cap X_2 \cap \ldots \cap X_k)$ is transferred to X, X_1, $\ldots$, X_k proportionally with respect to their corresponding column sums in the mass matrix.

1.10.3 Example for PCR4 versus minC

Let's consider $\Theta = \{A, B\}$, Shafer's model and the the two following bba's:

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27 \quad m_{12}(A \cup B) = 0.05$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.24$$

Applying PCR4 rule, one has the following proportional redistribution[11] to satisfy

$$\frac{x}{0.44} = \frac{y}{0.27} = \frac{0.24}{0.44 + 0.27} \approx 0.3380$$

from which, one deduces $x = 0.1487$ and $y = 0.0913$ and thus

$$\begin{aligned}
m_{PCR4}(A) &= 0.44 + 0.1487 = 0.5887 \\
m_{PCR4}(B) &= 0.27 + 0.0913 = 0.3613 \\
m_{PCR4}(A \cup B) &= 0.05
\end{aligned}$$

[11] x is the part of conflict redistributed to A, y is the part of conflict redistributed to B.

while applying minC (version a) and b)) are equivalent in this 2D case), one uses the following proportional redistribution[12]

$$\frac{x}{0.44} = \frac{y}{0.27} = \frac{z}{0.05} = \frac{0.24}{0.44 + 0.27 + 0.05} \approx 0.31578$$

Whence $x = 0.44 \cdot (0.24/0.76) \approx 0.138947$, $y = 0.27 \cdot (0.24/0.76) \approx 0.085263$, $z = 0.05 \cdot (0.24/0.76) \approx 0.015789$, so that

$$\begin{aligned}
m_{minC}(A) &\approx 0.44 + 0.138947 = 0.578948\\
m_{minC}(B) &\approx 0.27 + 0.085263 = 0.355263\\
m_{minC}(A \cup B) &\approx 0.05 + 0.015789 = 0.065789
\end{aligned}$$

Therefore, one sees clearly the difference between PCR4 and minC rules. It can be noted here that minC gives the same result as Dempster's rule, but the result drawn from minC and Dempster's rules is less exact in comparison to PCR4 because minC and Dempster's rules redistribute a fraction of the conflicting mass to $A \cup B$ too, although $A \cup B$ is not involved in any conflict (therefore $A \cup B$ doesn't deserve anything).

We can remark also that in the 2D Bayesian case, the PCR4, minC, and Dempster's rules give the same results. For example, let's take $\Theta = \{A, B\}$, Shafer's model and the two following bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0.4$$

$$m_2(A) = 0.1 \quad m_2(B) = 0.9$$

The conjunctive consensus yields $m_{12}(A) = 0.06$, $m_{12}(B) = 0.36$ with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.58$$

PCR4, MinC and Dempster's rules provide

$$\begin{aligned}
m_{PCR4}(A) &= m_{minC}(A) = m_{DS}(A) = 0.142857\\
m_{PCR4}(B) &= m_{minC}(B) = m_{DS}(B) = 0.857143
\end{aligned}$$

1.10.4 Example of neutral impact of VBA for PCR4

Let's consider the previous example with $\Theta = \{A, B\}$, Shafer's model and the the two following bba's:

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27 \quad m_{12}(A \cup B) = 0.05$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.24$$

[12] z is the part of conflict redistributed to $A \cup B$.

The canonical form $c(A \cap B) = A \cap B$, thus $k_{12} = m_{12}(A \cap B) = 0.24$ will be distributed to A and B only proportionally with respect to their corresponding $m_{12}(.)$, i.e. with respect to 0.44 and 0.27 respectively. One gets:

$$m_{PCR4|12}(A) = 0.5887 \quad m_{PCR4|12}(B) = 0.3613 \quad m_{PCR4|12}(A \cup B) = 0.05$$

Now let's introduce a third and vacuous belief assignment $m_v(A \cup B) = 1$ and combine altogether $m_1(.)$, $m_2(.)$ and $m_v(.)$ with the conjunctive consensus. One gets

$$m_{12v}(A) = 0.44 \quad m_{12v}(B) = 0.27 \quad m_{12v}(A \cup B) = 0.05 \quad m_{12v}(A \cap B \cap (A \cup B)) = 0.24$$

Since the canonical form $c(A \cap B \cap (A \cup B)) = A \cap B$, $m_{12v}(A \cap B \cap (A \cup B)) = 0.24$ will be distributed to A and B only proportionally with respect to their corresponding $m12v(.)$, i.e. with respect to 0.44 and 0.27 respectively, therefore exactly as above. Thus

$$m_{PCR4|12v}(A) = 0.5887 \quad m_{PCR4|12v}(B) = 0.3613 \quad m_{PCR4|12v}(A \cup B) = 0.05$$

In this example one sees that the neutrality property of VBA is effectively well satisfied by PCR4 rule since

$$m_{PCR4|12v}(.) = m_{PCR4|12}(.)$$

A general proof for neutrality of VBA within PCR4 is given in section 1.11.1.

1.10.5 A more complex example for PCR4

Let's consider now a more complex example involving some null masses (i.e. $m_{12}(A) = m_{12}(B) = 0$) in the conjunctive consensus between sources. So, let's consider $\Theta = \{A, B, C, D\}$, Shafer's model and the two following belief assignments:

$$m_1(A) = 0 \quad m_1(B) = 0.4 \quad m_1(C) = 0.5 \quad m_1(D) = 0.1$$

$$m_2(A) = 0.6 \quad m_2(B) = 0 \quad m_2(C) = 0.1 \quad m_2(D) = 0.3$$

The conjunctive consensus yields here $m_{12}(A) = m_{12}(B) = 0$, $m_{12}(C) = 0.05$, $m_{12}(D) = 0.03$ with the total conflicting mass

$$\begin{aligned} k_{12} &= m_{12}(A \cap B) + m_{12}(A \cap C) + m_{12}(A \cap D) \\ &\quad + m_{12}(B \cap C) + m_{12}(B \cap D) + m_{12}(C \cap D) \\ &= 0.24 + 0.30 + 0.06 + 0.04 + 0.12 + 0.16 = 0.92 \end{aligned}$$

Because $m_{12}(A) = m_{12}(B) = 0$, the denominator $m_{12}(A) + m_{12}(B) = 0$ and the transfer onto A and B should be done proportionally to $m_2(A)$ and $m_1(B)$, thus:

$$\frac{x}{0.6} = \frac{y}{0.4} = \frac{0.24}{0.6 + 0.4} = 0.24$$

whence $x = 0.144$, $y = 0.096$.

$m_{12}(A \cap C) = 0.30$ is transferred to A and C:

$$\frac{x}{0.6} = \frac{z}{0.5 + 0.1} = \frac{0.30}{1.2}$$

whence $x = z = 0.6 \cdot (0.30/1.2) = 0.15$.

$m_{12}(A \cap D) = 0.06$ is transferred to A and D:

$$\frac{x}{0.6} = \frac{w}{0.3 + 0.1} = \frac{0.06}{1}$$

whence $x = 0.6 \cdot (0.06) = 0.036$ and $w = 0.4 \cdot (0.06) = 0.024$.

$m_{12}(B \cap C) = 0.04$ is transferred to B and C:

$$\frac{y}{0.4} = \frac{z}{0.6} = \frac{0.04}{1}$$

whence $y = 0.4 \cdot (0.04) = 0.016$ and $z = 0.6 \cdot (0.04) = 0.024$.

$m_{12}(B \cap D) = 0.12$ is transferred to B and D:

$$\frac{y}{0.4} = \frac{w}{0.4} = \frac{0.12}{0.8} = 0.15$$

whence $y = 0.4 \cdot (0.15) = 0.06$ and $w = 0.4 \cdot (0.15) = 0.06$.

The partial conflict $m_{12}(C \cap D) = 0.16$ is proportionally redistributed to C and D only according to

$$\frac{z}{0.05} = \frac{w}{0.03} = \frac{0.16}{0.05 + 0.03} = 2$$

whence $z = 0.10$ and $w = 0.06$. Summing all redistributed partial conflicts, one finally gets:

$$\begin{aligned}
m_{PCR4}(A) &= 0 + 0.144 + 0.150 + 0.036 = 0.330\\
m_{PCR4}(B) &= 0 + 0.096 + 0.016 + 0.016 = 0.172\\
m_{PCR4}(C) &= 0.05 + 0.15 + 0.024 + 0.10 = 0.324\\
m_{PCR4}(D) &= 0.03 + 0.024 + 0.06 + 0.06 = 0.174
\end{aligned}$$

while minC provides[13]

$$m_{minC}(A) = m_{minC}(B) = m_{minC}(A \cup B) = 0.08$$

$$m_{minC}(C) = 0.490 \quad m_{minC}(D) = 0.270$$

The distinction between PCR4 and minC here is that minC transfers equally the 1/3 of conflicting mass $m_{12}(A \cap B) = 0.24$ onto A, B and $A \cup B$, while PCR4 redistributes it to A and B proportionally to their masses $m_2(A)$ and $m_1(B)$. Upon to authors opinions, the minC redistribution appears less exact than PCR4 since $A \cup B$ is not involved into the partial conflict $A \cap B$ and we don't see a reasonable justification on minC transfer onto $A \cup B$ in this case.

[13]It can be proven that versions a) and b) of minC provide here same result because in this specific example $m_{12}(A) = m_{12}(B) = m_{12}(A \cup B) = 0$.

1.11 The PCR5 rule

1.11.1 Principle of PCR5

Similarly to PCR2-4, PCR5 redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the canonical form of the partial conflict. PCR5 is the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule. But this is harder to implement. PCR5 satisfies the neutrality property of VBA also. In order to understand the principle of PCR5, let's start with examples going from the easiest to the more complex one.

Proof of neutrality of VBA for PCR2-PCR5: PCR2, PCR3, PCR4 and PCR5 rules preserve the neutral impact of the VBA because in any partial conflict, as well in the total conflict which is a sum of all partial conflicts, the canonical form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict), therefore Θ gets no mass after the redistribution of the conflicting mass. This general proof for neutrality of VBA works in dynamic or static cases for all PCR2-5, since the total ignorance, say I_t, can not escape the conjunctive normal form, i.e. the canonical form of $I_t \cap A$ is A, where A is any set included in D^Θ.

1.11.1.1 A two sources example 1 for PCR5

Suppose one has the frame of discernment $\Theta = \{A, B\}$ of exclusive elements, and 2 sources of evidences providing the following bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0 \quad m_1(A \cup B) = 0.4$$

$$m_2(A) = 0 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.7$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.42 \quad m_{12}(B) = 0.12 \quad m_{12}(A \cup B) = 0.28$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$$

Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively. Let x be the conflicting mass to be redistributed to A, and y the conflicting mass redistributed to B, then

$$\frac{x}{0.6} = \frac{y}{0.3} = \frac{x+y}{0.6+0.3} = \frac{0.18}{0.9} = 0.2$$

whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$. Thus:

$$\begin{aligned} m_{PCR5}(A) &= 0.42 + 0.12 = 0.54 \\ m_{PCR5}(B) &= 0.12 + 0.06 = 0.18 \\ m_{PCR5}(A \cup B) &= 0.28 \end{aligned}$$

This result is equal to that of PCR3 and even PCR2, but different from PCR1 and PCR4 in this specific example. PCR1 and PCR4 yield:

$$m_{PCR1}(A) = 0.42 + \frac{0.6+0}{2} \cdot 0.18 = 0.474$$
$$m_{PCR1}(B) = 0.12 + \frac{0+0.3}{2} \cdot 0.18 = 0.147$$
$$m_{PCR1}(A \cup B) = 0.28 + \frac{0.4+0.7}{2} \cdot 0.18 = 0.379$$

$$m_{PCR4}(A) = 0.42 + 0.42 \cdot \frac{0.18}{0.42+0.12} = 0.56$$
$$m_{PCR4}(B) = 0.12 + 0.12 \cdot \frac{0.18}{0.12+0.42} = 0.16$$
$$m_{PCR4}(A \cup B) = 0.28$$

In summary, here are the results obtained from Dempster's rule (DS), (DSmH), (PCR1), (PCR4) and (PCR5):

	A	B	$A \cup B$
m_{DS}	0.512	0.146	0.342
m_{DSmH}	0.420	0.120	0.460
m_{PCR1}	0.474	0.147	0.379
m_{PCR4}	0.560	0.160	0.280
m_{PCR5}	0.540	0.180	0.280

1.11.1.2 A two sources example 2 for PCR5

Now let's modify a little the previous example and consider now:

$$m_1(A) = 0.6 \quad m_1(B) = 0 \quad m_1(A \cup B) = 0.4$$
$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.50 \quad m_{12}(B) = 0.12 \quad m_{12}(A \cup B) = 0.20$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$$

The conflict k_{12} is the same as in previous example, which means that $m_2(A) = 0.2$ did not have any impact on the conflict; why?, because $m_1(B) = 0$. Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively. The mass $m_2(A) = 0.2$ is not considered to the weighting factors of the redistribution. Let x be the conflicting mass to be redistributed to A, and y the conflicting mass redistributed to B. By the same calculations one has:

$$\frac{x}{0.6} = \frac{y}{0.3} = \frac{x+y}{0.6+0.3} = \frac{0.18}{0.9} = 0.2$$

whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$. Thus, one gets now:

$$\begin{aligned}
m_{PCR5}(A) &= 0.50 + 0.12 = 0.62\\
m_{PCR5}(B) &= 0.12 + 0.06 = 0.18\\
m_{PCR5}(A \cup B) &= 0.20 + 0 = 0.20
\end{aligned}$$

We did not take into consideration the sum of masses of column A, i.e. $m_1(A) + m_2(A) = 0.6 + 0.2 = 0.8$, since clearly $m_2(A) = 0.2$ has no impact on the conflicting mass.

In this second example, the result obtained by PCR5 is different from WAO, PCR1, PCR2, PCR3 and PCR4 because

$$\begin{aligned}
m_{WAO}(A) &= 0.50 + \frac{0.6+0.2}{2} \cdot 0.18 = 0.572\\
m_{WAO}(B) &= 0.12 + \frac{0+0.3}{2} \cdot 0.18 = 0.147\\
m_{WAO}(A \cup B) &= 0.20 + \frac{0.4+0.5}{2} \cdot 0.18 = 0.281
\end{aligned}$$

$$\begin{aligned}
m_{PCR1}(A) &= 0.50 + \frac{0.6+0.2}{0.8+0.3+0.9} \cdot 0.18 = 0.572\\
m_{PCR1}(B) &= 0.12 + \frac{0+0.3}{0.8+0.3+0.9} \cdot 0.18 = 0.147\\
m_{PCR1}(A \cup B) &= 0.20 + \frac{0.4+0.5}{0.8+0.3+0.9} \cdot 0.18 = 0.281
\end{aligned}$$

$$\begin{aligned}
m_{PCR2}(A) &= 0.50 + \frac{0.6+0.2}{0.8+0.3} \cdot 0.18 \approx 0.631\\
m_{PCR2}(B) &= 0.12 + \frac{0+0.3}{0.8+0.3} \cdot 0.18 \approx 0.169\\
m_{PCR2}(A \cup B) &= 0.20
\end{aligned}$$

$$\begin{aligned}
m_{PCR3}(A) &= 0.50 + 0.8 \cdot [\frac{0.6 \cdot 0.3 + 0.2 \cdot 0}{0.8+0.3}] \approx 0.631\\
m_{PCR3}(B) &= 0.12 + 0.3 \cdot [\frac{0.6 \cdot 0.3 + 0.2 \cdot 0}{0.8+0.3}] \approx 0.169\\
m_{PCR3}(A \cup B) &= 0.20
\end{aligned}$$

$$\begin{aligned}
m_{PCR4}(A) &= 0.50 + 0.50 \cdot \frac{0.18}{0.50+0.12} \approx 0.645\\
m_{PCR4}(B) &= 0.12 + 0.12 \cdot \frac{0.18}{0.50+0.12} \approx 0.155\\
m_{PCR4}(A \cup B) &= 0.20
\end{aligned}$$

The results obtained with Dempster's rule (DS) and DSm Hybrid rule are:

$$m_{DS}(A) = 0.610$$
$$m_{DS}(B) = 0.146$$
$$m_{DS}(A \cup B) = 0.244$$

$$m_{DSmH}(A) = 0.500$$
$$m_{DSmH}(B) = 0.120$$
$$m_{DSmH}(A \cup B) = 0.380$$

Let's examine from this example the convergence of the PCR5 result by introducing a small positive increment on $m_1(B)$, i.e. one starts now with the PCR5 combination of the following bba's

$$m_1(A) = 0.6 \qquad m_1(B) = \epsilon \qquad m_1(A \cup B) = 0.4 - \epsilon$$
$$m_2(A) = 0.2 \qquad m_2(B) = 0.3 \qquad m_2(A \cup B) = 0.5$$

Then the conjunctive consensus yields: $m_{12}(A) = 0.50 - 0.2 \cdot \epsilon$, $m_{12}(B) = 0.12 + 0.5 \cdot \epsilon$, $m_{12}(A \cup B) = 0.20 - 0.5 \cdot \epsilon$ with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18 + 0.2 \cdot \epsilon$$

Applying the PCR5 rule for $\epsilon = 0.1$, $\epsilon = 0.01$,$\epsilon = 0.001$ and $\epsilon = 0.0001$ one gets the following result:

ϵ	$m_{PCR5}(A)$	$m_{PCR5}(B)$	$m_{PCR5}(A \cup B)$
0.1	0.613333	0.236667	0.15
0.01	0.619905	0.185095	0.195
0.001	0.619999	0.180501	0.1995
0.0001	0.62	0.180050	0.19995

Table 1.8: Convergence of PCR5

From Table 1.8, one can see that when ϵ tend towards zero, the results tends towards the previous result $m_{PCR5}(A) = 0.62$, $m_{PCR5}(B) = 0.18$ and $m_{PCR5}(A \cup B) = 0.20$. Let's explain now in details how this limit can be achieved formally. With PCR5, one redistributes the partial conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively, and also the partial conflicting mass $0.2 \cdot \epsilon$ to A and B proportionally with the masses $m_2(A)$ and $m_1(B)$ assigned to A and B respectively, thus one gets now two weighting factors in the redistribution for each corresponding set A and B. Let x_1 be the conflicting mass to be redistributed to A, and y_1 the conflicting mass redistributed to B from the first partial conflicting mass 0.18. This first partial proportional redistribution is then done according

$$\frac{x_1}{0.6} = \frac{y_1}{0.3} = \frac{x_1 + y_1}{0.6 + 0.3} = \frac{0.18}{0.9} = 0.2$$

whence $x_1 = 0.6 \cdot 0.2 = 0.12$, $y_1 = 0.3 \cdot 0.2 = 0.06$. Now let x_2 be the conflicting mass to be redistributed to A, and y_2 the conflicting mass redistributed to B from the second partial conflicting mass $0.2 \cdot \epsilon$. This first partial proportional redistribution is then done according

$$\frac{x_2}{0.2} = \frac{y_2}{\epsilon} = \frac{x_2 + y_2}{0.2 + \epsilon} = \frac{0.2 \cdot \epsilon}{0.2 + \epsilon}$$

whence $x_2 = 0.2 \cdot \frac{0.2 \cdot \epsilon}{0.2+\epsilon}$, $y_2 = \epsilon \frac{0.2 \cdot \epsilon}{0.2+\epsilon}$. Thus one gets the following result

$$\begin{aligned}
m_{PCR5}(A) &= m_{12}(A) + x_1 + x_2 = (0.50 - 0.2 \cdot \epsilon) + 0.12 + 0.2 \cdot \frac{0.2 \cdot \epsilon}{0.2 + \epsilon} \\
m_{PCR5}(B) &= m_{12}(B) + y_1 + y_2 = (0.12 + 0.5 \cdot \epsilon) + 0.06 + \epsilon \frac{0.2 \cdot \epsilon}{0.2 + \epsilon} \\
m_{PCR5}(A \cup B) &= m_{12}(A \cup B) = 0.20 - 0.5\epsilon
\end{aligned}$$

From these formal expressions of $m_{PCR5}(.)$, one sees directly that

$$\lim_{\epsilon \longrightarrow 0} m_{PCR5}(A) = 0.62 \qquad \lim_{\epsilon \longrightarrow 0} m_{PCR5}(B) = 0.18 \qquad \lim_{\epsilon \longrightarrow 0} m_{PCR5}(A \cup B) = 0.20$$

1.11.1.3 A two sources example 3 for PCR5

Let's go further modifying this time the previous example and considering:

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27 \quad m_{12}(A \cup B) = 0.05$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18 + 0.06 = 0.24$$

The conflict k_{12} is now different from the two previous examples, which means that $m_2(A) = 0.2$ and $m_1(B) = 0.3$ did make an impact on the conflict; why?, because $m_2(A)m_1(B) = 0.2 \cdot 0.3 = 0.06$ was added to the conflicting mass. Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the partial conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively, and also the partial conflicting mass 0.06 to A and B proportionally with the masses $m_2(A)$ and $m_1(B)$ assigned to A and B respectively, thus one gets two weighting factors of the redistribution for each corresponding set A and B respectively. Let x_1 be the conflicting mass to be redistributed to A, and y_1 the conflicting mass redistributed to B from the first partial conflicting mass 0.18. This first partial proportional redistribution is then done according

$$\frac{x_1}{0.6} = \frac{y_1}{0.3} = \frac{x_1 + y_1}{0.6 + 0.3} = \frac{0.18}{0.9} = 0.2$$

whence $x_1 = 0.6 \cdot 0.2 = 0.12$, $y_1 = 0.3 \cdot 0.2 = 0.06$. Now let x_2 be the conflicting mass to be redistributed to A, and y_2 the conflicting mass redistributed to B from second the partial conflicting mass 0.06. This second partial proportional redistribution is then done according

$$\frac{x_2}{0.2} = \frac{y_2}{0.3} = \frac{x_2 + y_2}{0.2 + 0.3} = \frac{0.06}{0.5} = 0.12$$

whence $x_2 = 0.2 \cdot 0.12 = 0.024$, $y_2 = 0.3 \cdot 0.12 = 0.036$. Thus:

$$\begin{aligned}
m_{PCR5}(A) &= 0.44 + 0.12 + 0.024 = 0.584\\
m_{PCR5}(B) &= 0.27 + 0.06 + 0.036 = 0.366\\
m_{PCR5}(A \cup B) &= 0.05 + 0 = 0.05
\end{aligned}$$

The result is different from PCR1, PCR2, PCR3 and PCR4 since one has[14]:

$$\begin{aligned}
m_{PCR1}(A) &= 0.536\\
m_{PCR1}(B) &= 0.342\\
m_{PCR1}(A \cup B) &= 0.122
\end{aligned}$$

$$\begin{aligned}
m_{PCR2}(A) &= m_{PCR3}(A) \approx 0.577\\
m_{PCR2}(B) &= m_{PCR3}(B) \approx 0.373\\
m_{PCR2}(A \cup B) &= m_{PCR3}(A \cup B) = 0.05
\end{aligned}$$

$$\begin{aligned}
m_{PCR4}(A) &\approx 0.589\\
m_{PCR4}(B) &\approx 0.361\\
m_{PCR4}(A \cup B) &= 0.05
\end{aligned}$$

Dempster's rule (DS) and DSm Hybrid rule (DSmH), give for this example:

$$m_{DS}(A) = \frac{0.44}{1 - 0.24} \approx 0.579 \quad m_{DS}(B) = \frac{0.27}{1 - 0.24} \approx 0.355 \quad m_{DS}(A \cup B) = \frac{0.05}{1 - 0.24} \approx 0.066$$

$$m_{DSmH}(A) = 0.440 \quad m_{DSmH}(B) = 0.270 \quad m_{DSmH}(A \cup B) = 0.290$$

One clearly sees that $m_{DS}(A \cup B)$ gets some mass from the conflicting mass although $A \cup B$ does not deserve any part of the conflicting mass since $A \cup B$ is not involved in the conflict (only A and B are involved in the conflicting mass). Dempster's rule appears to authors opinions less exact than PCR5 and Inagaki's rules [10] because it redistribute less exactly the conflicting mass than PCR5, even than PCR4 and minC, since Dempter's rule takes the total conflicting mass and redistributes it to all non-empty sets, even those not involved in the conflict. It can be shown [9] that Inagaki's fusion rule [10] (with an optimal choice of tuning parameters) can become in some cases very close to (PCR5) but upon our opinion (PCR5) result is more exact (at least less ad-hoc than Inagaki's one).

[14] The verification is left to the reader.

1.11.2 The PCR5 formula

Before explaining the general procedure to apply for PCR5 (see next section), we give here the PCR5 formula for $s = 2$ sources: $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \tag{1.32}$$

where all sets involved in the formula are in canonical form, $m_{12}(.)$ corresponds to the conjunctive consensus, i.e. $m_{12}(X) \triangleq \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)$ and where all denominators are *different from zero.* If a denominator is zero, that fraction is discarded.

Let $G = \{X_1, \ldots, X_n\} \neq \emptyset$ (G^Θ being either the power-set or hyper-power set depending on the model we want to deal with), $n \geq 2$, the general PCR5 formula for $s \geq 2$ sources is given by $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12\ldots s}(X) + \sum_{\substack{2 \leq t \leq s \\ 1 \leq r_1, \ldots, r_t \leq s \\ 1 \leq r_1 < r_2 < \ldots < r_{t-1} < (r_t = s)}} \sum_{\substack{X_{j_2}, \ldots, X_{j_t} \in G^\Theta \setminus \{X\} \\ \{j_2, \ldots, j_t\} \in \mathcal{P}^{t-1}(\{1, \ldots, n\}) \\ X \cap X_{j_2} \cap \ldots \cap X_{j_s} = \emptyset \\ \{i_1, \ldots, i_s\} \in \mathcal{P}^s(\{1, \ldots, s\})}}$$
$$\frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)^2) \cdot [\prod_{l=2}^{t} (\prod_{k_l = r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)) + [\sum_{l=2}^{t} (\prod_{k_l = r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]} \tag{1.33}$$

where i, j, k, r, s and t in (1.33) are integers. $m_{12\ldots s}(X)$ corresponds to the conjunctive consensus on X between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; $\mathcal{P}^k(\{1, 2, \ldots, n\})$ is the set of all subsets of k elements from $\{1, 2, \ldots, n\}$ (permutations of n elements taken by k), the order of elements doesn't count.

Let's prove here that (1.33) reduces to (1.32) when $s = 2$. Indeed, if one takes $s = 2$ in general PCR5 formula (1.33), let's note first that:

- $2 \leq t \leq s$ becomes $2 \leq t \leq 2$, thus $t = 2$.
- $1 \leq r_1, r_2 \leq (s = 2)$, or $r_1, r_2 \in \{1, 2\}$, but because $r_1 < r_2$ one gets $r_1 = 1$ and $r_2 = 2$.
- $m_{12\ldots s}(X)$ becomes $m_{12}(X)$
- $X_{j_2}, \ldots, X_{j_t} \in G^\Theta \setminus \{X\}$ becomes $X_{j_2} \in G^\Theta \setminus \{X\}$ because $t = 2$.
- $\{j_2, \ldots, j_t\} \in \mathcal{P}^{t-1}(\{1, \ldots, n\})$ becomes $j_2 \in \mathcal{P}^1(\{1, \ldots, n\}) = \{1, \ldots, n\}$
- the condition $X \cap X_{j_2} \cap \ldots \cap X_{j_s} = \emptyset$ becomes $X \cap X_{j_2} = \emptyset$
- $\{i_1, \ldots, i_s\} \in \mathcal{P}^s(\{1, \ldots, s\})$ becomes $\{i_1, i_2\} \in \mathcal{P}^2(\{1, 2\}) = \{\{1, 2\}, \{2, 1\}\}$

Thus (1.33) becomes when $s = 2$,

$$m_{PCR5}(X) = m_{12}(X) +$$

$$\sum_{\substack{t=2\\ r_1=1,r_2=2}} \sum_{\substack{X_{j_2}\in G^\Theta\setminus\{X\}\\ j_2\in\{1,\ldots,n\}\\ X\cap X_{j_2}=\emptyset\\ \{i_1,i_2\}\in\{\{1,2\},\{2,1\}\}}} \frac{(\prod_{k_1=1}^{1} m_{i_{k_1}}(X)^2)\cdot[\prod_{l=2}^{2}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{1} m_{i_{k_1}}(X))+[\sum_{l=2}^{2}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]} \tag{1.34}$$

After elementary algebraic simplification, it comes

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{X_{j_2}\in G^\Theta\setminus\{X\}\\ j_2\in\{1,\ldots,n\}\\ X\cap X_{j_2}=\emptyset\\ \{i_1,i_2\}\in\{\{1,2\},\{2,1\}\}}} \frac{m_{i_1}(X)^2\cdot[\prod_{k_2=2}^{2} m_{i_{k_2}}(X_{j_2})]}{m_{i_1}(X)+[\prod_{k_2=2}^{2} m_{i_{k_2}}(X_{j_2})]} \tag{1.35}$$

Since $\prod_{k_2=2}^{2} m_{i_{k_2}}(X_{j_2}) = m_{i_2}(X_{j_2})$ and condition "$X_{j_2}\in G^\Theta\setminus\{X\}$ and $j_2\in\{1,\ldots,n\}$" are equivalent to $X_{j_2}\in G^\Theta\setminus\{X\}$, one gets:

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{X_{j_2}\in G^\Theta\setminus\{X\}\\ X\cap X_{j_2}=\emptyset\\ \{i_1,i_2\}\in\{\{1,2\},\{2,1\}\}}} \frac{m_{i_1}(X)^2\cdot m_{i_2}(X_{j_2})}{m_{i_1}(X)+m_{i_2}(X_{j_2})} \tag{1.36}$$

This formula can also be written as (denoting X_{j_2} as Y)

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y\in G^\Theta\setminus\{X\}\\ X\cap Y=\emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}] \tag{1.37}$$

which is the same as formula (1.32). Thus the proof is completed.

1.11.3 The PCR5 formula for Bayesian beliefs assignments

For $\Theta = \{\theta_1,\theta_2,\ldots,\theta_n\}$ with Shafer's model and $s=2$ Bayesian equally reliable sources, i.e. when quantitative bba's $m_1(.)$ and $m_2(.)$ reduce to subjective probability measures $P_1(.)$ and $P_2(.)$, after elementary algebraic derivations, the (PCR5) formula for combination of two sources reduces to the following simple formula, $P_{12}^{PCR5}(\emptyset)=0$ and $\forall\theta_i\in\Theta$,

$$\begin{aligned} P_{12}^{PCR5}(\theta_i) &= P_1(\theta_i)\sum_{j=1}^{n}\frac{P_1(\theta_i)P_2(\theta_j)}{P_1(\theta_i)+P_2(\theta_j)} + P_2(\theta_i)\sum_{j=1}^{n}\frac{P_2(\theta_i)P_1(\theta_j)}{P_2(\theta_i)+P_1(\theta_j)}\\ &= \sum_{s=1,2} P_s(\theta_i)[\sum_{j=1}^{n}\frac{P_s(\theta_i)P_{s'\neq s}(\theta_j)}{P_s(\theta_i)+P_{s'\neq s}(\theta_j)}] \end{aligned} \tag{1.38}$$

This formula can be extended for $s>2$ sources. One can verify moreover that $P_{12}^{PCR5}(.)$ defines a subjective-combined probability measure satisfying all axioms of classical Probability Theory.

Proof: From (1.36), when replacing general bba $m_1(.)$ and $m_2(.)$ by probabilistic masses $P_1(.)$ and $P_2(.)$ one gets:

$$P_{12}(x_i) = P_1(x_i)P_2(x_i) + P_1(x_i)\sum_{j\neq i}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)} + P_2(x_i)\sum_{j\neq i}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}$$

By splitting $P_1(x_i)P_2(x_i)$ into two equal parts, one gets

$$P_{12}(x_i) = \frac{1}{2}P_1(x_i)P_2(x_i) + P_1(x_i)\sum_{j\neq i}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)} + \frac{1}{2}P_1(x_i)P_2(x_i) + P_2(x_i)\sum_{j\neq i}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}$$

$$P_{12}(x_i) = P_1(x_i)[\frac{1}{2}P_2(x_i) + \sum_{j\neq i}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_2(x_i)[\frac{1}{2}P_1(x_i) + \sum_{j\neq i}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}]$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)} - \frac{P_1(x_i)P_2(x_i)}{P_1(x_i)+P_2(x_i)} + \frac{1}{2}P_2(x_i)]$$
$$+ P_2(x_i)[\sum_{j=1}^{n}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)} - \frac{P_2(x_i)P_1(x_i)}{P_2(x_i)+P_1(x_i)} + \frac{1}{2}P_1(x_i)]$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)} - \frac{2P_1(x_i)P_2(x_i)}{2(P_1(x_i)+P_2(x_i))} + \frac{P_2(x_i)(P_1(x_i)+P_2(x_i))}{2(P_1(x_i)+P_2(x_i))}]$$
$$+ P_2(x_i)[\sum_{j=1}^{n}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)} - \frac{2P_2(x_i)P_1(x_i)}{2(P_2(x_i)+P_1(x_i))} + \frac{P_1(x_i)(P_2(x_i)+P_1(x_i))}{2(P_2(x_i)+P_1(x_i))}]$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_1(x_i)[\frac{P_2^2(x_i) - P_1(x_i)P_2(x_i)}{2(P_1(x_i)+P_2(x_i))}]$$
$$+ P_2(x_i)[\sum_{j=1}^{n}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}] + P_2(x_i)[\frac{P_1^2(x_i) - P_2(x_i)P_1(x_i)}{2(P_2(x_i)+P_1(x_i))}]$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_2(x_i)[\sum_{j=1}^{n}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}]$$
$$+ \frac{P_1(x_i)P_2^2(x_i) - P_1^2(x_i)P_2(x_i)}{2(P_1(x_i)+P_2(x_i))} + \frac{P_2(x_i)P_1^2(x_i) - P_2^2(x_i)P_1(x_i)}{2(P_2(x_i)+P_1(x_i))}$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n}\frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_2(x_i)[\sum_{j=1}^{n}\frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}]$$
$$+ \frac{P_1(x_i)P_2^2(x_i) - P_1^2(x_i)P_2(x_i) + P_2(x_i)P_1^2(x_i) - P_2^2(x_i)P_1(x_i)}{2(P_1(x_i)+P_2(x_i))}$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n} \frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_2(x_i)[\sum_{j=1}^{n} \frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}] + \frac{0}{2(P_1(x_i)+P_2(x_i))}$$

$$P_{12}(x_i) = P_1(x_i)[\sum_{j=1}^{n} \frac{P_1(x_i)P_2(x_j)}{P_1(x_i)+P_2(x_j)}] + P_2(x_i)[\sum_{j=1}^{n} \frac{P_2(x_i)P_1(x_j)}{P_2(x_i)+P_1(x_j)}]$$

which completes the proof. □□□

More concisely, the formula (1.38) can be rewritten as:

$$P_{12}(x_i) = \sum_{s=1,2} P_s(x_i)[\sum_{j=1}^{n} \frac{P_s(x_i)P_{s'\neq s}(x_j)}{P_s(x_i)+P_{s'\neq s}(x_j)}] \tag{1.39}$$

1.11.4 General procedure to apply the PCR5

Here is the general procedure to apply PCR5:

1. apply the conjunctive rule;
2. calculate all partial conflicting masses separately;
3. if $A \cap B = \emptyset$ then A, B are involved in the conflict; redistribute the mass $m_{12}(A \cap B) > 0$ to the non-empty sets A and B proportionally with respect to
   a) the non-zero masses $m_1(A)$ and $m_2(B)$ respectively,
   b) the non-zero masses $m_2(A)$ and $m_1(B)$ respectively, and
   c) other non-zero masses that occur in some products of the sum of $m_{12}(A \cap B)$;
4. if both sets A and B are empty, then the transfer is forwarded to the disjunctive form $u(A) \cup u(B)$, and if this disjunctive form is also empty, then the transfer is forwarded to the total ignorance in a closed world (or to the empty set if the open world approach is preferred); but if even the total ignorance is empty one considers an open world (i.e. new hypotheses might exist) and the transfer is forwarded to the empty set; if say $m_1(A) = 0$ or $m_2(B) = 0$, then the product $m_1(A)m_2(B) = 0$ and thus there is no conflicting mass to be transferred from this product to non-empty sets; if both products $m_1(A)m_2(B) = m_2(A)m_1(B) = 0$ then there is no conflicting mass to be transferred from them to non-empty sets; in a general case[15] , for $s \geq 2$ sources, the mass $m_{12\ldots s}(A_1 \cap A_2 \cap \ldots \cap \ldots A_r) > 0$, with $2 \leq r \leq s$, where $A_1 \cap A_2 \cap \ldots \cap A_r = \emptyset$, resulted from the application of the conjunctive rule, is a sum of many products; each non-zero particular product is proportionally redistributed to A_1, A_2, …, A_r with respect to the sub-products of masses assigned to A_1, A_2, …, A_r respectively by the sources; if both sets A_1, A_2, …, A_r are

[15] An easier calculation method, denoted *PCR5-approximate* for $s \geq 3$ bba's, which is an *approximation of PCR5*, is to first combine $s-1$ bba's altogether using the conjunctive rule, and the result to be again combined once more with the s-th bba also using the conjunctive rule; then the weighting factors will only depend on $m_{12\ldots(s-1)}(.)$ and $m_s(.)$ only - instead of depending on all bba's $m_1(.)$, $m_2(.)$, …, $m_s(.)$. PCR5-approximate result however depends on the chosen order of the sources.

empty, then the transfer is forwarded to the disjunctive form $u(A_1) \cup u(A_2) \cup \ldots \cup u(A_r)$, and if this disjunctive form is also empty, then the transfer is forwarded to the total ignorance in a closed world (or to the empty set if the open world approach is preferred); but if even the total ignorance is empty one considers an open world (i.e. new hypotheses might exist) and the transfer is forwarded to the empty set;

5. and so on until all partial conflicting masses are redistributed;
6. add the redistributed conflicting masses to each corresponding non-empty set involved in the conflict;
7. the sets not involved in the conflict do not receive anything from the conflicting masses (except some partial or total ignorances in degenerate cases).

The more hypotheses and more masses are involved in the fusion, the more difficult is to implement PCR5. Yet, it is easier to *approximate PCR5* by first combining $s-1$ bba's through the conjunctive rule, then by combining again the result with the s-th bba also using the conjunctive rule – in order to reduce very much the calculations of the redistribution of conflicting mass.

1.11.5 A 3-source example for PCR5

Let's see a more complex example using PCR5. Suppose one has the frame of discernment $\Theta = \{A, B\}$ of exclusive elements, and 3 sources such that:

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

$$m_3(A) = 0.4 \quad m_3(B) = 0.4 \quad m_3(A \cup B) = 0.2$$

Then the conjunctive consensus yields : $m_{123}(A) = 0.284$, $m_{123}(B) = 0.182$ and $m_{123}(A \cup B) = 0.010$ with the conflicting mass $k_{123} = m_{123}(A \cap B) = 0.524$, which is a sum of factors.

1. *Fusion based on PCR5*:

   In the long way, each product occurring as a term in the sum of the conflicting mass should be redistributed to the non-empty sets involved in the conflict proportionally to the masses (or sub-product of masses) corresponding to the respective non-empty set. For example, the product $m_1(A)m_3(B)m_2(A \cup B) = 0.6 \cdot 0.4 \cdot 0.5 = 0.120$ occurs in the sum of k_{123}, then 0.120 is proportionally distributed to the sets involved in the conflict; because $c(A \cap B \cap (A \cup B)) = A \cap B$ the transfer is done to A and B with respect to 0.6 and 0.4. Whence:

   $$\frac{x}{0.6} = \frac{y}{0.4} = \frac{0.12}{0.6 + 0.4}$$

   whence $x = 0.6 \cdot 0.12 = 0.072$, $y = 0.4 \cdot 0.12 = 0.048$, which will be added to the masses of A and B respectively. Another example, the product $m_2(A)m_1(B)m_3(B) = 0.2 \cdot 0.3 \cdot 0.4 = 0.024$ occurs in the sum of k_{123}, then 0.024 is proportionally distributed to A, B with respect to 0.20 and $0.3 \cdot 0.4 = 0.12$ respectively. Whence:

   $$\frac{x}{0.20} = \frac{y}{0.12} = \frac{0.024}{0.32} = 0.075$$

whence $x = 0.20 \cdot \frac{0.024}{0.32} = 0.015$ and $y = 0.12 \cdot \frac{0.024}{0.32} = 0.009$, which will be added to the masses of A, and B respectively.

But this procedure is more difficult, that's why we can use the following crude approach:

2. *Fusion based on PCR5-approximate*:

   If s sources are involved in the fusion, then first combine using the conjunctive rule $s-1$ sources, and the result will be combined with the remaining source.

We resolve now this 3-source example by combining the first two sources

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_2(B) = 0.3 \quad m_2(A \cup B) = 0.5$$

with the DSm classic rule (i.e. the conjunctive consensus on hyper-power set D^Θ) to get

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27$$

$$m_{12}(A \cup B) = 0.05 \quad m_{12}(A \cap B) = 0.24$$

Then one combines $m_{12}(.)$ with $m_3(.)$ still with the DSm classic rule and one gets as preliminary step for PCR5-version b just above-mentioned

$$m_{123}(A) = 0.284 \qquad m_{123}(B) = 0.182$$

$$m_{123}(A \cup B) = 0.010 \quad m_{123}(A \cap B) = 0.524$$

The conflicting mass has been derived from

$$\begin{aligned} m_{123}(A \cap B) = {} & [m_{12}(A)m_3(B) + m_3(A)m_{12}(B)] + [m_3(A)m_{12}(A \cap B) + m_3(B)m_{12}(A \cap B) \\ & + m_3(A \cup B)m_{12}(A \cap B)] \\ = {} & [0.44 \cdot 0.4 + 0.4 \cdot 0.27] + [0.4 \cdot 0.24 + 0.4 \cdot 0.24 + 0.2 \cdot 0.24] = 0.524 \end{aligned}$$

But in the last brackets $A \cap B = \emptyset$, therefore the masses of $m_3(A)m_{12}(A \cap B) = 0.096$, $m_3(B)m_{12}(A \cap B) = 0.096$, and $m_3(A \cap B)m_{12}(A \cap B) = 0.048$ are transferred to A, B, and $A \cup B$ respectively. In the first brackets, $0.44 \cdot 0.4 = 0.176$ is transferred to A and B proportionally to 0.44 and 0.4 respectively:

$$\frac{x}{0.44} = \frac{y}{0.40} = \frac{0.176}{0.84}$$

whence

$$x = 0.44 \cdot \frac{0.176}{0.84} = 0.09219 \qquad y = 0.40 \cdot \frac{0.176}{0.84} = 0.08381$$

Similarly, $0.4 \cdot 0.27 = 0.108$ is transferred to A and B proportionally to 0.40 and 0.27 and one gets:

$$\frac{x}{0.40} = \frac{y}{0.27} = \frac{0.108}{0.67}$$

whence

$$x = 0.40 \cdot \frac{0.108}{0.67} = 0.064478 \qquad y = 0.27 \cdot \frac{0.108}{0.67} = 0.043522$$

Adding all corresponding masses, one gets the final result with PCR5 (version b), denoted here with index $PCR5b|\{12\}3$ to emphasize that one has applied the version b) of PCR5 for the combination of the 3 sources by combining first the sources 1 and 2 together :

$$m_{PCR5b|\{12\}3}(A) = 0.536668 \qquad m_{PCR5b|\{12\}3}(B) = 0.405332 \qquad m_{PCR5b|\{12\}3}(A\cup B) = 0.058000$$

1.11.6 On the neutral impact of VBA for PCR5

Let's take again the example given in section 1.11.1.3 with $\Theta = \{A, B\}$, Shafer's model and the two bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.2 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.5$$

Then the conjunctive consensus yields :

$$m_{12}(A) = 0.44 \quad m_{12}(B) = 0.27 \quad m_{12}(A \cup B) = 0.05$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18 + 0.06 = 0.24$$

The canonical form $c(A\cap B) = A\cap B$, thus $m_{12}(A\cap B) = 0.18+0.06 = 0.24$ will be distributed to A and B only proportionally with respect to their corresponding masses assigned by $m1(.)$ and $m2(.)$, i.e: 0.18 redistributed to A and B proportionally with respect to 0.6 and 0.3 respectively, and 0.06 redistributed to A and B proportionally with respect to 0.2 and 0.3 respectively. One gets as computed above (see also section 1.11.1.3):

$$m_{PCR5|12}(A) = 0.584 \qquad m_{PCR5|12}(B) = 0.366 \qquad m_{PCR5|12}(A \cup B) = 0.05$$

Now let's introduce a third and vacuous belief assignment $m_v(A\cup B) = 1$ and combine altogether $m_1(.)$, $m_2(.)$ and $m_v(.)$ with the conjunctive consensus. One gets

$$m_{12v}(A) = 0.44 \quad m_{12v}(B) = 0.27 \qquad m_{12v}(A \cup B) = 0.05 \qquad m_{12v}(A \cap B \cap (A \cup B)) = 0.24$$

Since the canonical form $c(A\cap B\cap(A\cup B)) = A\cap B$, $m_{12v}(A\cap B\cap(A\cup B)) = 0.18+0.06 = 0.24$ will be distributed to A and B only (therefore nothing to $A \cup B$) proportionally with respect to their corresponding masses assigned by $m_1(.)$ and $m_2(.)$ (because $m_v(.)$ is not involved since all its masses assigned to A and B are zero: $m_v(A) = m_v(B) = 0$), i.e: 0.18 redistributed to A and B proportionally with respect to 0.6 and 0.3 respectively, and 0.06 redistributed to A and B proportionally with respect to 0.2 and 0.3 respectively, therefore exactly as above. Thus

$$m_{PCR5|12v}(A) = 0.584 \qquad m_{PCR5|12v}(B) = 0.366 \qquad m_{PCR5|12v}(A \cup B) = 0.05$$

In this example one sees that the neutrality property of VBA is effectively well satisfied by PCR5 rule since

$$m_{PCR5|12v}(.) = m_{PCR5|12}(.)$$

A general proof for neutrality of VBA within PCR5 is given in section 1.11.1.

1.11.7 PCR6 as alternative to PCR5 when $s > 2$

In this volume, Arnaud Martin and Christophe Osswald have proposed the following alternative rule to PCR5 for combining more than two sources altogether (i.e. $s \geq 3$). This new rule denoted PCR6 does not follow back on the track of conjunctive rule as PCR5 general formula does, but it gets better intuitive results. For $s = 2$ PCR5 and PCR6 coincide. The general formula for PCR6[16] is:

$$m_{PCR6}(\emptyset) = 0,$$

and $\forall A \in G^\Theta \setminus \emptyset$

$$\begin{aligned} m_{\text{PCR6}}(A) &= m_{12\ldots s}(A) + \\ & \sum_{i=1}^{s} m_i(A)^2 \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)} \cap A \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(s-1)}) \in (G^\Theta)^{s-1}}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \end{aligned}$$

with $m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$ and where $m_{12\ldots s}(.)$ is the conjunctive consensus rule and σ_i counts from 1 to s avoiding i, i.e.:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j + 1 & \text{if } j \geq i, \end{cases}$$

A detailed presentation of PCR6 and application of this rule can be found in Chapters 2 and 11.

1.11.8 Imprecise PCR5 fusion rule (imp-PCR5)

The (imp-PCR5) formula is a direct extension of (PCR5) formula (1.33) using addition, multiplication and division operators on sets [18]. It is given for the combination of $s \geq 2$ sources by $m^I_{PCR5}(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$:

$$\begin{aligned} m^I_{PCR5}(X) = [& \boxed{\sum}_{\substack{X_1, X_2, \ldots, X_s \in G^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_s) = X}} \boxed{\prod}_{i=1,\ldots,s} m^I_i(X_i)] \\ \boxplus [& \boxed{\sum}_{\substack{2 \leq t \leq s \\ 1 \leq r_1, \ldots, r_t \leq s \\ 1 \leq r_1 < r_2 < \ldots < r_{t-1} < (r_t = s)}} \boxed{\sum}_{\substack{X_{j_2}, \ldots, X_{j_t} \in G^\Theta \setminus \{X\} \\ \{j_2, \ldots, j_t\} \in \mathcal{P}^{t-1}(\{1, \ldots, n\}) \\ X \cap X_{j_2} \cap \ldots \cap X_{j_s} = \emptyset \\ \{i_1, \ldots, i_s\} \in \mathcal{P}^s(\{1, \ldots, s\})}} [Num^I(X) \boxslash Den^I(X)]] \end{aligned} \qquad (1.40)$$

[16]Two extensions of PCR6 (i.e. PCR6f and PCR6g) are also proposed by A. Martin and C. Osswald in [13].

where all sets are in canonical form and where $Num^I(X)$ and $Den^I(X)$ are defined by

$$Num^I(X) \triangleq [\boxed{\prod}_{k_1=1,\ldots,r_1} m^I_{i_{k_1}}(X)^2] \boxdot [\boxed{\prod}_{l=2,\ldots,t} (\boxed{\prod}_{k_l=r_{l-1}+1,\ldots,r_l} m^I_{i_{k_l}}(X_{j_l})] \tag{1.41}$$

$$Den^I(X) \triangleq [\boxed{\prod}_{k_1=1,\ldots,r_1} m^I_{i_{k_1}}(X)] \boxplus [\boxed{\sum}_{l=2,\ldots,t} (\boxed{\prod}_{k_l=r_{l-1}+1,\ldots,r_l} m^I_{i_{k_l}}(X_{j_l})] \tag{1.42}$$

where all denominators-sets $Den^I(X)$ involved in (1.40) are different from zero. If a denominator-set $Den^I(X)$ is such that $\inf(Den^I(X)) = 0$, then the fraction is discarded. When $s = 2$ (fusion of only two sources), the previous (imp-PCR5) formula reduces to its simple following fusion formula: $m^I_{PCR5}(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$\begin{aligned} m^I_{PCR5}(X) = m^I_{12}(X) + & \\ & \boxed{\sum}_{\substack{Y\in G^\Theta\setminus\{X\}\\ X\cap Y=\emptyset}} [(m^I_1(X)^2 m^I_2(Y)) \boxslash (m^I_1(X) + m^I_2(Y))] \boxplus \\ & [(m^I_2(X)^2 m^I_1(Y)) \boxslash (m^I_2(X) + m^I_1(Y))] \end{aligned} \tag{1.43}$$

with

$$m^I_{12}(X) \triangleq \boxed{\sum}_{\substack{X_1,X_2\in G^\Theta\\ X_1\cap X_2=X}} m^I_1(X_1) \boxdot m^I_2(X_2)$$

1.11.9 Examples for imprecise PCR5 (imp-PCR5)

Example no 1:

Let's consider $\Theta = \{\theta_1, \theta_2\}$, Shafer's model and two independent sources with the same imprecise admissible bba as those given in the table below, i.e.

$m^I_1(\theta_1) = [0.1, 0.2] \cup \{0.3\}$	$m^I_1(\theta_2) = (0.4, 0.6) \cup [0.7, 0.8]$
$m^I_2(\theta_1) = [0.4, 0.5]$	$m^I_2(\theta_2) = [0, 0.4] \cup \{0.5, 0.6\}$

Working with sets, one gets for the conjunctive consensus

$$m^I_{12}(\theta_1) = [0.04, 0.10] \cup [0.12, 0.15] \qquad m^I_{12}(\theta_2) = [0, 0.40] \cup [0.42, 0.48]$$

while the conflicting imprecise mass is given by

$$k^I_{12} \equiv m^I_{12}(\theta_1 \cap \theta_2) = [m^I_1(\theta_1) \boxdot m^I_2(\theta_2)] \boxplus [m^I_1(\theta_2) \boxdot m^I_2(\theta_1)] = (0.16, 0.58]$$

Using the PCR5 rule for Proportional Conflict redistribution,

- one redistributes the partial imprecise conflicting mass $m_1^I(\theta_1) \boxdot m_2^I(\theta_2)$ to θ_1 and θ_2 proportionally to $m_1^I(\theta_1)$ and $m_2^I(\theta_2)$. Using the fraction bar symbol instead of $\boxslash$ for convenience to denote the division operator on sets, one has

$$\begin{aligned}
\frac{x_1^I}{[0.1,0.2]\cup\{0.3\}} = \frac{y_1^I}{[0,0.4]\cup\{0.5,0.6\}} &= \frac{([0.1,0.2]\cup\{0.3\})\boxdot([0,0.4]\cup\{0.5,0.6\})}{([0.1,0.2]\cup\{0.3\})\boxplus([0,0.4]\cup\{0.5,0.6\})}\\
&= \big[[0,0.08]\cup[0.05,0.10]\cup[0.06,0.12]\cup[0,0.12]\cup\{0.15,0.18\}\big]\\
&\quad \boxslash \big[[0.1,0.6]\cup[0.6,0.7]\cup[0.7,0.8]\cup[0.3,0.7]\cup\{0.8,0.9\}\big]\\
&= \frac{[0,0.12]\cup\{0.15,0.18\}}{[0.1,0.8]\cup\{0.9\}}
\end{aligned}$$

whence

$$\begin{aligned}
x_1^I &= [\frac{[0,0.12]\cup\{0.15,0.18\}}{[0.1,0.8]\cup\{0.9\}}]\boxdot([0.1,0.2]\cup\{0.3\})\\
&= \frac{[0,0.024]\cup[0.015,0.030]\cup[0.018,0.036]\cup[0,0.036]\cup\{0.045,0.048\}}{[0.1,0.8]\cup\{0.9\}}\\
&= \frac{[0,0.036]\cup\{0.045,0.048\}}{[0.1,0.8]\cup\{0.9\}}\\
&= [\frac{0}{0.8},\frac{0.036}{0.1}]\cup[\frac{0}{0.9},\frac{0.036}{0.9}]\cup[\frac{0.045}{0.8},\frac{0.045}{0.1}]\cup[\frac{0.048}{0.8},\frac{0.048}{0.1}]\\
&= [0,0.36]\cup[0,0.04]\cup[0.05625,0.45000]\cup[0.06,0.48] = [0,0.48]
\end{aligned}$$

$$\begin{aligned}
y_1^I &= [\frac{[0,0.12]\cup\{0.15,0.18\}}{[0.1,0.8]\cup\{0.9\}}]\boxdot(0,0.4]\cup\{0.5,0.6\})\\
&= \big[[0,0.048]\cup[0,0.060]\cup[0,0.072]\cup[0,0.6]\cup[0,0.072]\\
&\quad \cup\{0,075,0.090,0.090,0.108\}\big]\boxslash[0.1,0.8]\cup\{0.9\}\\
&= \frac{[0,0.072]\cup\{0,075,0.090,0.108\}}{[0.1,0.8]\cup\{0.9\}}\\
&= [\frac{0}{0.8},\frac{0.072}{0.1}]\cup[\frac{0}{0.9},\frac{0.072}{0.9}]\cup[\frac{0.075}{0.8},\frac{0.075}{0.1}]\\
&\quad \cup[\frac{0.090}{0.8},\frac{0.090}{0.1}]\cup[\frac{0.108}{0.8},\frac{0.108}{0.1}]\cup\{\frac{0.075}{0.9},\frac{0.090}{0.9},\frac{0.108}{0.9}\}\\
&= [0,0.72]\cup[0,0.08]\cup[0.09375,0.75]\cup[0.1125,0.9]\cup[0.135,1.08]\\
&\quad \cup\{0.083333,0.1,0.12\}\\
&= [0,1.08]\approx[0,1]
\end{aligned}$$

- one redistributes the partial imprecise conflicting mass $m_1^I(\theta_2) \boxdot m_2^I(\theta_1)$ to θ_1 and θ_2 proportionally to $m_1^I(\theta_2)$ and $m_2^I(\theta_1)$. One gets now the following proportionalization

$$\begin{aligned}
\frac{x_2^I}{[0.4,0.5]} = \frac{y_2^I}{(0.4,0.6)\cup[0.7,0.8]} &= \frac{([0.4,0.5]\boxdot((0.4,0.6)\cup[0.7,0.8])}{([0.4,0.5]\boxplus((0.4,0.6)\cup[0.7,0.8])}\\
&= \frac{(0.16,0.30)\cup[0.28,0.40]}{(0.8,1.1)\cup[1.1,1.3]} = \frac{(0.16,0.40]}{(0.8,1.3]}
\end{aligned}$$

whence

$$x_2^I = \frac{(0.16, 0.40]}{(0.8, 1.3]} \boxdot [0.4, 0.5] = \frac{(0.064, 0.200]}{(0.8, 1.3]} = (\frac{0.064}{1.3}, \frac{0.200}{0.8}) = (0.049231, 0.250000)$$

$$\begin{aligned} y_2^I &= \frac{(0.16, 0.40]}{(0.8, 1.3]} \boxdot (0.4, 0.6) \cup [0.7, 0.8] = \frac{(0.064, 0.240) \cup (0.112, 0.320]}{(0.8, 1.3]} \\ &= \frac{(0.064, 0.320]}{(0.8, 1.3]} = (\frac{0.064}{1.3}, \frac{0.320}{0.8}) = (0.049231, 0.400000) \end{aligned}$$

Hence, one finally gets with imprecise PCR5,

$$\begin{aligned} m_{PCR5}^I(\theta_1) &= m_{12}^I(\theta_1) \boxplus x_1^I \boxplus x_2^I \\ &= ([0.04, 0.10] \cup [0.12, 0.15]) \boxplus [0, 0.48] \boxplus (0.049231, 0.250000) \\ &= ([0.04, 0.10] \cup [0.12, 0.15]) \boxplus (0.049231, 0.73) \\ &= (0.089231, 0.83) \cup (0.169231, 0.88) = (0.089231, 0.88) \\ m_{PCR5}^I(\theta_2) &= m_{12}^I(\theta_2) \boxplus y_1^I \boxplus y_2^I \\ &= ([0, 0.40] \cup [0.42, 0.48]) \boxplus [0, 1] \boxplus (0.049231, 0.400000) \approx [0, 1] \\ m_{PCR5}^I(\theta_1 \cap \theta_2) &= 0 \end{aligned}$$

Example no 2:

Let's consider a more simple example with $\Theta = \{\theta_1, \theta_2\}$, Shafer's model and two independent sources with the following imprecise admissible bba

$m_1^I(\theta_1) = (0.2, 0.3)$	$m_1^I(\theta_2) = [0.6, 0.8]$
$m_2^I(\theta_1) = [0.4, 0.7]$	$m_2^I(\theta_2) = (0.5, 0.6]$

Working with sets, one gets for the conjunctive consensus

$$m_{12}^I(\theta_1) = (0.08, 0.21) \qquad m_{12}^I(\theta_2) = (0.30, 0.48)$$

The total (imprecise) conflict between the two imprecise quantitative sources is given by

$$\begin{aligned} k_{12}^I \equiv m_{12}^I(\theta_1 \cap \theta_2) &= [m_1^I(\theta_1) \boxdot m_2^I(\theta_2)] \boxplus [m_1^I(\theta_2) \boxdot m_2^I(\theta_1)] \\ &= ((0.2, 0.3) \boxdot (0.5, 0.6]) \boxplus ([0.4, 0.7] \boxdot [0.6, 0.8]) \\ &= (0.10, 0.18) \boxplus [0.24, 0.56) = (0.34, 0.74) \end{aligned}$$

Using the PCR5 rule for Proportional Conflict redistribution of partial (imprecise) conflict $m_1^I(\theta_1) \boxdot m_2^I(\theta_2)$, one has

$$\frac{x_1^I}{(0.2, 0.3)} = \frac{y_1^I}{(0.5, 0.6]} = \frac{(0.2, 0.3) \boxdot (0.5, 0.6]}{(0.2, 0.3) \boxplus (0.5, 0.6]} = \frac{(0.10, 0.18)}{(0.7, 0.9)}$$

whence

$$x_1^I = \frac{(0.10, 0.18)}{(0.7, 0.9)} \boxdot (0.2, 0.3) = \frac{(0.02, 0.054)}{(0.7, 0.9)} = (\frac{0.02}{0.9}, \frac{0.054}{0.7}) = (0.022222, 0.077143)$$

$$y_1^I = \frac{(0.10, 0.18)}{(0.7, 0.9)} \boxdot (0.5, 0.6] = \frac{(0.050, 0.108)}{(0.7, 0.9)} = (\frac{0.050}{0.9}, \frac{0.108}{0.7}) = (0.055556, 0.154286)$$

Using the PCR5 rule for Proportional Conflict redistribution of partial (imprecise) conflict $m_1^I(\theta_2) \boxdot m_2^I(\theta_1)$, one has

$$\frac{x_2^I}{[0.4, 0.7)} = \frac{y_2^I}{[0.6, 0.8]} = \frac{[0.4, 0.7) \boxdot [0.6, 0.8]}{[0.4, 0.7) \boxplus [0.6, 0.8]} = \frac{[0.24, 0.56)}{[1, 1.5)}$$

whence

$$x_2^I = \frac{[0.24, 0.56)}{[1, 1.5)} \boxdot [0.4, 0.7) = \frac{[0.096, 0.392)}{[1, 1.5)} = (\frac{0.096}{1.5}, \frac{0.392}{1}) = (0.064, 0.392)$$

$$y_2^I = \frac{[0.24, 0.56)}{[1, 1.5)} \boxdot [0.6, 0.8] = \frac{[0.144, 0.448)}{[1, 1.5)} = (\frac{0.144}{1.5}, \frac{0.448}{1}) = (0.096, 0.448)$$

Hence, one finally gets with imprecise PCR5,

$$\begin{aligned}
m_{PCR5}^I(\theta_1) &= m_{12}^I(\theta_1) \boxplus x_1^I \boxplus x_2^I \\
&= (0.08, 0.21) \boxplus (0.022222, 0.077143) \boxplus (0.064, 0.392) \\
&= (0.166222, 0.679143) \\
m_{PCR5}^I(\theta_2) &= m_{12}^I(\theta_2) \boxplus y_1^I \boxplus y_2^I \\
&= (0.30, 0.48) \boxplus (0.055556, 0.154286) \boxplus (0.096, 0.448) \\
&= (0.451556, 1.08229) \approx (0.451556, 1] \\
m_{PCR5}^I(\theta_1 \cap \theta_2) &= 0
\end{aligned}$$

1.12 More numerical examples and comparisons

In this section, we present some numerical examples and comparisons of PCR rules with other rules proposed in literature.

1.12.1 Example 1

Let's consider the frame of discernment $\Theta = \{A, B, C\}$, Shafer's model (i.e. all intersections empty), and the 2 following Bayesian bba's

$$m_1(A) = 0.6 \quad m_1(B) = 0.3 \quad m_1(C) = 0.1$$

$$m_2(A) = 0.4 \quad m_2(B) = 0.4 \quad m_2(C) = 0.2$$

Then the conjunctive consensus yields : $m_{12}(A) = 0.24$, $m_{12}(B) = 0.12$ and $m_{12}(C) = 0.02$ with the conflicting mass $k_{12} = m_{12}(A \cap B) + m_{12}(A \cap C) + m_{12}(B \cap C) = 0.36 + 0.16 + 0.10 = 0.62$, which is a sum of factors.

From the PCR1 and PCR2 rules, one gets

$$m_{PCR1}(A) = 0.550 \qquad m_{PCR2}(A) = 0.550$$

$$m_{PCR1}(B) = 0.337 \qquad m_{PCR2}(B) = 0.337$$

$$m_{PCR1}(C) = 0.113 \qquad m_{PCR2}(C) = 0.113$$

And from the PCR3 and PCR5 rules, one gets

$$m_{PCR3}(A) = 0.574842 \qquad m_{PCR5}(A) = 0.574571$$

$$m_{PCR3}(B) = 0.338235 \qquad m_{PCR5}(B) = 0.335429$$

$$m_{PCR3}(C) = 0.086923 \qquad m_{PCR5}(C) = 0.090000$$

Dempster's rule is a particular case of proportionalization, where the conflicting mass is redistributed to the non-empty sets A_1, A_2, ... proportionally to $m_{12}(A_1)$, $m_{12}(A_2)$, ... respectively (for the case of 2 sources) and similarly for n sources, i.e.

$$\frac{x}{0.24} = \frac{y}{0.12} = \frac{z}{0.02} = \frac{0.62}{0.38}$$

whence $x = 0.24 \cdot \frac{0.62}{0.38} = 0.391579$, $y = 0.12 \cdot \frac{0.62}{0.38} = 0.195789$, $z = 0.02 \cdot \frac{0.62}{0.38} = 0.032632$. Dempster's rule yields

$$m_{DS}(A) = 0.24 + 0.391579 = 0.631579$$

$$m_{DS}(B) = 0.12 + 0.195789 = 0.315789$$

$$m_{DS}(C) = 0.02 + 0.032632 = 0.052632$$

Applying PCR4 for this example, one has

$$\frac{x_1}{0.24} = \frac{y_1}{0.12} = \frac{0.36}{0.24 + 0.12}$$

therefore $x_1 = 0.24$ and $y_1 = 0.12$;

$$\frac{x_2}{0.24} = \frac{z_1}{0.02} = \frac{0.16}{0.24 + 0.02} = \frac{0.16}{0.26}$$

therefore $x_2 = 0.24(0.16/0.26) = 0.147692$ and $z_1 = 0.02(0.16/0.26) = 0.012308$:

$$\frac{y_2}{0.12} = \frac{z_2}{0.02} = \frac{0.10}{0.12 + 0.02} = \frac{0.10}{0.14}$$

therefore $y_2 = 0.12(0.10/0.14) = 0.085714$ and $z_2 = 0.02(0.10/0.14) = 0.014286$. Summing all of them, one gets finally:

$$m_{PCR4}(A) = 0.627692 \qquad m_{PCR4}(B) = 0.325714 \qquad m_{PCR4}(C) = 0.046594$$

It can be shown that minC combination provides same result as PCR4 for this example.

1.12.2 Example 2

Let's consider the frame of discernment $\Theta = \{A, B\}$, Shafer's model (i.e. all intersections empty), and the following two bba's:

$$m_1(A) = 0.7 \qquad m_1(B) = 0.1 \qquad m_1(A \cup B) = 0.2$$
$$m_2(A) = 0.5 \qquad m_2(B) = 0.4 \qquad m_2(A \cup B) = 0.1$$

Then the conjunctive consensus yields $m_{12}(A) = 0.52$, $m_{12}(B) = 0.13$ and $m_{12}(A \cup B) = 0.02$ with the total conflict $k_{12} = m_{12}(A \cap B) = 0.33$.

From PCR1 and PCR2 rules, one gets:

$$\begin{aligned} m_{PCR1}(A) &= 0.7180 & m_{PCR2}(A) &= 0.752941 \\ m_{PCR1}(B) &= 0.2125 & m_{PCR2}(B) &= 0.227059 \\ m_{PCR1}(A \cup B) &= 0.0695 & m_{PCR2}(A \cup B) &= 0.02 \end{aligned}$$

From PCR3 and PCR5 rules, one gets

$$\begin{aligned} m_{PCR3}(A) &= 0.752941 & m_{PCR5}(A) &= 0.739849 \\ m_{PCR3}(B) &= 0.227059 & m_{PCR5}(B) &= 0.240151 \\ m_{PCR3}(A \cup B) &= 0.02 & m_{PCR5}(A \cup B) &= 0.02 \end{aligned}$$

From Dempster's rule:

$$m_{DS}(A) = 0.776119 \qquad m_{DS}(B) = 0.194030 \qquad m_{DS}(A \cup B) = 0.029851$$

From PCR4, one has

$$\frac{x}{0.52} = \frac{y}{0.13} = \frac{0.33}{0.52 + 0.13} = \frac{0.33}{0.65}$$

therefore $x = 0.52(0.33/0.65) = 0.264$ and $y = 0.13(0.33/0.65) = 0.066$. Summing, one gets:

$$m_{PCR4}(A) = 0.784 \qquad m_{PCR4}(B) = 0.196 \qquad m_{PCR4}(A \cup B) = 0.02$$

From minC, one has

$$\frac{x}{0.52} = \frac{y}{0.13} = \frac{z}{0.02} = \frac{0.33}{0.52 + 0.13 + 0.02} = \frac{0.33}{0.67}$$

therefore $x = 0.52(0.33/0.67) = 0.256119$, $y = 0.13(0.33/0.67) = 0.064030$ and $z = 0.02(0.33/0.02) = 0.009851$. Summing, one gets same result as with the Demspter's rule in this second example:

$$m_{minC}(A) = 0.776119 \qquad m_{minC}(B) = 0.194030 \qquad m_{minC}(A \cup B) = 0.029851$$

1.12.3 Example 3 (Zadeh's example)

Let's consider the famous Zadeh's example[17] [31] with $\Theta = \{A, B, C\}$, Shafer's model and the two following belief assignments

$$m_1(A) = 0.9 \qquad m_1(B) = 0 \qquad m_1(C) = 0.1$$
$$m_2(A) = 0 \qquad m_2(B) = 0.9 \qquad m_2(C) = 0.1$$

[17] A detailed discussion on this example can be found in [18] (Chap. 5, p. 110).

The conjunctive consensus yields for this case, $m_{12}(A) = m_{12}(b) = 0$, $m_{12}(C) = 0.01$. The masses committed to partial conflicts are given by

$$m_{12}(A \cap B) = 0.81 \qquad m_{12}(A \cap C) = m_{12}(B \cap C) = 0.09$$

and the conflicting mass by

$$k_{12} = m_1(A)m_2(B) + m_1(A)m_2(C) + m_2(B)m_1(C) = 0.81 + 0.09 + 0.09 = 0.99$$

The first partial conflict $m_{12}(A \cap B) = 0.9 \cdot 0.9 = 0.81$ is proportionally redistributed to A and B according to

$$\frac{x_1}{0.9} = \frac{y_1}{0.9} = \frac{0.81}{0.9 + 0.9}$$

whence $x_1 = 0.405$ and $y_1 = 0.405$.

The second partial conflict $m_{12}(A \cap C) = 0.9 \cdot 0.1 = 0.09$ is proportionally redistributed to A and C according to

$$\frac{x_2}{0.9} = \frac{y_2}{0.1} = \frac{0.09}{0.9 + 0.1}$$

whence $x_2 = 0.081$ and $y_2 = 0.009$.

The third partial conflict $m_{12}(B \cap C) = 0.9 \cdot 0.1 = 0.09$ is proportionally redistributed to B and C according to

$$\frac{x_3}{0.9} = \frac{y_3}{0.1} = \frac{0.09}{0.9 + 0.1}$$

whence $x_3 = 0.081$ and $y_3 = 0.009$.

After summing all proportional redistributions of partial conflicts to corresponding elements with PCR5, one finally gets:

$$\begin{aligned}
m_{PCR5}(A) &= 0 + 0.405 + 0.081 = 0.486 \\
m_{PCR5}(B) &= 0 + 0.405 + 0.081 = 0.486 \\
m_{PCR5}(C) &= 0.01 + 0.009 + 0.009 = 0.028
\end{aligned}$$

The fusion obtained from other rules yields:

- with Dempster's rule based on Shafer's model, one gets the counter-intuitive result

$$m_{DS}(C) = 1$$

- with Smets' rule based on Open-World model, one gets

$$m_S(\emptyset) = 0.99 \qquad m_S(C) = 0.01$$

- with Yager's rule based on Shafer's model, one gets

$$m_Y(A \cup B \cup C) = 0.99 \qquad m_{DS}(C) = 0.01$$

- with Dubois & Prade's rule based on Shafer's model, one gets

$$m_{DP}(A \cup B) = 0.81 \qquad m_{DP}(A \cup C) = 0.09 \qquad m_{DP}(B \cup C) = 0.09 \qquad m_{DP}(C) = 0.01$$

- with the classic DSm rule based on the free-DSm model, one gets

$$m_{DSmC}(A \cap B) = 0.81 \qquad m_{DSmC}(A \cap C) = 0.09$$

$$m_{DSmC}(B \cap C) = 0.09 \qquad m_{DSmC}(C) = 0.01$$

- with the hybrid DSm rule based on Shafer's model, one gets same as with Dubois & Prade (in this specific example)

$$m_{DSmH}(A \cup B) = 0.81 \qquad m_{DSmH}(A \cup C) = 0.09$$

$$m_{DSmH}(B \cup C) = 0.09 \qquad m_{DSmH}(C) = 0.01$$

- with the WAO rule based on Shafer's model, one gets

$$m_{WAO}(A) = 0 + \frac{0.9+0}{2} \cdot 0.99 = 0.4455$$
$$m_{WAO}(B) = 0 + \frac{0+0.9}{2} \cdot 0.99 = 0.4455$$
$$m_{WAO}(C) = 0.01 + \frac{0.1+0.1}{2} \cdot 0.99 = 0.1090$$

- with the PCR1 rule based on Shafer's model, one gets (same as with WAO)

$$m_{PCR1}(A) = 0 + \frac{0.9}{0.9+0.9+0.2} \cdot 0.99 = 0.4455$$
$$m_{PCR1}(B) = 0 + \frac{0.9}{0.9+0.9+0.2} \cdot 0.99 = 0.4455$$
$$m_{PCR1}(C) = 0.01 + \frac{0.2}{0.9+0.9+0.2} \cdot 0.99 = 0.1090$$

- with the PCR2 rule based on Shafer's model, one gets in this example the same result as with WAO and PCR1.

- with the PCR3 rule based on Shafer's model, one gets

$$m_{PCR3}(A) = 0 + 0.9 \cdot [\frac{0 \cdot 0 + 0.9 \cdot 0.9}{0.9+0.9} + \frac{0.1 \cdot 0 + 0.9 \cdot 0.1}{0.9+0.2}] \approx 0.478636$$
$$m_{PCR3}(B) = 0 + 0.9 \cdot [\frac{0 \cdot 0 + 0.9 \cdot 0.9}{0.9+0.9} + \frac{0.1 \cdot 0 + 0.9 \cdot 0.1}{0.9+0.2}] \approx 0.478636$$
$$m_{PCR3}(C) \approx 0.042728$$

- With the PCR4 rule based on Shafer's model, $m_{12}(A \cap B) = 0.81$ is distributed to A and B with respect to their $m_{12}(.)$ masses, but because $m_{12}(A)$ and $m_{12}(B)$ are zero, it is distributed to A and B with respect to their corresponding column sum of masses, i.e. with respect to $0.9 + 0 = 0.9$ and $0 + 0.9 = 0.9$;

$$\frac{x_1}{0.9} = \frac{y_1}{0.9} = \frac{0.81}{0.09+0.09}$$

whence $x_1 = 0.405$ and $y_1 = 0.405$.

$m(A \cap C) = 0.09$ is redistributed to A and C proportionally with respect to their corresponding column sums, i.e. 0.9 and 0.2 respectively:

$$x/0.9 = z/0.2 = 0.09/1.1$$

wence $x = 0.9 \cdot (0.09/1.1) = 0.073636$ and $z = 0.2 \cdot (0.09/1.1) = 0.016364$.

$m(B \cap C) = 0.09$ is redistributed to B and C proportionally with respect to their corresponding column sums, i.e. 0.9 and 0.2 respectively:

$$y/0.9 = z/0.2 = 0.09/1.1$$

wence $y = 0.9 \cdot (0.09/1.1) = 0.073636$ and $z = 0.2 \cdot (0.09/1.1) = 0.016364$.

Summing one gets:

$$m_{PCR4}(A) = 0.478636 \qquad m_{PCR4}(B) = 0.478636 \qquad m_{PCR4}(C) = 0.042728$$

- With the minC rule based on Shafer's model, one gets:

$$m_{minC}(A) = 0.405 \qquad m_{minC}(B) = 0.405 \qquad m_{minC}(C) = 0.190$$

- With the PCR5 rule based on Shafer's model, the mass $m_{12}(A \cap B) = 0.9 \cdot 0.9 = 0.81$ is proportionalized according to

$$\frac{x}{0.9} = \frac{y}{0.9} = \frac{0.81}{0.9 + 0.9}$$

whence $x = 0.405$ and $y = 0.405$. Similarly, $m_{12}(A \cap C) = 0.09$ is proportionalized according to

$$\frac{x}{0.9} = \frac{z}{0.9} = \frac{0.09}{0.9 + 0.1}$$

whence $x = 0.081$ and $z = 0.009$; Similarly, $m_{12}(B \cap C) = 0.09$ is proportionalized according to

$$\frac{y}{0.9} = \frac{z}{0.1} = \frac{0.09}{0.9 + 0.1}$$

whence $y = 0.081$ and $z = 0.009$. Summing one gets:

$$m_{PCR5}(A) = 0 + 0.405 + 0.081 = 0.486$$
$$m_{PCR5}(B) = 0 + 0.405 + 0.081 = 0.486$$
$$m_{PCR5}(C) = 0.01 + 0.009 + 0.009 = 0.028$$

1.12.4 Example 4 (hybrid model)

Let's consider a hybrid model on $\Theta = \{A, B, C\}$ where $A \cap B = \emptyset$, while $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. This model corresponds to a hybrid model [18]. Then only the mass $m_{12}(A \cap B)$ of partial conflict $A \cap B$ will be transferred to other non-empty sets, while the masses $m_{12}(A \cap C)$ stays on $A \cap C$ and $m_{12}(B \cap C)$ stays on $B \cap C$. Let's consider two sources of evidence with the following basic belief assignments

$$m_1(A) = 0.5 \quad m_1(B) = 0.4 \quad m_1(C) = 0.1$$

$$m_2(A) = 0.6 \quad m_2(B) = 0.2 \quad m_2(C) = 0.2$$

Using the table representation, one has

	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$
m_1	0.5	0.4	0.1			
m_2	0.6	0.2	0.2			
m_{12}	0.3	0.08	0.02	0.34	0.16	0.10

Thus, the conjunctive consensus yields

$$m_{12}(A) = 0.30 \qquad m_{12}(B) = 0.08 \qquad m_{12}(C) = 0.02$$

$$m_{12}(A \cap B) = 0.34 \qquad m_{12}(A \cap C) = 0.16 \qquad m_{12}(B \cap C) = 0.10$$

- with the PCR1 rule, $m_{12}(A \cap B) = 0.34$ is the only conflicting mass, and it is redistributed to A, B and C proportionally with respect to their corresponding columns' sums: $0.5 + 0.6 = 1.1$, $0.4 + 0.2 = 0.6$ and $0.1 + 0.2 = 0.3$. The sets $A \cap C$ and $B \cap C$ don't get anything from the conflicting mass 0.34 since their columns' sums are zero. According to proportional conflict redistribution of PCR1, one has

$$\frac{x}{1.1} = \frac{y}{0.6} = \frac{z}{0.3} = \frac{0.34}{1.1 + 0.6 + 0.3} = 0.17$$

  Therefore, one gets the proportional redistributions for A, B and C

$$x = 1.1 \cdot 0.17 = 0.187 \qquad y = 0.6 \cdot 0.17 = 0.102 \qquad z = 0.3 \cdot 0.17 = 0.051$$

  Thus the final result of PCR1 is given by

$$\begin{aligned} m_{PCR1}(A) &= 0.30 + 0.187 = 0.487 \\ m_{PCR1}(B) &= 0.08 + 0.102 = 0.182 \\ m_{PCR1}(C) &= 0.02 + 0.051 = 0.071 \\ m_{PCR1}(A \cap C) &= 0.16 \\ m_{PCR1}(B \cap C) &= 0.10 \end{aligned}$$

- with the PCR2 rule, $m_{12}(A \cap B) = 0.34$ is redistributed to A and B only with respect to their corresponding columns' sums: $0.5 + 0.6 = 1.1$ and $0.4 + 0.2 = 0.6$. The set C doesn't get anything since C was not involved in the conflict. According to proportional conflict redistribution of PCR2, one has

$$\frac{x}{1.1} = \frac{y}{0.6} = \frac{0.34}{1.1 + 0.6} = 0.2$$

Therefore, one gets the proportional redistributions for A and B

$$x = 1.1 \cdot 0.2 = 0.22 \qquad y = 0.6 \cdot 0.2 = 0.12$$

Thus the final result of PCR2 is given by

$$\begin{aligned}
m_{PCR2}(A) &= 0.30 + 0.22 = 0.52\\
m_{PCR2}(B) &= 0.08 + 0.12 = 0.20\\
m_{PCR2}(C) &= 0.02\\
m_{PCR2}(A \cap C) &= 0.16\\
m_{PCR2}(B \cap C) &= 0.10
\end{aligned}$$

- PCR3 gives the same result like PCR2 since there is only a partial conflicting mass which coincides with the total conflicting mass.

- with the PCR4 rule, $m_{12}(A \cap B) = 0.34$ is redistributed to A and B proportionally with respect to $m_{12}(A) = 0.30$ and $m_{12}(B) = 0.08$. According to proportional conflict redistribution of PCR4, one has

$$\frac{x}{0.30} = \frac{y}{0.08} = \frac{0.34}{0.30 + 0.08}$$

Therefore, one gets the proportional redistributions for A and B

$$x = 0.30 \cdot (0.34/0.38) \approx 0.26842 \qquad y = 0.08 \cdot (0.34/0.38) \approx 0.07158$$

Thus the final result of PCR4 is given by

$$\begin{aligned}
m_{PCR4}(A) &= 0.30 + 0.26842 = 0.56842\\
m_{PCR4}(B) &= 0.08 + 0.07158 = 0.15158\\
m_{PCR4}(C) &= 0.02\\
m_{PCR4}(A \cap C) &= 0.16\\
m_{PCR4}(B \cap C) &= 0.10
\end{aligned}$$

- with the PCR5 rule, $m_{12}(A \cap B) = 0.34$ is redistributed to A and B proportionally with respect to $m_1(A) = 0.5$, $m_2(B) = 0.2$ and then with respect to $m_2(A) = 0.6$, $m_1(B) = 0.4$. According to proportional conflict redistribution of PCR5, one has

$$\frac{x_1}{0.5} = \frac{y_1}{0.2} = \frac{0.10}{0.5 + 0.2} = 0.10/0.7 \qquad \frac{x_2}{0.6} = \frac{y_2}{0.4} = \frac{0.24}{0.6 + 0.4} = 0.24$$

Therefore, one gets the proportional redistributions for A and B

$$\begin{aligned}
x_1 &= 0.5 \cdot (0.10/0.7) = 0.07143 \qquad & y_1 &= 0.2 \cdot (0.10/0.7) = 0.02857\\
x_2 &= 0.6 \cdot 0.24 = 0.144 \qquad & y_2 &= 0.4 \cdot 0.24 = 0.096
\end{aligned}$$

Thus the final result of PCR5 is given by

$$\begin{aligned}
m_{PCR5}(A) &= 0.30 + 0.07143 + 0.144 = 0.51543\\
m_{PCR5}(B) &= 0.08 + 0.02857 + 0.096 = 0.20457\\
m_{PCR5}(C) &= 0.02\\
m_{PCR5}(A \cap C) &= 0.16\\
m_{PCR5}(B \cap C) &= 0.10
\end{aligned}$$

1.12.5 Example 5 (Target ID tracking)

This example is drawn from Target ID (identification) tracking application pointed out by Dezert and al. in [5]. The problem consists in updating bba on ID of a target based on a sequence of uncertain attribute measurements expressed as sensor's bba. In such case, a problem can arise when the fusion rule of the predicted ID bba with the current observed ID bba yields to commit certainty on a given ID of the frame Θ (the set of possible target IDs under consideration). If this occurs once, then the ID bba remains unchanged by all future observations, whatever the value they can take ! By example, at a given time the ID system finds with "certainty" that a target is a truck, and then during next, say 1000 scans, all the sensor reports claim with high belief that target is a car, but the ID system is unable to doubt itself of his previous ID assessment (certainty state plays actually the role of an absorbing/black hole state). Such behavior of a fusion rule is what we feel drastically dangerous, specially in defence applications and better rules than the classical ones have to be used to avoid such severe drawback. We provide here a simple numerical example and we compare the results for the new rules presented in this chapter. So let's consider here Shafer's model, a 2D frame $\Theta = \{A, B\}$ and two bba $m_1(.)$ and $m_2(.)$ with

	A	B	$A \cup B$
m_1	1	0	0
m_2	0.1	0.9	0

$m_1(.)$ plays here the role of a prior (or predicted) target ID bba for a given time step and $m_2(.)$ is the observed target ID bba drawn from some attribute measurement for the time step under consideration. The conjunctive operator of the prior bba and the observed bba is then

$$m_{12}(A) = 0.1 \qquad m_{12}(A \cap B) = 0.9$$

Because we are working with Shafer's model, one has to redistribute the conflicting mass $m_{12}(A \cap B) = 0.9$ in some manner onto the non conflicting elements of power-set. Once the fusion/update is obtained at a given time, we don't keep in memory $m_1(.)$ and $m_2(.)$ but we only use the fusion result as new prior[18] bba for the fusion with the next observation, and this process is reitered at every observation time. Let's examine the result of the rule after at first observation time (when only $m_2(.)$ comes in).

- **With minC rule**: minC rule distributes the whole conflict to A since $m_{12}(B) = 0$, thus:

$$m_{minC|12}(A) = 1$$

[18]For simplicity, we don't introduce a prediction ID model here and we just consider as predicted bba for time $k+1$, the updated ID bba available at time k (i.e. the ID state transition matrix equals identity matrix).

- **With PCR1-PCR4 rules**: Using PCR1-4, they all coincide here. One has $x/1.1 = y/0.9 = 0.9/2 = 0.45$, whence $x = 1.1 \cdot (0.45) = 0.495$ and $y = 0.9 \cdot (0.45) = 0.405$. Hence

$$m_{PCR1-4|12}(A) = 0.595 \qquad m_{PCR1-4|12}(B) = 0.405$$

- **With PCR5 rule**: One gets $x/1 = y/0.9 = 0.9/1.9$, whence $x = 1 \cdot (0.9/1.9) = 0.473684$ and $y = 0.9 \cdot (0.9/1.9) = 0.426316$. Hence

$$m_{PCR5|12}(A) = 0.573684 \qquad m_{PCR5|12}(B) = 0.426316$$

Suppose a new observation, expressed by $m_3(.)$ comes in at next scan with

$$m_3(A) = 0.4 \qquad m_3(B) = 0.6$$

and examine the result of the new target ID bba update based on the fusion of the previous result with $m_3(.)$.

- **With minC rule**: The conjunctive operator applied on $m_{minC|12}(.)$ and $m_3(.)$ yields now

$$m_{(minC|12)3}(A) = 0.4 \qquad m_{(minC|12)3}(A \cap B) = 0.6$$

  Applying minC rule again, one distributes the whole conflict 0.6 to A and one finally gets[19]:

$$m_{minC|(12)3}(A) = 1$$

  Therefore, minC rule does not respond to the new tracking ID observations.

- **With PCR1-PCR4 rules**: The conjunctive operator applied on $m_{PCR1-4|12}(.)$ and $m_3(.)$ yields now

$$m_{(PCR1-4|12)3}(A) = 0.238 \quad m_{(PCR1-4|12)3}(B) = 0.243 \quad m_{(PCR1-4|12)3}(A \cap B) = 0.519$$

  - **For PCR1-3**: $x/0.995 = y/1.005 = 0.519/2 = 0.2595$, so that $x = 0.995 \cdot (0.2595) = 0.258203$ and $y = 1.005 \cdot (0.2595) = 0.260797$. Hence:

$$m_{PCR1-3|(12)3}(A) = 0.496203 \quad m_{PCR1-3|(12)3}(B) = 0.503797$$

    Therefore PCR1-3 rules do respond to the new tracking ID observations.

  - **For PCR4**: $x/0.238 = y/0.243 = 0.519/(0.238 + 0.243) = 0.519/0.481$, so that $x = 0.238 \cdot (0.519/0.481) = 0.256802$ and $y = 0.243 \cdot (0.519/0.481) = 0.262198$. Hence:

$$m_{PCR4|(12)3}(A) = 0.494802 \quad m_{PCR4|(12)3}(B) = 0.505198$$

  Therefore PCR4 rule does respond to the new tracking ID observations.

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[19]For convenience, we use the notation $m_{minC|(12)3}(A)$ instead of $m_{minC|(minC|12)3}(.)$, and similarly with PCR indexes.

- **With PCR5 rule**: The conjunctive operator applied on $m_{PCR5|12}(.)$ and $m_3(.)$ yields now

  $m_{(PCR5|12)3}(A) = 0.229474 \quad m_{(PCR5|12)3}(B) = 0.255790 \quad m_{(PCR5|12)3}(A \cap B) = 0.514736$

  Then: $x/0.573684 = y/0.6 = (0.573684 \cdot 0.6)/(0.573684 + 0.6) = 0.293273$, so that $x = 0.573684 \cdot 0.293273 = 0.168246$ and $y = 0.6 \cdot 0.293273 = 0.175964$. Also: $x/0.4 = y/0.426316 = (0.4 \cdot 0.426316)/(0.4 + 0.426316) = 0.206369$, so that $x = 0.4 \cdot 0.206369 = 0.082548$ and $y = 0.426316 \Delta 0.206369 = 0.087978$. Whence:

  $$m_{PCR5|(12)3}(A) = 0.480268 \quad m_{PCR5|(12)3}(B) = 0.519732$$

  Therefore PCR5 rule does respond to the new tracking ID observations.

It can moreover be easily verified that Dempster's rule gives the same results as minC here, hence does not respond to new observations in target ID tracking problem.

1.13 On Ad-Hoc-ity of fusion rules

Each fusion rule is more or less ad-hoc. Same thing for PCR rules. There is up to the present no rule that fully satisfies everybody. Let's analyze some of them.

Dempster's rule transfers the total conflicting mass to non-empty sets proportionally with their resulting masses. What is the reasoning for doing this? Just to swell the masses of non-empty sets in order to sum up to 1 and preserve associativity?

Smets' rule transfers the conflicting mass to the empty set. Why? Because, he says, we consider on open world where unknown hypotheses might be. This approach does not make difference between all origins of conflicts since all different conflicting masses are committed with the same manner to the empty set. Not convincing. And what about real closed worlds?

Yager's rule transfers all the conflicting mass only to the total ignorance. Should the internal structure of partial conflicting mass be ignored?

Dubois-Prade's rule and *DSm hybrid rule* transfer the conflicting mass to the partial and total ignorances upon the principle that between two conflicting hypotheses one is right. Not completely justified either. What about the case when no hypothesis is right?

PCR rules are based on total or partial conflicting masses, transferred to the corresponding sets proportionally with respect to some functions (weighting coefficients) depending on their corresponding mass matrix columns. But other weighting coefficients can be found.

Inagaki [10], Lefèvre-Colot-Vannoorenberghe [12] proved that there are infinitely many fusion rules based on the conjunctive rule and then on the transfer of the conflicting mass, all of them depending on the weighting coefficients/factors that transfer that conflicting mass. How to choose them, what parameters should they rely on – that's the question! There is not a precise measure for this. In authors' opinion, neither DSm hybrid rule nor PCR rules are not more ad-hoc than other fusion rules.

1.14 On quasi-associativity and quasi-Markovian properties

1.14.1 Quasi-associativity property

Let $m_1(.), m_2(.), m_3(.) : G^\Theta \mapsto [0,1]$ be any three bba's, and a fusion rule denoted by $\oplus$ operating on these masses. One says that this fusion rule is associative if and only if:

$$\forall A \in G^\Theta, \qquad ((m_1 \oplus m_2) \oplus m_3)(A) = (m_1 \oplus (m_2 \oplus m_3))(A) \tag{1.44}$$

which is also equal to $(m_1 \oplus m_2 \oplus m_3)(A)$.

Only three fusion rules based on the conjunctive operator are known associative: Dempster's rule in DST, Smets' rule (conjunctive consensus based on the open-world assumption), and the DSm classic rule on free DSm model. All alternative rules developed in literature so far do not hold the associativity. Although, some rules such as Yager's, Dubois & Prade's, DSm hybrid, WAO, minC, PCR rules, which are not associative become *quasi-associative* if one stores the result of the conjunctive rule at each time when a new bba arises in the combination process. Instead of combining it with the previous result of the rule, we combine the new bba with the stored conjunctive rule's result.

1.14.2 Quasi-Markovian property

Let $m_1(.), m_2(.), \ldots, m_n(.) : G^\Theta \mapsto [0,1]$ be any $n \geq 3$ masses, and a fusion rule denoted by $\oplus$ operating on these masses. One says that this fusion rule satisfies Markovian property or Markovian requirement (according to Ph. Smets) if and only if:

$$\forall A \in G^\Theta, \text{ and } n \geq 3, \qquad (m_1 \oplus m_2 \oplus \ldots \oplus m_n)(A) = ((m_1 \oplus m_2 \oplus \ldots \oplus m_{n-1}) \oplus m_n)(A) \tag{1.45}$$

Similarly, only three fusion rules derived from the conjunctive rule are known satisfying the Markovian requirement, i.e. Dempster's rule, Smets' TBM's rule, and the DSm classic rule on free DSm model. In an analoguous way as done for quasi-associativity, we can transform a non-Markovian fusion rule based on conjunctive rule into a Markovian fusion rule by keeping in the computer's memory the results of the conjunctive rule - see next section.

1.14.3 Algorithm for Quasi-Associativity and Quasi-Markovian Requirement

The following algorithm will help transform a fusion rule into an associative and Markovian fusion rule. Let's call a rule which first uses the conjunctive rule and then the transfer of the conflicting mass to empty or non-empty sets *quasi-conjunctive* rule. the following algorithm is proposed in order to restore the associativity and Markovian requirements to any quasi-conjunctive based rules.

Let's consider a rule $\circledR$ formed by using: first the conjunctive rule, noted by $\copyright$, and second the transfer/redistribution of the conflicting mass to empty or non-empty sets, noted by the operator $O(.)$ (no matter how the transfer is done, either proportionally with some parameters, or transferred to partial or total ignorances and/or to the empty set; if all conflicting mass is transferred to the empty set, as in Smets' rule, there is no need for transformation into an associative or Markovian rule since Smets' rule has already these properties). Clearly $\circledR \equiv O(\copyright)$.

The idea is simple; we store the conjunctive rule's result (before doing the transfer) and, when a new mass arises, one combines this new mass with the conjunctive rule's result, not with the result after the transfer of conflicting mass.

Let's have two bba's $m_1(.)$, $m_2(.)$ defined as previously.

a) One applies the conjunctive rule to $m_1(.)$ and $m_2(.)$ and one stores the result[20]:

$$m_{c(1,2)}(.) \triangleq [m_1 \copyright m_2](.) = [m_2 \copyright m_1](.).$$

b) One applies the operator $O(.)$ of transferring conflicting mass to the non-empty sets, i.e. $O(m_{c(1,2)}(.))$. This calculation completely does the work of our fusion rule, i.e. $[m_1 \circledR m_2](.) = O(m_{c(1,2)}(.))$ that we compute for decision-making purpose.

c) When a new bba, $m_3(.)$, arises, we combine using the conjunctive rule this $m_3(.)$ with the previous conjunctive rule's result $m_{c(1,2)}(.)$, not with $O(m_{c(1,2)}(.))$. Therefore (by notation): $[m_{c(1,2)} \copyright m_3](.) = m_{c(c(1,2),3)}(.)$. One stores this results, while deleting the previous one stored.

d) Now again we apply the operator $O(.)$ to transfer the conflicting mass, i.e. compute $O(m_{c(c(1,2),3)}(.))$ needed for decision-making.

e) ... And so on the algorithm is continued for any number $n \geq 3$ of bba's.

The properties of the conjunctive rule, i.e. associativity and satisfaction of the Markovian requirement, are passed on to the fusion rule $\circledR$ too. One remarks that the algorithm gives the same result if one applies the rule $\circledR$ to all $n \geq 3$ bba's together, and then one does the transfer of conflicting mass based on the conjunctive rule's result only.

For each rule we may adapt our algorithm and store, besides the conjunctive rule's result, more information if needed. For example, for the PCR1-3 rules we also need the sum of column masses to be stored. For PCR5-6 we need to store all bba's in a mass matrix.

Generalization: The previous algorithm can be extended in a similar way if one considers instead of the conjunctive rule applied first, any associative (respectively Markovian) rule applied first and next the transfer of masses.

In this section we have proposed a fusion algorithm that transforms a quasi-conjunctive fusion rule (which first uses the conjunctive rule and then the transfer of conflicting masses to non-empty sets, except for Smets' rule) to an associative and Markovian rule. This is very important in information fusion since the order of combination of masses should not matter, and for the Markovian requirement the algorithm allows the storage of information of all previous masses into the last result (therefore not necessarily to store all the masses), which later will be combined with the new mass. In DSmT, using this fusion algorithm for $n \geq 3$ sources, the DSm hybrid rule and PCRi become commutative, associative and Markovian. Some numerical examples of the application of this algorithm can be found in [19].

[20] where the symbol $\triangleq$ means *by definition*.

1.15 Conclusion

We have presented in this chapter five versions of the Proportional Conflict Redistribution rule of combination in information fusion, which are implemented as follows: first one uses the conjunctive rule, then one redistribute the conflicting mass to non-empty sets proportionally with respect to either the non-zero column sum of masses (for PCR1, PCR2, PCR3) or with respect to the non-zero masses (of the corresponding non-empty set) that enter in the composition of each individual product in the partial conflicting masses (PCR5). PCR1 restricted from the hyper-power set to the power set and without degenerate cases gives the same result as WAO as pointed out by P. Smets in a private communication. PCR1 and PCR2 redistribute the total conflicting mass, while PCR3 and PCR5 redistribute partial conflicting masses. PCR1-3 uses the proportionalization with respect to the sum of mass columns, PCR4 with respect to the results of the conjunctive rule, and PCR5 with respect to the masses entered in the sum products of the conflicting mass. PCR4 is an improvement of minC and Dempster's rules. From PCR1 to PCR2, PCR3, PCR4, PCR5 one increases the complexity of the rules and also the exactitude of the redistribution of conflicting masses. All the PCR rules proposed in this chapter preserve the neutral impact of the vacuous belief assignment but PCR1 and work for any hybrid DSm model (including Shafer's model). For the free DSm model, i.e. when all intersections not empty, there is obviously no need for transferring any mass since there is no conflicting mass, the masses of the intersections stay on them. Thus only DSm classic rule is applied, no PCR1-5, no DSm hybrid rule and no other rule needed to apply. In this chapter, PCR, minC and Dempster's rules are all compared with respect to the conjunctive rule (i.e. the conjunctive rule is applied first, then the conflicting mass is redistributed following the way the conjunctive rule works). Therefore, considering the way each rule works, the rule which works closer to the conjunctive rule in redistributing the conflicting mass is considered better than other rule. This is not a subjective comparison between rules, but only a mathematical one.

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Chapter 2

A new generalization of the proportional conflict redistribution rule stable in terms of decision

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Abstract: *In this chapter, we present and discuss a new generalized proportional conflict redistribution rule. The Dezert-Smarandache extension of the Dempster-Shafer theory has relaunched the studies on the combination rules especially for the management of the conflict. Many combination rules have been proposed in the last few years. We study here different combination rules and compare them in terms of decision on didactic example and on generated data. Indeed, in real applications, we need a reliable decision and it is the final results that matter. This chapter shows that a fine proportional conflict redistribution rule must be preferred for the combination in the belief function theory.*

2.1 Introduction

Many fusion theories have been studied for the combination of the experts opinions such as voting rules [10, 25], possibility theory [7, 27], and belief function theory [2, 15]. We can divide all these fusion approaches into four steps: the modelization, the parameters estimation depending on the model (not always necessary), the combination, and the decision. The most difficult step is presumably the first one. If both possibility and probability-based theories can modelize imprecise and uncertain data at the same time, in a lot of applications, experts can express their certitude on their perception of the reality. As a result, probabilities theory such as the belief function theory is more adapted. In the context of the belief function theory, the Dempster-Shafer theory (DST) [2, 15] is based on the use of functions defined on the power set 2^Θ (that is the set of all the disjunctions of the elements of Θ). Hence the experts can express their opinion not only on Θ but also on 2^Θ as in the probabilities theory. The extension of this power set into the hyper-power set D^Θ (that is the set of all the disjunctions and conjunctions of the elements of Θ) proposed by Dezert and Smarandache [3], gives more freedom to the expert. This extension of the DST is called Dezert-Smarandache Theory (DSmT).

This extension has relaunched the studies on the combination rules. The combination of multiple sources of information has still been an important subject of research since the proposed combination rule given by Dempster [2]. Hence, many solutions have been studied in order to manage the conflict [6, 8, 9, 11, 12, 18, 22, 23, 26]. These combination rules are the most of time compared following the properties of the operator such as associativity, commutativity, linearity, anonymity and on special and simple cases of experts responses [1, 22, 24].

In real applications, we need a reliable decision and it is the final results that matter. Hence, for a given application, the best combination rule is the rule given the best results. For the decision step, different functions such as credibility, plausibility and pignistic probability [4, 15, 20] are usually used.

In this chapter, we discuss and compare different combination rules especially managing the conflict. First, the principles of the DST and DSmT are recalled. We present the formalization of the belief function models, different rules of combination and decision. The combination rule (PCR5) proposed by [18] for two experts is mathematically one of the best for the proportional redistribution of the conflict applicable in the context of the DST and the DSmT. In the section 2.3, we propose a new extension of this rule for more experts, the PCR6 rule. This new rule is compared to the generalized PCR5 rule given in [5], in the section 2.4. Then this section presents a comparison of different combination rules in terms of decision in a general case, where the experts opinions are randomly simulated. We demonstrate also that some combination rules are different in terms of decision, in the case of two experts and two classes, but most of them are equivalent.

2.2 Theory bases

2.2.1 Belief Function Models

The belief functions or basic belief assignments m are defined by the mapping of the power set 2^Θ onto $[0, 1]$, in the DST , and by the mapping of the hyper-power set D^Θ onto $[0, 1]$, in the DSmT, with:

$$m(\emptyset) = 0, \tag{2.1}$$

and

$$\sum_{X \in 2^\Theta} m(X) = 1, \tag{2.2}$$

in the DST , and

$$\sum_{X \in D^\Theta} m(X) = 1, \tag{2.3}$$

in the DSmT.

The equation (2.1) is the hypothesis at a closed world [15, 16]. We can define the belief function only with:

$$m(\emptyset) > 0, \tag{2.4}$$

and the world is open [20]. In a closed world, we can also add one element in order to propose an open world.

These simple conditions in equation (2.1) and (2.2) or (2.1) and (2.3), give a large panel of definitions of the belief functions, which is one the difficulties of the theory. The belief functions must therefore be chosen according to the intended application.

2.2.2 Combination rules

Many combination rules have been proposed in the last few years in the context of the belief function theory ([6, 16, 18, 20, 22, 26], *etc.*). In the context of the DST , the combination rule most used today seems to be the conjunctive rule given by [20] for all $X \in 2^\Theta$ by:

$$m_{\mathrm{c}}(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j), \tag{2.5}$$

where $Y_j \in 2^\Theta$ is the response of the expert j, and $m_j(Y_j)$ the associated belief function.

However, the conflict can be redistributed on partial ignorance like in the Dubois and Prade rule [6], a mixed conjunctive and disjunctive rule given for all $X \in 2^\Theta$, $X \neq \emptyset$ by:

$$m_{\mathrm{DP}}(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) + \sum_{\substack{Y_1 \cup \ldots \cup Y_M = X \\ Y_1 \cap \ldots \cap Y_M = \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \tag{2.6}$$

where $Y_j \in 2^\Theta$ is the response of the expert j, and $m_j(Y_j)$ the associated belief function value. The corresponding algorithm, building the whole belief function, is algorithm 1 provided in appendix.

In the context of the DSmT, the conjunctive rule can be used for all $X \in D^\Theta$ and $Y \in D^\Theta$. The rule given by the equation (2.6), called DSmH [16], can be written in D^Θ for all $X \in D^\Theta$, $X \not\equiv \emptyset$ [1] by:

$$\begin{aligned} m_H(X) = & \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) \; + \sum_{\substack{Y_1 \cup \ldots \cup Y_M = X \\ Y_1 \cap \ldots \cap Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) \; + \\ & \sum_{\substack{\{u(Y_1) \cup \ldots \cup u(Y_M) = X\} \\ Y_1, \ldots, Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) \; + \sum_{\substack{\{u(Y_1) \cup \ldots \cup u(Y_M) \equiv \emptyset \text{ and } X = \Theta\} \\ Y_1, \ldots, Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \end{aligned} \tag{2.7}$$

where $Y_j \in D^\Theta$ is the response of the expert j, $m_j(Y_j)$ the associated belief function, and $u(Y)$ is the function giving the union of the terms that compose Y [17]. For example if $Y = (A \cap B) \cup (A \cap C)$, $u(Y) = A \cup B \cup C$.

If we want to take the decision only on the elements in Θ, some rules propose to redistribute the conflict on these elements. The most accomplished is the PCR5 given in [18] for two experts

[1] The notation $X \not\equiv \emptyset$ means that $X \neq \emptyset$ and following the chosen model in D^Θ, X is not one of the elements of D^Θ defined as $\emptyset$. For example, if $\Theta = \{A, B, C\}$, we can define a model for which the expert can provide a mass on $A \cap B$ and not on $A \cap C$, so $A \cap B \not\equiv \emptyset$ and $A \cap B \equiv \emptyset$

and for $X \in D^\Theta$, $X \neq \emptyset$ by:

$$m_{PCR5}(X) = m_c(X) + \sum_{\substack{Y \in D^\Theta \\ X \cap Y \equiv \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right), \tag{2.8}$$

where $m_c(.)$ is the conjunctive rule given by the equation (2.5).

Note that more rules managing the conflict have been proposed [8, 9, 11, 12, 18, 26]. The comparison of all the combination rules is not the scope of this paper.

2.2.3 Decision rules

The decision is a difficult task. No measures are able to provide the best decision in all the cases. Generally, we consider the maximum of one of the three functions: credibility, plausibility, and pignistic probability.

In the context of the DST, the credibility function is given for all $X \in 2^\Theta$ by:

$$\mathrm{bel}(X) = \sum_{Y \in 2^X, Y \neq \emptyset} m(Y). \tag{2.9}$$

The plausibility function is given for all $X \in 2^\Theta$ by:

$$\mathrm{pl}(X) = \sum_{Y \in 2^\Theta, Y \cap X \neq \emptyset} m(Y) = \mathrm{bel}(\Theta) - \mathrm{bel}(X^c), \tag{2.10}$$

where X^c is the complementary of X. The pignistic probability, introduced by [21], is here given for all $X \in 2^\Theta$, with $X \neq \emptyset$ by:

$$\mathrm{betP}(X) = \sum_{Y \in 2^\Theta, Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)}. \tag{2.11}$$

Generally the maximum of these functions is taken on the elements in Θ, but we will give the values on all the focal elements.

In the context of the DSmT the corresponding generalized functions have been proposed [4, 16]. The generalized credibility Bel is defined by:

$$\mathrm{Bel}(X) = \sum_{Y \in D^\Theta, Y \subseteq X, Y \not\equiv \emptyset} m(Y) \tag{2.12}$$

The generalized plausibility Pl is defined by:

$$\mathrm{Pl}(X) = \sum_{Y \in D^\Theta, X \cap Y \not\equiv \emptyset} m(Y) \tag{2.13}$$

The generalized pignistic probability is given for all $X \in D^\Theta$, with $X \neq \emptyset$ is defined by:

$$\mathrm{GPT}(X) = \sum_{Y \in D^\Theta, Y \not\equiv \emptyset} \frac{\mathcal{C}_\mathcal{M}(X \cap Y)}{\mathcal{C}_\mathcal{M}(Y)} m(Y), \tag{2.14}$$

where $\mathcal{C}_\mathcal{M}(X)$ is the DSm cardinality corresponding to the number of parts of X in the Venn diagram of the problem [4, 16].

If the credibility function provides a pessimist decision, the plausibility function is often too optimist. The pignistic probability is often taken as a compromise. We present the three functions for our models.

2.3 The generalized PCR rules

In the equation (2.8), the PCR5 is given for two experts only. Two extensions for three experts and two classes are given in [19], and the equation for M experts for $X \in D^\Theta$, $X \not\equiv \emptyset$ is given in [5] and implemented in algorithm 2.

$$m_{\mathrm{PCR5}}(X) = m_{\mathrm{c}}(X) +$$
$$\sum_{i=1}^{M} m_i(X) \sum_{\substack{(Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)})\in(D^\Theta)^{M-1} \\ \bigcap_{k=1}^{M-1} Y_{\sigma_i(k)}\cap X\equiv\emptyset}} \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\mathbb{1}_{j>i}\right)\prod_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{\sum_{Z\in\{X,Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}} \prod_{Y_{\sigma_i(j)}=Z}\left(m_{\sigma_i(j)}(Y_{\sigma_i(j)}).T(X{=}Z, m_i(X))\right)}, \quad (2.15)$$

where σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j+1 & \text{if } j \geq i, \end{cases} \quad (2.16)$$

and:

$$\begin{cases} T(B,x) = x & \text{if } B \text{ is true}, \\ T(B,x) = 1 & \text{if } B \text{ is false}, \end{cases} \quad (2.17)$$

We propose another generalization of the equation (2.8) for M experts for $X \in D^\Theta$, $X \neq \emptyset$, implemented in algorithm 3. This defines the rule PCR6.

$$\begin{aligned} m_{\mathrm{PCR6}}(X) &= m_{\mathrm{c}}(X) + \\ & \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)}\cap X\equiv\emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)})\in(D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X)+\sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}\right), \end{aligned} \quad (2.18)$$

where σ is defined like in (2.16).

As Y_i is a focal element of expert i, $m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$; the belief function m_c is the conjunctive consensus rule given by the equation (2.5).

We can propose two more general rules given by:

$$\begin{aligned} m_{\mathrm{PCR6f}}(X) &= m_{\mathrm{c}}(X) + \\ & \sum_{i=1}^{M} m_i(X) f(m_i(X)) \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)}\cap X\equiv\emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)})\in(D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{f(m_i(X))+\sum_{j=1}^{M-1} f(m_{\sigma_i(j)}(Y_{\sigma_i(j)}))}\right), \end{aligned} \quad (2.19)$$

with the same notations that in the equation (2.18), and f an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$.

The second generalized rule is given by:

$$m_{\text{PCR6g}}(X) = m_c(X) + \sum_{i=1}^{M} \sum_{\substack{\overset{M-1}{\underset{k=1}{\cap}} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}}$$

$$m_i(X) \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)\left(\prod_{Y_{\sigma_i(j)}=X} \mathbb{1}_{j>i}\right) g\left(m_i(X) + \sum_{Y_{\sigma_i(j)}=X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})\right)}{\sum_{Z\in\{X,Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}} g\left(\sum_{Y_{\sigma_i(j)}=Z} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) + m_i(X)\mathbb{1}_{X=Z}\right)}, \tag{2.20}$$

with the same notations that in the equation (2.18), and g an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$. These rules are implemented in algorithms 4 and 5.

For instance, we can choose $f(x) = g(x) = x^\alpha$, with $\alpha \in \mathbb{R}^+$.

Algorithms for Dubois and Prade's rule (equation (2.6)), the PCR5 (equation (2.15)), the PCR6 (equation (2.18)), the PCR6f (equation (2.19)), and the PCR6g (equation (2.20)) combinations are given in appendix.

Remarks on the generalized PCR rules

- $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X \equiv \emptyset$ means that $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X$ is considered as a conflict by the model: $m_i(X) \prod_{k=1}^{M-1} m_{\sigma_i(k)}(Y_{\sigma_i(k)})$ has to be redistributed on X and the Y_k.

- The second term of the equation (2.18) is null if $\overset{M-1}{\underset{k=1}{\cap}} Y_k \cap X \not\equiv \emptyset$, hence in a general model in D^Θ for all X and Y in $D^\Theta \setminus \{\emptyset\}$, $X \cap Y \neq \emptyset$. The PCR5 and PCR6 are exactly the conjunctive rule: there is never any conflict. However in 2^{2^Θ}, there exists X and Y such that $X \cap Y = \emptyset$.

- One of the principal problem of the PCR5 and PCR6 rules is the non associativity. That is a real problem for dynamic fusion. Take for example three experts and two classes giving:

	$\emptyset$	A	B	Θ
Expert 1	0	1	0	0
Expert 2	0	0	1	0
Expert 3	0	0	1	0

If we fuse the expert 1 and 2 and then 3, the PCR5 and the PCR6 rules give:

$$\begin{cases} m_{12}(A) = 0.5, & m_{12}(B) = 0.5, \\ \text{and} & \\ m_{(12)3}(A) = 0.25, & m_{(12)3}(B) = 0.75. \end{cases} \tag{2.21}$$

Now if we fuse the experts 2 and 3 and then 1, the PCR5 and the PCR6 rules give:

$$\begin{cases} m_{23}(A) = 0, & m_{23}(B) = 1, \\ \text{and} & \\ m_{1(23)}(A) = 0.5, & m_{1(23)}(B) = 0.5, \end{cases} \tag{2.22}$$

and the result is not the same.

With the generalized PCR6 rule we obtain:

$$m_{(123)}(A) = 1/3, \; m_{(123)}(B) = 2/3, \tag{2.23}$$

a more intuitive and expected result.

- The conflict is not only redistributed on singletons. For example if three experts give:

	$A \cup B$	$B \cup C$	$A \cup C$	Θ
Expert 1	0.7	0	0	0.3
Expert 2	0	0	0.6	0.4
Expert 3	0	0.5	0	0.5

The conflict is given here by 0.7×0.6×0.5=0.21, with the generalized PCR6 rule we obtain:

$$\begin{aligned} m_{(123)}(A) &= 0.21, \\ m_{(123)}(B) &= 0.14, \\ m_{(123)}(C) &= 0.09, \\ m_{(123)}(A \cup B) &= 0.14 + 0.21.\frac{7}{18} \simeq 0.2217, \\ m_{(123)}(B \cup C) &= 0.06 + 0.21.\frac{5}{18} \simeq 0.1183, \\ m_{(123)}(A \cup C) &= 0.09 + 0.21.\frac{6}{18} = 0.16, \\ m_{(123)}(\Theta) &= 0.06. \end{aligned} \tag{2.24}$$

2.4 Discussion on the decision following the combination rules

In order to compare the previous rules in this section, we study the decision on the basic belief assignments obtained by the combination. Hence, we consider here the induced order on the singletons given by the plausibility, credibility, pignistic probability functions, or directly by the masses. Indeed, in order to compare the combination rules, we think that the study on the induced order of these functions is more informative than the obtained masses values. All the combination rules presented here are not idempotent, for instance for the conjunctive non-normalized rule:

	$\emptyset$	A	B	C
m_1	0	0.6	0.3	0.1
m_1	0	0.6	0.3	0.1
m_{11}	0.54	0.36	0.09	0.01

So, if we only compare the rules by looking at the obtained masses, we have to normalize them with the auto-conflict given by the combination of a mass with itself. However, if $m_1(A) > m_1(B)$, then $m_{11}(A) > m_{11}(B)$.

2.4.1 Extending the PCR rule for more than two experts

In [19], two approaches are presented in order to extend the PCR5 rule. The second approach suggests to fuse the first two experts and then fuse the third expert. However the solution depend on the order of the experts because of the non-associativity of the rule, and so it is not satisfying.

The first approach proposed in [19], that is the equation (2.15) proposes to redistribute the conflict about the singleton, *e.g.* if we have $m_1(A)m_3(B)m_2(A\cup B)$, the conflict is redistributed on A and B proportionally to $m_1(A)$ and $m_3(B)$. But this approach do not give solution if we have for instance $m_1(A\cup B)m_2(B\cup C)m_3(A\cup C)$ where the conflict is $A\cap B\cap C$ and we have no idea on the masses for A, B and C.

Moreover, if we have $m_1(A)m_2(B)m_3(B)$ the proposed solution distributes the conflict to A and B with respect to $m_1(A)$ and $m_2(B)m_3(B)$ and not $m_2(B)+m_3(B)$ that is more intuitive. For example, if $m_1(A) = m_2(B) = m_3(B) = 0.5$, 0.0833 and 0.0416 is added to the masses A and B respectively, while there is more consensus on B than on A and we would expected the contrary: 0.0416 and 0.0833 could be added to the masses A and B respectively.

What is more surprising are the results given by PCR5 and PCR6 on the following example:

	A	B	C	D	E	F	G
Expert 1	0.0	0.57	0.43	0.0	0.0	0.0	0.0
Expert 2	0.58	0.0	0.0	0.42	0.0	0.0	0.0
Expert 3	0.58	0.0	0.0	0.0	0.42	0.0	0.0
Expert 4	0.58	0.0	0.0	0.0	0.0	0.42	0.0
Expert 5	0.58	0.0	0.0	0.0	0.0	0.0	0.42

As all the masses are on singletons, neither PCR5 nor PCR6 can put any mass on total or partial ignorance. So the fusion result is always a probability, and $\text{bel}(X) = \text{betP}(X) = \text{pl}(X)$.

Conflict is total: conjunctive rule does not provide any information. PCR5 and PCR6 give the following results:

	A	B	C	D	E	F	G
PCR5	0.1915	**0.2376**	0.1542	0.1042	0.1042	0.1042	0.1042
PCR6	**0.5138**	0.1244	0.0748	0.0718	0.0718	0.0718	0.0718

So decision is "A" according to PCR6, and decision is "B" according to PCR5. However, for any subset of 2, 3 or 4 experts, decision is "A" for any of these combination rules.

2.4.2 Stability of decision process

The space where experts can define their opinions on which n classes are present in a given tile is a part of $[0,1]^n$: $\mathcal{E} = [0,1]^n \cap \left\{(x_1, ..., x_n) \in \mathbb{R} / \sum_{i=1}^{n} x_i \leq 1\right\}$. In order to study the different combination rules, and the situations where they differ, we use a Monte Carlo method, considering the masses given on each class (a_X) by each expert, as uniform variables, filtering them by the condition $\sum_{X \in \Theta} a_X \leq 1$ for one expert.

Thus, we measure the proportion of situations where decision differs between the conjunctive combination rule, and the PCR, where conflict is proportionally distributed.

We can not choose $A \cap B$, as the measure of $A \cap B$ is always lower (or equal with probability 0) than the measure of A or B. In the case of two classes, $A \cup B$ is the total ignorance, and is usually excluded (as it always maximizes bel, pl, betP, Bel, Pl and GPT). We restrict the possible choices to singletons, A, B, etc. Therefore, it is equivalent to tag the tile by the most credible class (maximal for bel), the most plausible (maximal for pl), the most probable (maximal for betP) or the heaviest (maximal for m), as the only focal elements are singletons, Θ and $\emptyset$.

The only situation where the total order induced by the masses m on singletons can be modified is when the conflict is distributed on the singletons, as is the case in the PCR method.

Thus, for different numbers of classes, the decision obtained by fusing the experts' opinions is much less stable:

number of classes	2	3	4	5	6	7
decision change in the two experts case						
PCR/DST	0.61%	5.51%	9.13%	12.11%	14.55%	16.7%
PCR/DP	0.61%	2.25%	3.42%	4.35%	5.05%	5.7%
DP/DST	0.00%	3.56%	6.19%	8.39%	10.26%	11.9%
decision change in the three experts case						
PCR6/DST	1.04%	8.34%	13.90%	18.38%	21.98%	25.1%
PCR6/DP	1.04%	5.11%	7.54%	9.23%	10.42%	11.3%
DP/DST	0.00%	4.48%	8.88%	12.88%	16.18%	19.0%

Therefore, the specificity of PCR6 appears mostly with more than two classes, and the different combination rules are nearly equivalent when decision must be taken within two possible classes.

For two experts and two classes, the mixed rule (DP) and the conjunctive rule are equivalent. For three experts, we use the generalized PCR6 (2.18).

The percentage of decision differences defines a distance between fusion methods:

$$d(\text{PCR6,DST}) \leq d(\text{PCR6,DP}) + d(\text{DP,DST}).$$

The two other triangular inequalities are also true. As we have $d(PCR6, DST) \geq d(\text{PCR6,DP})$ and $d(\text{PCR,DST}) \geq d(\text{DP,DST})$ for any number of experts or classes, we can conclude that the mixed rule lies between the PCR6 method and the conjunctive rule.

The figure 2.1 shows the density of conflict within $\mathcal{E}$. The left part shows the conflict for two random experts and a number of classes of 2, 3 or 7. Plain lines show conflict when there

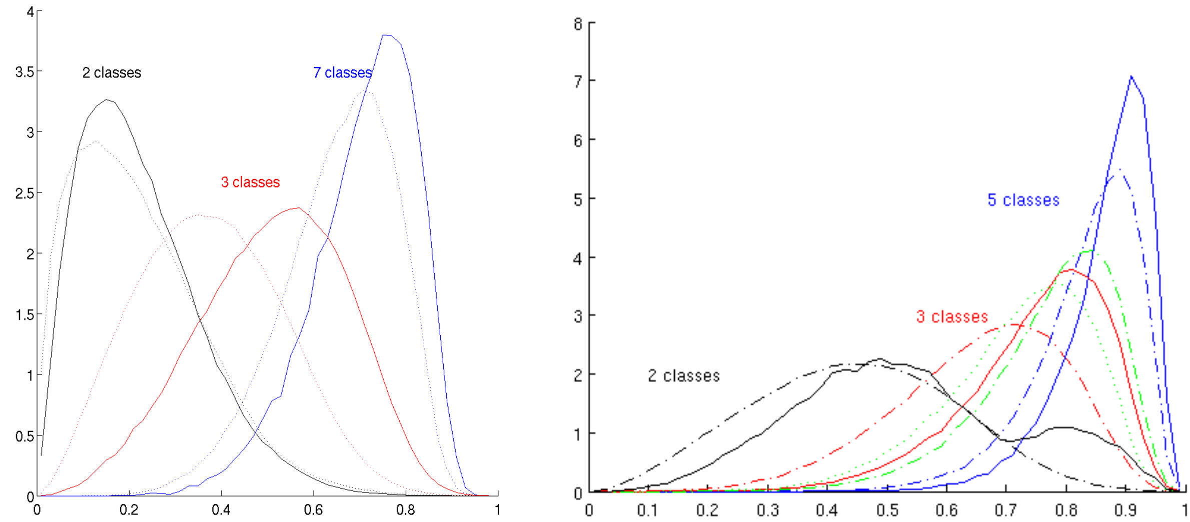

Figure 2.1: Density of conflict for (left) two uniform random experts and (right) three uniform random experts; with and without decision change

is difference between decisions, and dashed lines show the overall conflict. Right part shows the conflict values for three experts; plain lines show the conflict where there is a difference between the PCR rule and the conjunctive rule.

Conflict is more important in this subspace where decision changes with the method used, mostly because a low conflict often means a clear decision. The measure on the best class is then very different than measure on the second best class.

Dashed green line represents the conflict density for 3 classes when there is a difference between conjunctive rule and mixed rule. Dotted green line represents the conflict density for 3 classes when there is a difference between PCR6 rule and mixed rule. We can see that an high conflict level emphasizes mostly a decision change between conjunctive and mixed rule.

2.4.3 Calculi for two experts and two classes

For the "two experts and two classes" case, it is difficult to characterize analytically the stability of the decision process between the conjunctive rule and the PCRrule (the PCR5 and PCR6 rules are the same in the two experts case). Note that in this case the DSmH rule given by the equation (2.7), the mixed rule given by the equation (2.6) and the conjunctive rule given by the equation (2.5) are equal. However, we can easily resolve few cases where the final decision does not depend on the chosen combination rule.

Standard repartition of expert's opinions is given by this table:

	$\emptyset$	A	B	Θ
Expert 1	0	a_1	b_1	$1 - a_1 - b_1$
Expert 2	0	a_2	b_2	$1 - a_2 - b_2$

The conjunctive rule gives:

$$m_c(\emptyset) = a_1 b_2 + a_2 b_1,$$

$$m_c(A) = a_1 + a_2 - a_1 a_2 - a_1 b_2 - a_2 b_1 = a_1 + a_2 - a_1 a_2 - m_c(\emptyset),$$

$$m_c(B) = b_1 + b_2 - b_1b_2 - a_1b_2 - a_2b_1 = b_1 + b_2 - b_1b_2 - m_c(\emptyset),$$

$$m_c(\Theta) = (1 - a_1 - b_1)(1 - a_2 - b_2).$$

PCR gives:

$$m_{PCR}(A) = m(A) + \frac{a_1^2 b_2}{a_1 + b_2} + \frac{a_2^2 b_1}{a_2 + b_1},$$

$$m_{PCR}(B) = m(B) + \frac{a_1 b_2^2}{a_1 + b_2} + \frac{a_2 b_1^2}{a_2 + b_1},$$

$$m_{PCR}(\emptyset) = 0 \text{ and } m_{PCR}(\Theta) = m_c(\Theta).$$

The stability of the decision is reached if we do not have:

$$\begin{cases} m_c(A) > m_c(B) \text{ and } m_{PCR}(A) < m_{PCR}(B) \\ \text{or} \\ m_c(A) < m_c(B) \text{ and } m_{PCR}(A) > m_{PCR}(B) \end{cases} \tag{2.25}$$

That means for all a_1, a_2, b_1 and $b_2 \in [0,1]$:

$$\begin{array}{l} \begin{cases} a_2 + a_1(1-a_2) - b_1(b_2-1) - b_2 > 0 \\ a_1(1-a_2) + a_2\left((1 + b_1\left(1 - \frac{2}{(1+a_2/b_1)}\right)\right) - b_1(1-b_2) \\ \ -b_2\left(1 + a_1\left((1 - \frac{2}{(1+b_2/a_1))}\right)\right) < 0 \\ a_1 + b_1 \in [0,1] \\ a_2 + b_2 \in [0,1] \end{cases} \\ \text{or} \\ \begin{cases} a_2 + a_1(1-a_2) - b_1(b_2-1) - b_2 < 0 \\ a_1(1-a_2) + a_2\left((1 + b_1\left(1 - \frac{2}{(1+a_2/b_1)}\right)\right) - b_1(1-b_2) \\ \ -b_2\left(1 + a_1\left((1 - \frac{2}{(1+b_2/a_1))}\right)\right) > 0 \\ a_1 + b_1 \in [0,1] \\ a_2 + b_2 \in [0,1] \end{cases} \end{array} \tag{2.26}$$

This system of inequalities is difficult to solve, but with the help of a Monte Carlo method, considering the weights a_1, a_2, b_1 and b_2, as uniform variables we can estimate the proportion of points (a_1, a_2, b_1, b_2) solving this system.

We note that absence of solution in spaces where $a_1 + b_1 > 1$ or $a_2 + b_2 > 1$ comes from the two last conditions of the system. Also there is no solution if $a_1 = b_1$ (or $a_2 = b_2$ by symmetry) and if $a_1 = b_2$ (or $a_2 = b_1$ by symmetry). This is proved analytically.

2.4.3.1 Case $a_1 = b_1$

In this situation, expert 1 considers that the data unit is equally filled with classes A and B:

	$\emptyset$	A	B	Θ
Expert 1	0	x	x	$1-2x$
Expert 2	0	y	z	$1-y-z$

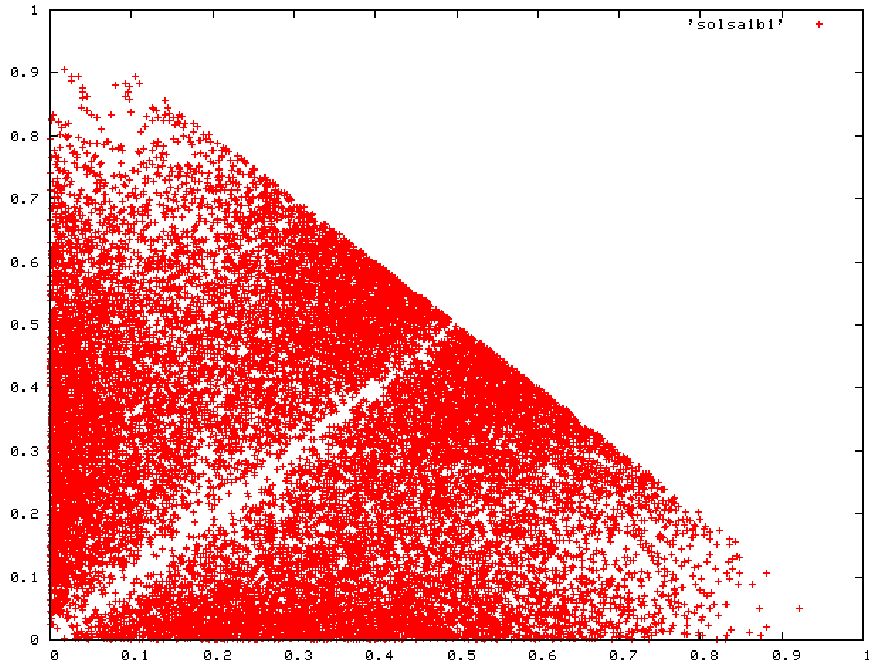

Figure 2.2: Decision changes, projected on the plane a_1, b_1.

The conjunctive rule yields:

$$m_c(\emptyset) = 2xy,$$
$$m_c(A) = x + y - 2xy - xz = x - m_c(\emptyset) + y(1-x),$$
$$m_c(B) = x + y - xy - 2xz = x - m_c(\emptyset) + z(1-x),$$
$$m_c(\Theta) = 1 - 2x - y - z + 2xy + 2xz.$$

Therefore, as $1 - x \geq 0$:

$$m_c(A) > m_c(B) \iff y > z.$$

The PCR yields:

$$m_{PCR}(\emptyset) = 0$$
$$m_{PCR}(A) = x - m_c(\emptyset) + y(1-x) + \frac{x^2 z}{x+z} + \frac{xy^2}{x+y},$$
$$m_{PCR}(B) = x - m_c(\emptyset) + z(1-x) + \frac{xz^2}{x+z} + \frac{x^2 y}{x+y},$$
$$m_{PCR}(\Theta) = 1 - 2x - y - z + 2xy + 2xz.$$

So, we have:

$$\begin{aligned}(m_{PCR}(A) + m_c(\emptyset) - x)(x+y)(x+z) &= y(1-x)(x+z)(x+y) \\ &\quad + x^2 z(x+y) + y^2 x(x+z) \\ &= y(x+y)(x+z) + x^3(z-y)\end{aligned}$$

$$(m_{PCR}(B) + m_c(\emptyset) - x)(x+y)(x+z) = z(x+y)(x+z) - x^3(z-y),$$

$$m_{PCR}(A) > m_{PCR}(B) \iff (y-z)((x+y)(x+y) - 2x^3) > 0.$$

As $0 \leq x \leq \frac{1}{2}$, we have $2x^3 \leq x^2 \leq (x+y)(x+z)$. So $m_{PCR}(A) > m_{PCR}(B)$ if and only if $y > z$.

That shows that the stability of the decision is reached if $a_1 = b_1$ for all a_2 and $b_2 \in [0,1]$ or by symmetry if $a_2 = b_2$ for all a_1 and $b_1 \in [0,1]$.

2.4.3.2 Case $a_1 = b_2$

In this situation, expert 1 believes A and the expert 2 believes B with the same weight:

	$\emptyset$	A	B	Θ
Expert 1	0	x	y	$1-x-y$
Expert 2	0	z	x	$1-x-z$

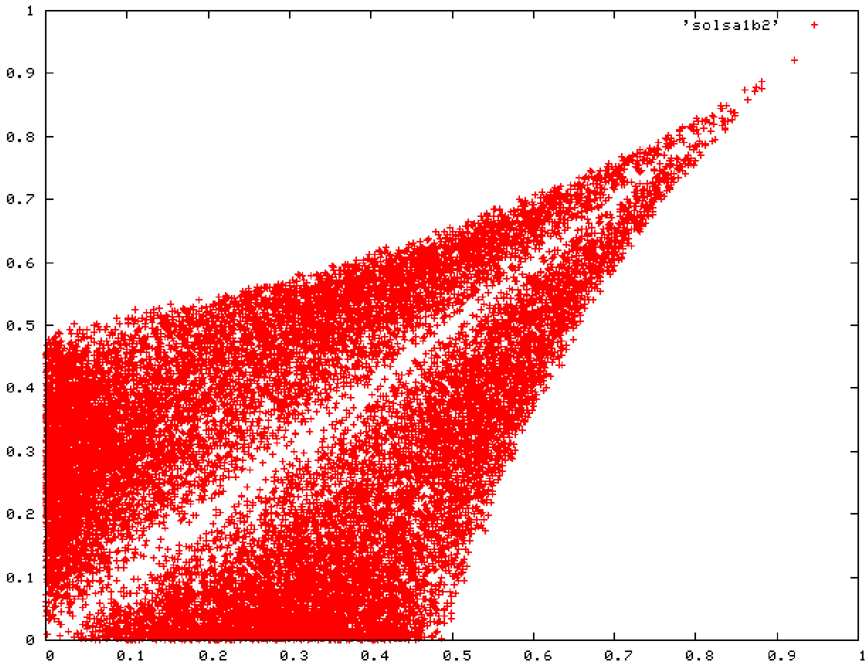

Figure 2.3: Decision changes, projected on the plane a_1, b_2.

The conjunctive rule yields:

$$m_c(\emptyset) = x^2 + yz,$$

$$m_c(A) = x + z - xz - m_c(\emptyset) = -x^2 + x(1-z) + z(1-y),$$

$$m_c(B) = x + y - xy - m_c(\emptyset) = -x^2 + x(1-y) + y(1-z),$$

$$m_c(\Theta) = 1 + m_c(\emptyset) - 2x - y - z + x(y+z).$$

Therefore:

$$m_c(A) > m_c(B) \iff (x-1)(y-z) > 0,$$

as $1 - x \geq 0$:

$$m_c(A) > m_c(B) \iff y > z.$$

The PCR yields:

$$m_{PCR}(\emptyset) = 0,$$

$$m_{PCR}(A) = x + z - xz - m_c(\emptyset) = -x^2 + x(1-z) + z(1-y) + \frac{x^3}{2x} + \frac{yz^2}{y+z},$$

$$m_{PCR}(B) = x + y - xy - m_c(\emptyset) = -x^2 + x(1-y) + y(1-z) + \frac{x^3}{2x} + \frac{y^2 z}{y+z},$$

$$m_{PCR}(\Theta) = 1 + m_c(\emptyset) - 2x - y - z + x(y+z).$$

Therefore:

$$m_{PCR}(A) > m_{PCR}(B) \iff (y-z)\left((x-1)(y+z) - yz\right) > 0,$$

as $(x-1) \leq 0$, $(x-1)(y+z) - yz \leq 0$ and:

$$m_{PCR}(A) > m_{PCR}(B) \iff y > z.$$

That shows that the stability of the decision is reached if $a_1 = b_2$ for all a_2 and $b_1 \in [0,1]$ or by symmetry if $a_2 = b_1$ for all a_1 and $b_2 \in [0,1]$.

2.4.3.3 Case $a_2 = 1 - a_1$

We can notice that if $a_1 + a_2 > 1$, no change occurs. In this situation, we have $b_1 + b_2 < 1$, but calculus is still to be done.

In this situation, if $a_2 = 1 - a_1$:

	$\emptyset$	A	B	Θ
Expert 1	0	x	y	$1-x-y$
Expert 2	0	$1-x$	z	$x-z$

The conjunctive rule yields:

$$m_c(\emptyset) = xz + (1-x)y,$$

$$m_c(A) = 1 + x^2 - x - y + xy - xz,$$

$$m_c(B) = z - yz + xy - xz,$$

$$m_c(\Theta) = -x^2 + x + xz - xy + yz - z.$$

Therefore:

$$m_c(A) > m_c(B) \iff 1 + x^2 - x > y + z - yz,$$

$$\iff x(1-x) > (1-y)(1-z),$$

as $z < x$ and $x < 1 - y$, $m_c(A) > m_c(B)$ is always true.

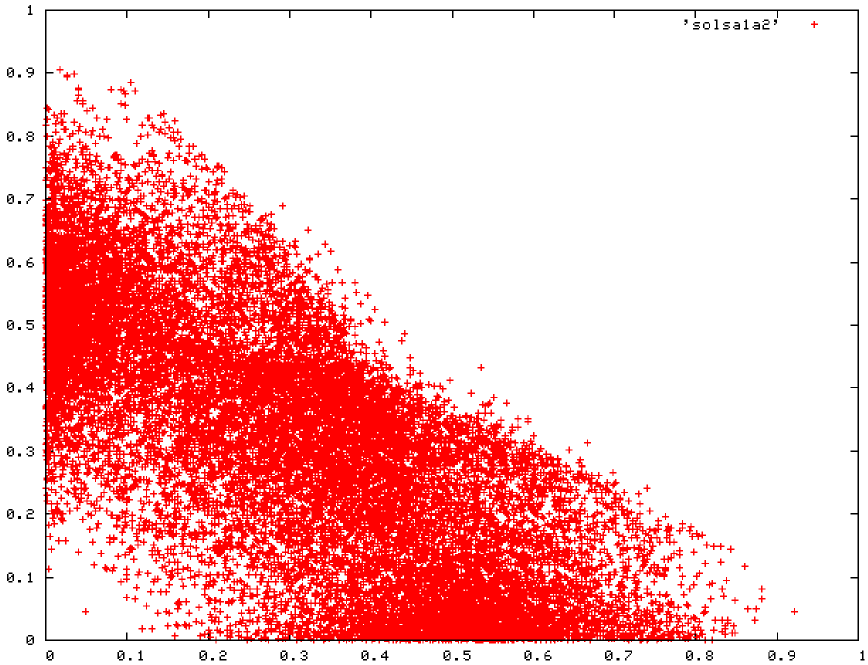

Figure 2.4: Decision changes, projected on the plane a_1, a_2.

The PCR yields:

$$m_{PCR}(\emptyset) = 0,$$

$$m_{PCR}(A) = m_c(A) + \frac{x^2 z}{x+z} + \frac{(1-x)^2 y}{1-x+y},$$

$$m_{PCR}(B) = m_c(B) + \frac{x z^2}{x+z} + \frac{(1-x) y^2}{1-x+y},$$

$$m_{PCR}(\Theta) = m_c(\Theta).$$

Therefore:

$$m_{PCR}(A) > m_{PCR}(B)$$

is always true.

Indeed $m_c(A) > m_c(B)$ is always true and:

$$\frac{x^2 z}{x+z} > \frac{x z^2}{x+z}$$

because $x > z$ and:

$$\frac{(1-x)^2 y}{1-x+y} > \frac{(1-x) y^2}{1-x+y}$$

because $1 - x > y$.

That shows that the stability of the decision is reached if $a_2 = 1 - a_1$ for all a_2 and $a_1 \in [0, 1]$ or by symmetry if $a_1 = 1 - a_2$ for all a_1 and $a_2 \in [0, 1]$.

2.5 Conclusion

In this chapter, we have proposed a study of the combination rules compared in term of decision. A new generalized proportional conflict redistribution (PCR6) rule have been proposed and discussed. We have presented the pro and con of this rule. The PCR6 rule is more intuitive than the PCR5. We have shown on randomly generated data, that there is a difference of decision following the choice of the combination rule (for the non-normalized conjunctive rule, the mixed conjunctive and disjunction rule of Dubois and Prade, the PCR5 rule and the PCR6 rule). We have also proven, on a two experts and two classes case, the changes following the values of the basic belief assignments. This difference can be very small in percentage and we can not say on these data if it is a significant difference. We have conducted this comparison on real data in the chapter [14].

All this discussion comes from a fine proportional conflict distribution initiated by the consideration of the extension of the discernment space in D^Θ. The generalized PCR6 rule can be used on 2^Θ or D^Θ.

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2.7 Appendix: Algorithms

An input belief function e is an association of a list of focal classes and their masses. We write $size(e)$ the number of its focal classes. The focal classes are $e[1]$, $e[2]$, ..., $e[size(e)]$. The mass associated to a class c is $e(c)$, written with parenthesis.

The principle of the algorithms is to use the variable *ind* to build all the n-uples of focal elements of the n input belief functions. Then, if the intersection of these is not $\emptyset$ or equivalent to $\emptyset$, the corresponding conjunctive mass (the product of all the masses of the focal elements in the n-uple) is put on the intersection; otherwise, this mass is put on the disjunction (Dubois and Prade algorithm) or redistributed over the input focal elements.

Algorithm 1: Conflict replaced on partial ignorance, by Dubois and Prade or DSmH

```
n experts ex: ex[1]...ex[n] Fusion of ex by Dubois-Prade method : edp  for i = 1 to n do
    foreach c in ex[i] do  Append c to cl[i];
foreach ind in [1, size(cl[1])] × [1, size(cl[2])] × ... × [1, size(cl[n])] do
    s ← Θ;
    for i = 1 to n do  s ← s ∩ cl[i][ind[i]];
    if s = ∅ then
        lconf ← 1;
        u ← ∅;
        for i = 1 to n do
            u ← p ∪ cl[i][ind[i]];
            lconf ← lconf × ex[i](cl[i][ind[i]]);
        edp(u) ← edp(u) + lconf;
    else
        lconf ← 1;
        for i = 1 to n do  lconf ← lconf × ex[i](cl[i][ind[i]]);
        edp(s) ← edp(s) + lconf;
```

Algorithm 2: Conflict redistributed by the PRC5 combination rule

---

n experts ex: $ex[1] \dots ex[n]$ Fusion of ex by PCR5 method : ep **for** i = *1 to* n **do**
  foreach c *in* $ex[i]$ **do** Append c to $cl[i]$;

foreach ind *in [1, size*($cl[1]$)*] × [1, size*($cl[2]$)*] × ... × [1, size*($cl[n]$)*]* **do**
  $s \leftarrow \Theta$;
  for i = *1 to* n **do** $s \leftarrow s \cap cl[i][ind[i]]$;
  if $s \equiv \emptyset$ **then**
    $lconf \leftarrow 1$; el is an empty expert;
    for i = *1 to* n **do**
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
      if $cl[i][ind[i]]$ *in* el **then** $el(cl[i][ind[i]]) \leftarrow el(cl[i][ind[i]]) * ex[i](cl[i][ind[i]])$;
      else $el(cl[i][ind[i]]) \leftarrow ex[i](cl[i][ind[i]])$;
    for c *in* el **do** $sum \leftarrow sum + el(c)$;
    for c *in* el **do** $ep(c) \leftarrow ep(c) + g(el(c)) * lconf/sum$;
  else
    $lconf \leftarrow 1$;
    for i = *1 to* n **do** $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $ep(s) \leftarrow ep(s) + lconf$;

Algorithm 3: Conflict redistributed by the PRC6 combination rule

---

n experts ex: $ex[1] \dots ex[n]$ Fusion of ex by PCR6 method : ep **for** i = *1 to* n **do**
  foreach c *in* $ex[i]$ **do** Append c to $cl[i]$;

foreach ind *in [1, size*($cl[1]$)*] × [1, size*($cl[2]$)*] × ... × [1, size*($cl[n]$)*]* **do**
  $s \leftarrow \Theta$;
  for i = *1 to* n **do** $s \leftarrow s \cap cl[i][ind[i]]$;
  if $s \equiv \emptyset$ **then**
    $lconf \leftarrow 1$; $sum \leftarrow 0$;
    for i = *1 to* n **do**
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
      $sum \leftarrow sum + ex[i](cl[i][ind[i]])$;
    for i = *1 to* n **do**
      $ep(ex[i][ind[i]]) \leftarrow ep(ex[i][ind[i]]) + ex[i](cl[i][ind[i]]) * lconf/sum$;
  else
    $lconf \leftarrow 1$;
    for i = *1 to* n **do** $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $ep(s) \leftarrow ep(s) + lconf$;

Algorithm 4: Conflict redistributed by the PRC6 combination rule, with a function f applied on masses before redistribution

n experts ex: $ex[1] \dots ex[n]$ A non-decreasing positive function f Fusion of ex by PCR6$_f$ method : ep **for** i = *1 to n* **do**
  foreach c *in* $ex[i]$ **do** Append c to $cl[i]$;

foreach *ind in [1, size(cl[1])]* $\times$ *[1, size(cl[2])]* $\times \dots \times$ *[1, size(cl[n])]* **do**
  $s \leftarrow \Theta$;
  for i = *1 to n* **do** $s \leftarrow s \cap cl[i][ind[i]]$;
  if $s \equiv \emptyset$ **then**
    $lconf \leftarrow 1$; $sum \leftarrow 0$;
    for i = *1 to n* **do**
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
      $sum \leftarrow sum + f(ex[i](cl[i][ind[i]]))$;
    for i = *1 to n* **do**
      $ep(ex[i][ind[i]]) \leftarrow ep(ex[i][ind[i]]) + f(ex[i](cl[i][ind[i]])) * lconf/sum$;
  else
    $lconf \leftarrow 1$;
    for i = *1 to n* **do** $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $ep(s) \leftarrow ep(s) + lconf$;

Algorithm 5: Conflict redistributed by the PRC6 combination rule, with a function g applied on masses sums

n experts ex: $ex[1] \dots ex[n]$ A non-decreasing positive function g Fusion of ex by PCR6$_g$ method : ep **for** i = *1 to n* **do**
  foreach c *in* $ex[i]$ **do** Append c to $cl[i]$;

foreach *ind in [1, size(cl[1])]* $\times$ *[1, size(cl[2])]* $\times \dots \times$ *[1, size(cl[n])]* **do**
  $s \leftarrow \Theta$;
  for i = *1 to n* **do**
    $s \leftarrow s \cap cl[i][ind[i]]$;
  if $s \equiv \emptyset$ **then**
    $lconf \leftarrow 1$; el is an empty expert;
    for i = *1 to n* **do**
      $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
      $el(cl[i][ind[i]]) \leftarrow el(cl[i][ind[i]]) + ex[i](cl[i][ind[i]])$;
    $sum \leftarrow 0$;
    for c *in el* **do** $sum \leftarrow sum + g(el(c))$;
    for c *in el* **do** $ep(c) \leftarrow ep(c) + g(el(c)) * lconf/sum$;
  else
    $lconf \leftarrow 1$;
    for i = *1 to n* **do** $lconf \leftarrow lconf \times ex[i](cl[i][ind[i]])$;
    $ep(s) \leftarrow ep(s) + lconf$;

Chapter 3

Classical Combination Rules Generalized to DSm Hyper-power Sets and their Comparison with the Hybrid DSm Rule

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Abstract: *Dempster's rule, non-normalized conjunctive rule, Yager's rule and Dubois-Prade's rule for belief functions combination are generalized to be applicable to hyper-power sets according to the DSm theory. A comparison of the rules with DSm rule of combination is presented. A series of examples is included.*

3.1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Belief functions were originally introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [13].

For a combination of beliefs Dempster's rule of combination is used in DST. Under strict probabilistic assumptions, its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. It is not uncommon to find examples where the assumptions are not fulfilled and where results of

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Dempster's rule are counter-intuitive, e.g. see [1, 2, 14], thus a rule with more intuitive results is required in such situations.

Hence, a series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are Dubois and Prade's rule [9] and Yager's rule of belief combination [17]. Others include a wide class of weighted operators [12] and an analogous idea proposed in [11], the Transferable Belief Model (TBM) using the so-called non-normalized Dempster's rule [16], disjunctive (or dual Dempster's) rule of combination [4, 8], combination 'per elements' with its special case — minC combination, see [3], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [10].

A brand new approach performs the Dezert-Smarandache (or Dempster-Shafer modified) theory (DSmT) with its DSm rule of combination. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general; mathematically it means that belief functions are not defined on the power set of the frame, but on a so-called hyper-power set, i.e., on the Dedekind lattice defined by the frame; 2) a new combination mechanism which overcomes problems with conflict among the combined beliefs and which also enables a dynamic fusion of beliefs.

As the classical Shafer's frame of discernment may be considered the special case of a so-called hybrid DSm model, the DSm rule of combination is compared with the classic rules of combination in the publications about DSmT [7, 14].

Unfortunately, none of the classical combination rules has been formally generalized to hyper-power sets, thus their comparison with the DSm rule is not fully objective until now.

This chapter brings a formal generalization of the classical Dempster's, non-normalized conjunctive, Dubois-Prade's, and Yager's rules to hyper-power sets. These generalizations perform a solid theoretical background for a serious objective comparison of the DSm rule with the classical combination rules.

The classic definitions of Dempster's, Dubois-Prade's, and Yager's combination rules are briefly recalled in Section 3.2, basic notions of DSmT and its state which is used in this text (Dedekind lattice, hyper-power set, DSm models, and DSmC and DSmH rules of belief combination) are recalled in Section 3.3.

A generalization of Dempster's rule both in normalized and non-normalized versions is presented in Section 3.4, and a generalization of Yager's rule in Section 3.5. Both these classic rules are straightforwardly generalized as their ideas work on hyper-power sets simply without any problem.

More interesting and more complicated is the case of Dubois-Prade's rule. The nature of this rule is closer to DSm rule, but on the other hand the generalized Dubois-Prade's rule is not compatible with a dynamic fusion in general. It works only for a dynamic fusion without non-existential constraints, whereas a further extension of the generalized rule is necessary in the case of a dynamic fusion with non-existential constraints.

Section 3.7 presents a brief comparison of the rules. There is a series of examples included. All the generalized combination rules are applied to belief functions from examples from the DSmT book Vol. 1 [14]. Some open problems for a future research are mentioned in Section 3.8 and the concluding Section 3.9 closes the chapter.

3.2 Classic definitions

All the classic definitions assume an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0,1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$, the values of bba are called *basic belief masses (bbm).* The value $m(A)$ is called the *basic belief mass*[1] *(bbm) of A*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^Θ. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$. If a focal element is a one-element subset of Θ, we are referring to a *singleton.*

Let us start with the classic definition of Dempster's rule. *Dempster's (conjunctive) rule of combination* $\oplus$ is given as $(m_1 \oplus m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} K m_1(X)m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, with $\kappa = \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [13]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\odot$, see e. g. [16].

Yager's rule of combination $\circledY$, see [17], is given as
$(m_1 \circledY m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subset \Theta$,
$(m_1 \circledY m_2)(\Theta) = m_1(\Theta)m_2(\Theta) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset} m_1(X)m_2(Y)$,
and $(m_1 \circledY m_2)(\emptyset) = 0$;

Dubois-Prade's rule of combination $\circledD$ is given as
$(m_1 \circledD m_2)(A) = \sum_{X,Y\subseteq\Theta,\, X\cap Y=A} m_1(X)m_2(Y) + \sum_{X,Y\subseteq\Theta,\, X\cap Y=\emptyset, X\cup Y=A} m_1(X)m_2(Y)$ for $\emptyset \neq A \subseteq \Theta$, and $(m_1 \circledD m_2)(\emptyset) = 0$, see [9].

3.3 Introduction to the DSm theory

Because DSmT is a new theory which is in permanent dynamic evolution, we have to note that this text is related to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [14]. Rapid development of the theory is demonstrated by appearing of the current second volume of the book. For new advances of DSmT see other chapters of this volume.

3.3.1 Dedekind lattice, basic DSm notions

Dempster-Shafer modified Theory or Dezert-Smarandache Theory (DSmT) by J. Dezert and F. Smarandache [7, 14] allows mutually overlapping elements of a frame of discernment. Thus, a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example, we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[2]. DSmT allows that an object can have 2 or 3

[1] $m(\emptyset) = 0$ *is often assumed in accordance with Shafer's definition [13]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive* $m(\emptyset)$ *as it assumes* $m(\emptyset) \geq 0$.

[2] www.gallup.unm.edu/∼smarandache/DSmT.htm

colours at the same time: e.g. it can be both red and blue, or red and green and blue in the same time, it corresponds to a composition of the colours from the 3 basic ones.

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), but they are defined on a so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions, they are called generalized basic belief assignments and generalized basic belief functions.

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ now, we have $D^\Theta = \{\alpha_0, \alpha_1, ...\alpha_{18}\}$, where $\alpha_0 = \emptyset, \alpha_1 = \theta_1 \cap \theta_2 \cap \theta_3, \alpha_2 = \theta_1 \cap \theta_2, \alpha_3 = \theta_1 \cap \theta_3, ..., \alpha_{17} = \theta_2 \cup \theta_3, \alpha_{18} = \theta_1 \cup \theta_2 \cup \theta_3$, i.e., $|D^\Theta| = 19$ for $|\Theta| = 3$.

A *generalized basic belief assignment (gbba)* m is a mapping $m : D^\Theta \longrightarrow [0,1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *generalized basic belief mass (gbbm) of* A. A *generalized belief function (gBF) Bel* is a mapping $Bel : D^\Theta \longrightarrow [0,1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$, generalized belief function Bel uniquely corresponds to gbba m and vice-versa.

3.3.2 DSm models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i.e., all elements of an exhaustive frame of discernment can mutually overlap themselves, we refer to the *free DSm model* $\mathcal{M}^f(\Theta)$, i.e., about the DSm model free of constraints.

In general it is possible to add exclusivity or non-existential constraints into DSm models, we speak about *hybrid DSm models* in such cases.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows them: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case all the elements of the 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model as *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings additional information about a frame of discernment saying that θ_3 is impossible; it forces all the gbbm's of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination which is used in a dynamic fusion.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ (*vacuous DSm model* in [14]) we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only case where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_{\mathcal{M}}, \emptyset\}$, where $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of all elements of D^{Θ} which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set[3].

For a given DSm model we can define (in addition to [14]) $\Theta_{\mathcal{M}} = \{\theta_i | \theta_i \in \Theta, \theta_i \not\in \boldsymbol{\emptyset}_{\mathcal{M}}\}$, $\Theta_{\mathcal{M}} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_{\mathcal{M}} = \bigcup_{\theta_i \in \Theta_{\mathcal{M}}} \theta_i$, i.e. $I_{\mathcal{M}} \overset{\mathcal{M}}{\equiv} I_t$, $I_{\mathcal{M}} = I_t \cap \Theta_{\mathcal{M}}$, $I_{\mathcal{M}_{\emptyset}} = \emptyset$. $D^{\Theta_{\mathcal{M}}}$ is a hyper-power set on the DSm frame of discernment $\Theta_{\mathcal{M}}$, i.e., on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. It holds $\Theta_{\mathcal{M}} = \Theta$, $D^{\Theta_{\mathcal{M}}} = D^{\Theta}$and $I_{\mathcal{M}} = I_t$ for any DSm model without non-existential constraint. Whereas *reduced (or constrained) hyper-power set* $D^{\Theta}_{\mathcal{M}}$ (or $D^{\Theta}(\mathcal{M})$) from Chapter 4 in [14] arises from D^{Θ} by identifying of all $\mathcal{M}$-equivalent elements. $D^{\Theta}_{\mathcal{M}^0}$ corresponds to classic power set 2^{Θ}.

3.3.3 The DSm rules of combination

The *classic DSm rule DSmC* is defined on the free DSm models as it follows[4]:
$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledast m_2)(A) = \sum_{X,Y \in D^{\Theta}, X \cap Y = A} m_1(X) m_2(Y)$.
Since D^{Θ} is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classic DSm rule guarantees that $(m_1 \circledast m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [14].

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the nature of the problem under consideration, which requires us to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_{\emptyset}(\Theta)$, for $k \geq 2$ independent sources of information is defined as: $m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledast m_2 \circledast ... \circledast m_k)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)]$, where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \not\in \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined for two sources (for n-ary versions see [14]) as it follows:
$S_1(A) = \sum_{X,Y \in D^{\Theta}, X \cap Y = A} m_1(X) m_2(Y)$,
$S_2(A) = \sum_{X,Y \in \boldsymbol{\emptyset}, [\mathcal{U}=A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]} m_1(X) m_2(Y)$,
$S_3(A) = \sum_{X,Y \in D^{\Theta}, X \cup Y = A, X \cap Y \in \boldsymbol{\emptyset}} m_1(X) m_2(Y)$ with $\mathcal{U} = u(X) \cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are supposed to be in some canonical form, e.g. CNF. Unfortunately no mention about the canonical form is included in [14]. $S_1(A)$ corresponds to the classic DSm rule on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets in both the input gbba's, which arises due to non-existential constraints and is transferred to the total or relative ignorance; and $S_3(A)$ transfers the sum of masses of relatively and absolutely empty sets, which arise as conflicts of the input gbba's, to the non-empty union of input sets[5].

On the degenerated DSm model $\mathcal{M}_{\emptyset}$ it must be $m_{\mathcal{M}_{\emptyset}}(\emptyset) = 1$ and $m_{\mathcal{M}_{\emptyset}}(A) = 0$ for $A \neq \emptyset$.

The hybrid DSm rule generalizes the classic DSm rule to be applicable to any DSm model. The hybrid DSm rule is commutative but not associative. It is the reason the n-ary version

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[3] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_{\mathcal{M}}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_{\mathcal{M}} \cup \{\emptyset\}$.

[4] To distinguish the DSm rule from Dempster's rule, we use $\circledast$ instead of $\oplus$ for the DSm rule in this text.

[5] As a given DSm model $\mathcal{M}$ is used a final compression step must be applied, see Chapter 4 in [14], which is part of Step 2 of the hybrid DSm combination mechanism and "consists in gathering (summing) all masses corresponding to same proposition because of the constraints of the model". I.e., gbba's of $\mathcal{M}$-equivalent elements of D^{Θ} are summed. Hence the final gbba m is computed as $m(A) = \sum_{X \equiv A} m_{\mathcal{M}(\Theta)}(X)$; it is defined on the reduced hyper-power set $D^{\Theta}_{\mathcal{M}}$.

of the rule should be used in practical applications. For the n-ary version of $S_i(A)$, see [14]. For easier comparison with generalizations of the classic rules of combination we suppose all formulas in CNF, thus we can include the compression step into formulas $S_i(A)$ as it follows[6]:
$S_1(A) = \sum_{X\equiv A,\ X\in D^\Theta} m_{\mathcal{M}^f(\Theta)}(X) = \sum_{X\cap Y\equiv A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_2(A) = \sum_{X,Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}\equiv A]\vee[(\mathcal{U}\in\ \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_3(A) = \sum_{X,Y\in D^\Theta,\ (X\cup Y)\equiv A,\ X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}})} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_i(A) = 0$ for $A = \emptyset$, and for $A \notin D^\Theta_{\mathcal{M}}$ (where $\mathcal{U}$ is as it is above).

We can further rewrite the DSmH rule to the following equivalent form:
$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledast m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y) +$
$\sum_{X,Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}_{\mathcal{M}}=A]\vee[(\mathcal{U}\in\ \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y) +$
$\sum_{X,Y\in D^\Theta,\ X\cup Y=A\ X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$ for all $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$m_{\mathcal{M}(\Theta)}(\emptyset) = 0$ and $m_{\mathcal{M}(\Theta)}(A) = 0$ for $A \in (D^\Theta \backslash D^\Theta_{\mathcal{M}})$.

3.4 A generalization of Dempster's rule

Let us assume all elements X from D^Θ to be in CNF in the rest of this contribution, unless another form of X is explicitly specified. With $X = Y$ we mean that the formulas X and Y have the same CNF. With $X \equiv Y$ ($X \overset{\mathcal{M}}{\equiv} Y$) we mean that the formulas X and Y are equivalent in DSm model $\mathcal{M}$, i.e. their DNFs are the same up to unions with some constrained conjunctions of elements of Θ.

Let us also assume non-degenerated hybrid DSm models, i.e., $\Theta_{\mathcal{M}} \neq \emptyset$, $I_{\mathcal{M}} \notin\ \boldsymbol{\emptyset}_{\mathcal{M}}$. Let us denote $\boldsymbol{\emptyset} =\ \boldsymbol{\emptyset}_{\mathcal{M}} \cup \{\emptyset\}$, i.e. set of set of all elements of D^Θ which are forced to be empty trough the constraints of DSm model $\mathcal{M}$ extended with classic empty set $\emptyset$, hence we can write $X \in\ \boldsymbol{\emptyset}_{\mathcal{M}}$ for all $\emptyset \neq X \overset{\mathcal{M}}{\equiv} \emptyset$ or $X \in\ \boldsymbol{\emptyset}$ for all $X \overset{\mathcal{M}}{\equiv} \emptyset$ including $\emptyset$.

The classic Dempster's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ (the rule adds it to $(m_1 \oplus m_2)(X \cap Y)$) whenever it is non-empty, otherwise the mass is normalized. In the free DSm model all the intersections of non-empty elements are always non-empty, thus no normalization is necessary and Dempster's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ coincides with the classic DSm rule: $(m_1 \oplus m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y=A} m_1(X)m_2(Y) = (m_1 \circledast m_2)(A) = m_{\mathcal{M}^f(\Theta)}(A)$. It follows the fact that the classic DSm rule (DSmC rule) is in fact the conjunctive combination rule generalized to the free DSm model. Hence, Dempster's rule generalized to the free DSm model is defined for any couple of belief functions.

Empty intersections can appear in a general hybrid model $\mathcal{M}$ due to the model's constraints, thus positive gbbm's of constrained elements (i.e equivalent to empty set) can appear, hence the normalization should be used to meet the DSm assumption $m(X) = 0$ for $X \overset{\mathcal{M}}{\equiv} \emptyset$. If we sum together all the gbbm's $m_{\mathcal{M}^f(\Theta)}(X)$ which are assigned to constrained elements of the

[6]We can further simplify the formulas for DSmH rule by using a special canonical form related to the used hybrid DSm model, e.g. $CNF_{\mathcal{M}}(X) = X_{\mathcal{M}} \in D^\Theta_{\mathcal{M}}$ such that $CNF(X) \equiv X_{\mathcal{M}}$. Thus all subexpressions '$\equiv A$' can be replaced with '$= A$' in the definitions of $S_i(A)$ and '$S_i(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$' can be removed from the definition. Hence we obtain a similar form to that published in DSmT book Vol. 1:
$S_1(A) = \sum_{X\cap Y=A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$,
$S_2(A) = \sum_{X,Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\ \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$,
$S_3(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$.
Hence all the necessary assumptions of the definitions of $S_i(A)$ have been formalized.

hyper-power set ($X \in \Theta$, $X \stackrel{\mathcal{M}}{\equiv} \emptyset$) and assign the resulting sum to $m(\emptyset)$ (or more precisely to $m_{\mathcal{M}}(\emptyset)$), we obtain the non-normalized generalized conjunctive rule of combination. If we redistribute this sum of gbbm's among non-constrained elements of D^Θ using normalization as it is used in the classic Dempster's rule, we obtain the generalized Dempster's rule which meets DSm assumption $m(\emptyset) = 0$.

3.4.1 The generalized non-normalized conjunctive rule

The *generalized non-normalized conjunctive rule of combination* $\circledcirc$ is given as
$(m_1 \circledcirc m_2)(A) = \sum_{X,Y \in D^\Theta, X\cap Y \equiv A} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$(m_1 \circledcirc m_2)(\emptyset) = \sum_{X,Y \in D^\Theta, X\cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$,
and $(m_1 \circledcirc m_2)(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$.

We can easily rewrite it as

$$(m_1 \circledcirc m_2)(A) = \sum_{X,Y \in D^\Theta, X\cap Y \equiv A} m_1(X)m_2(Y)$$

for $A \in D^\Theta_{\mathcal{M}}$ ($\emptyset$ including), $(m_1 \circledcirc m_2)(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$.

Similarly to the classic case of the non-normalized conjunctive rule, its generalized version is defined for any couple of generalized belief functions. But we have to keep in mind that positive gbbm of the classic empty set ($m(\emptyset) > 0$) is not allowed in DSmT[7].

3.4.2 The generalized Dempster's rule

To eliminate positive gbbm's of empty set we have to relocate or redistribute gbbm's $m_{\mathcal{M}^f(\Theta)}(X)$ for all $X \stackrel{\mathcal{M}}{\equiv} \emptyset$. The normalization of gbbm's of non-constrained elements of D^Θ is used in the case of the Dempster's rule.

The *generalized Dempster's rule of combination* $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X,Y \in D^\Theta, X\cap Y \equiv A} K m_1(X)m_2(Y)$$

for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X,Y \in D^\Theta, X\cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \notin D^\Theta_{\mathcal{M}}$.

Similarly to the classic case, the generalized Dempster's rule is not defined in fully contradictive cases[8] in hybrid DSm models, i.e. whenever $\kappa = 1$. Specially the generalized Dempster's rule is not defined (and it cannot be defined) on the degenerated DSm model $\mathcal{M}_\emptyset$.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Dempster's rule to the following equivalent form: $(m_1 \oplus m_2)(A) = \phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)]$,

[7]The examples, which compare DSmH rule with the classic combination rules in Chapter 1 of DSmT book Vol. 1. [14], include also the non-normalized conjunctive rule (called Smets' rule there). To be able to correctly compare all that rules on the generalized level in Section 3.7 of this chapter, we present, here, also a generalization of the non-normalized conjunctive rule, which does not respect the DSm assumption $m(\emptyset) = 0$.

[8]Note that in a static combination it means a full conflict/contradiction between input BF's. Whereas in the case of a dynamic combination it could be also a full conflict between mutually non-conflicting or partially conflicting input BF's and constraints of a used hybrid DSm model. E.g. $m_1(\theta_1 \cup \theta_2) = 1$, $m_2(\theta_2 \cup \theta_3) = 1$, where θ_2 is constrained in a used hybrid model.

where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \emptyset$ and $\phi(A) = 0$ otherwise, $S_1^{\oplus}(A)$, $S_2^{\oplus}(A)$, and $S_3^{\oplus}(A)$ are defined by
$S_1^{\oplus}(A) = S_1(A) = \sum_{X,Y \in D^\Theta,\, X\cap Y \equiv A} m_1(X)m_2(Y)$,
$S_2^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^\Theta \;\, Z\notin\emptyset} S_1(Z)} \sum_{X,Y\in\; \emptyset_{\mathcal{M}}} m_1(X)m_2(Y)$,
$S_3^{\oplus}(A) = \frac{S_1(A)}{\sum_{Z\in D^\Theta \;\, Z\notin\emptyset} S_1(Z)} \sum_{X,Y\in D^\Theta,\; X\cup Y\notin\, \emptyset,\; X\cap Y\in\; \emptyset_{\mathcal{M}}} m_1(X)m_2(Y)$.

For proofs see Appendix 3.11.1.

$S_1^{\oplus}(A)$ corresponds to a non-conflicting belief mass, $S_3^{\oplus}(A)$ includes all classic conflicting masses and the cases where one of X, Y is excluded by a non-existential constraint, and $S_2^{\oplus}(A)$ corresponds to the cases where both X and Y are excluded by (a) non-existential constraint(s).

It is easy verify that the generalized Dempster's rule coincides with the classic one on Shafer's model $\mathcal{M}^0$, see Appendix 3.11.1. Hence, the above definition of the generalized Dempster's rule is really a generalization of the classic Dempster's rule. Similarly, we can notice that the rule works also on the free DSm model $\mathcal{M}^f$ and its results coincide with those by DSmC rule. We can define n-ary version of the generalized Dempster's rule, analogically to n-ary versions of DSm rules, but because of its associativity it is not necessary in the case of the Dempster's rule.

3.5 A generalization of Yager's rule

The classic Yager's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ whenever it is non-empty, otherwise the mass is added to $m(\Theta)$. As all the intersections are non-empty in the free DSm model, nothing should be added to $m_1(\Theta)m_2(\Theta)$ and Yager's rule generalized to the free DSm model $\mathcal{M}^f(\Theta)$ also coincides with the classic DSm rule.

$$(m_1 \circledY m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y = A} m_1(X)m_2(Y) = (m_1 \circledS m_2)(A).$$

The *generalized Yager's rule of combination* $\circledY$ for a general hybrid DSm model $\mathcal{M}$ is given as

$$(m_1 \circledY m_2)(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y \equiv A} m_1(X)m_2(Y)$$

for $A \notin\; \emptyset,\; \Theta_{\mathcal{M}} \neq A \in D^\Theta_{\mathcal{M}}$,

$$(m_1 \circledY m_2)(\Theta_{\mathcal{M}}) = \sum_{\substack{X,Y\in D^\Theta \\ X\cap Y \equiv \Theta_{\mathcal{M}}}} m_1(X)m_2(Y) + \sum_{\substack{X,Y\in D^\Theta \\ X\cap Y \in\; \emptyset_{\mathcal{M}}}} m_1(X)m_2(Y)$$

and $(m_1 \circledY m_2)(A) = 0$ otherwise, i.e. for $A \in\; \emptyset$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}})$.

It is obvious that the generalized Yager's rule of combination is defined for any couple of belief functions which are defined on hyper-power set D^Θ.

To be easily comparable with the DSm rule, we can rewrite the definition of the generalized Yager's rule to an equivalent form: $(m_1 \circledY m_2)(A) = \phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)]$, where $S_1^{\circledY}(A)$, $S_2^{\circledY}(A)$, and $S_3^{\circledY}(A)$ are defined by:

$$S_1^{\circledY}(A) = S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y \equiv A} m_1(X)m_2(Y)$$

$$S_2^{\circledY}(\Theta_{\mathcal{M}}) = \sum_{X,Y \in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$$

$$S_2^{\circledY}(A) = 0 \quad \text{for } A \neq \Theta_{\mathcal{M}}$$

$$S_3^{\circledY}(\Theta_{\mathcal{M}}) = \sum_{\substack{X,Y \in D^\Theta,\\ X \cup Y \notin \boldsymbol{\emptyset},\\ X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}}} m_1(X)m_2(Y)$$

$$S_3^{\circledY}(A) = 0 \quad \text{for } A \neq \Theta_{\mathcal{M}}.$$

For proofs see Appendix 3.11.2.

Analogically to the case of the generalized Dempster's rule, $S_1^{\circledY}(A)$ corresponds to non-conflicting belief mass, $S_3^{\circledY}(A)$ includes all classic conflicting masses and the cases where one of X, Y is excluded by a non-existential constraint, and $S_2^{\circledY}(A)$ corresponds to the cases where both X and Y are excluded by (a) non-existential constraint(s).

It is easy to verify that the generalized Yager's rule coincides with the classic one on Shafer's model $\mathcal{M}^0$. Hence the definition of the generalized Yager's rule is really a generalization of the classic Yager's rule, see Appendix 3.11.2.

Analogically to the generalized Dempster's rule, we can observe that the formulas for the generalized Yager's rule work also on the free DSm model and that their results really coincide with those by DSmC rule. If we admit also the degenerated (vacuous) DSm model $\mathcal{M}_\emptyset$, i.e., $\Theta_{\mathcal{M}_\emptyset} = \emptyset$, it is enough to modify conditions for $(m_1 \circledY m_2)(A) = 0$, so that it holds for $\Theta_{\mathcal{M}} \neq A \in \boldsymbol{\emptyset}$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}})$. Then the generalized Yager's rule works also on $\mathcal{M}_\emptyset$; and because of the fact that there is the only bba $m_\emptyset(\emptyset) = 1$, $m_\emptyset(X) = 0$ for any $X \neq \emptyset$ on $\mathcal{M}_\emptyset$, the generalized Yager's rule coincides with the DSmH rule there.

3.6 A generalization of Dubois-Prade's rule

The classic Dubois-Prade's rule puts belief mass $m_1(X)m_2(Y)$ to $X \cap Y$ whenever it is non-empty, otherwise the mass $m_1(X)m_2(Y)$ is added to $X \cup Y$ which is always non-empty in the DST.

In the free DSm model all the intersections of non-empty elements are always non-empty, thus nothing to be added to unions and Dubois-Prade's rule generalized to the free model $\mathcal{M}^f(\Theta)$ also coincides with the classic DSm rule

$$(m_1 \circledD m_2)(A) = \sum_{X,Y \in D^\Theta, X \cap Y = A} m_1(X)m_2(Y) = (m_1 \circledplus m_2)(A).$$

In the case of a static fusion, only exclusivity constraints are used, thus all the unions of $X_i \in D^\Theta$, $X \notin \boldsymbol{\emptyset}$ are also out of $\boldsymbol{\emptyset}$. Thus we can easily generalize Dubois-Prade's rule as $(m_1 \circledD m_2)(A) = \sum_{X,Y \in D^\Theta, X \cap Y \equiv A} m_1(X)m_2(Y) + \sum_{X,Y \in D^\Theta, X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}, X \cup Y \equiv A} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$, and $(m_1 \circledD m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ or $A \notin D^\Theta_{\mathcal{M}}$.

The situation is more complicated in the case of a dynamic fusion, where non-existential constraints are used. There are several sub-cases how $X \cap Y \in \boldsymbol{\emptyset}$ arises.

There is no problem if both X, Y are out of $\emptyset$, because their union $X \cup Y \notin \emptyset$. Similarly if at the least one of X, Y is out of $\emptyset$ then their union is also out of $\emptyset$.

On the other hand if both X, Y are excluded by a non-existential constraint or if they are subsets of elements of D^Θ excluded by non-existential constraints then their union is also excluded by the constraints and the idea of Dubois-Prade's rule is not sufficient to solve this case. Thus the generalized Dubois-Prade rule should be extended to cover also such cases.

Let us start with a simple solutions. As there is absolutely no reason to prefer any of non-constrained elements of D^Θ, the mass $m_1(X)m_2(Y)$ should be either normalized as in Dempster's rule or added to $m(\Theta_\mathcal{M})$ as in Yager's rule. Another option — division of $m_1(X)m_2(Y)$ to k same parts — does not keep a nature of beliefs represented by input belief functions. Because $m_1(X)m_2(Y)$ is always assigned to subsets of X, Y in the case of intersection or to supersets of X, Y in the case of union, addition of $m_1(X)m_2(Y)$ to $m(\Theta)$ is closer to Dubois-Prade's rule nature as $X, Y \subset \Theta$. Whereas the normalization assigns parts of $m_1(X)m_2(Y)$ also to sets which can be disjoint with both of X, Y.

To find a more sophisticated solution, we have to turn our attention to the other cases, where $X \cap Y, X \cup Y \in \emptyset$, and where a simple application of the idea of Dubois-Prade's rule also does not work. Let us assume a fixed hybrid DSm model $\mathcal{M}(\Theta)$ now. Let us further assume that neither X nor Y is a part of a set of elements which are excluded with a non-existential constraint, i.e., $X \cup Y \nsubseteq \bigcup Z_i$ where Z_is are excluded by a non-existential constraint[9]. Let us transfer both X and Y into disjunctive normal form (a union of intersections / a disjunction of conjunctions). Thus, $X \cup Y$ is also in disjunctive form (DNF we obtain by simple elimination of repeating conjuncts/intersections) and at the least one of the conjuncts, let say $W = \theta_{1w} \cap \theta_{2w} \cap ... \cap \theta_{iw}$, contains θ_{jw} non-equivalent to empty set in the given DSm model $\mathcal{M}(\Theta)$. Thus it holds that $\theta_{1w} \cup \theta_{2w} \cup ... \cup \theta_{jw} \notin \emptyset$. Hence we can assign belief masses to $\theta_{1w} \cup \theta_{2w} \cup ... \cup \theta_{jw}$ or to some of its supersets. This idea fully corresponds to Dubois-Prade's rule as the empty intersections are substituted with unions. As we cannot prefer any of the conjuncts — we have to substitute $\cap$s with $\cup$s in all conjuncts of the disjunctive normal form of $X \cup Y$ — we obtain a union $U_{X \cup Y}$ of elements of Θ. The union $U_{X \cup Y}$ includes θ_{jw}; thus it is not equivalent to the empty set and we can assign $m_1(X)m_2(Y)$ to $U_{X \cup Y} \cap I_\mathcal{M} \notin \emptyset$ [10].

Thus we can now formulate a definition of the generalized Dubois-Prade rule. We can distinguish three cases of input generalized belief functions: (i) all inputs satisfy all the constraints of a hybrid DSm model $\mathcal{M}(\Theta)$ which is used (a static belief combination), (ii) inputs do not satisfy the constraints of $\mathcal{M}(\Theta)$ (a dynamic belief combination), but no non-existential constraint is used, (iii) completely general inputs which do not satisfy the constraints, and non-existential constraints are allowed (a more general dynamic combination). According to these three cases, we can formulate three variants of the generalized Dubois-Prade rule.

[9]Hence $X \cup Y$ has had to be excluded by dynamically added exclusivity constraints, e.g. $X = \theta_1 \cap \theta_2$, $Y = \theta_2 \cap \theta_3 \cap \theta_4$ $X \cup Y = (\theta_1 \cap \theta_2) \cup (\theta_2 \cap \theta_3 \cap \theta_4)$, and all $\theta_1, \theta_2, \theta_3, \theta_4$ are forced to be exclusive by added exclusivity constraints, thus $X \cap Y, X \cup Y \in \emptyset_\mathcal{M}$.

[10]We obtain $(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) \cap I_\mathcal{M}$ in the example from the previous footnote.

The *simple generalized Dubois-Prade rule of combination* $\circledast$ is given as[11]

$$(m_1 \circledast m_2)(A) \;=\; \sum_{X\cap Y\equiv A} m_1(X)\; m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)\; m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and $(m_1 \circledast m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$.

The *generalized Dubois-Prade rule of combination* $\circledast$ is given as

$$(m_1 \circledast m_2)(A) = \sum_{X\cap Y\equiv A} m_1(X)m_2(Y) + \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)m_2(Y) + \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv A}} m_1(X)m_2(Y)$$

for $\emptyset \neq A \in D^{\Theta}_{\mathcal{M}}$, and
$(m_1 \circledast m_2)(A) = 0$ otherwise, i.e., for $A = \emptyset$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$,
where $U_{X\cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

The *extended generalized Dubois-Prade rule of combination* $\circledast$ is given as

$$\begin{aligned}(m_1 \circledast m_2)(A) \;=\; &\sum_{X\cap Y\equiv A} m_1(X)\; m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv A}} m_1(X)\; m_2(Y)\\ &+ \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv A}} m_1(X)\; m_2(Y)\end{aligned}$$

for $\emptyset \neq A \neq \Theta_{\mathcal{M}},\; A \in D^{\Theta}_{\mathcal{M}}$,

$$\begin{aligned}(m_1 \circledast m_2)(\Theta_{\mathcal{M}}) \;=\; &\sum_{X\cap Y\equiv \Theta_{\mathcal{M}}} m_1(X)\; m_2(Y) \;+\; \sum_{\substack{X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ X\cup Y\equiv \Theta_{\mathcal{M}}}} m_1(X)\; m_2(Y)\\ &+ \sum_{\substack{X\cup Y\in \boldsymbol{\emptyset}_{\mathcal{M}}\\ U_{X\cup Y}\equiv \Theta_{\mathcal{M}}}} m_1(X)\; m_2(Y) \;+\; \sum_{U_{X\cup Y}\in \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)\; m_2(Y),\end{aligned}$$

and

$$(m_1 \circledast m_2)(A) = 0$$

otherwise, i.e., for $A \in \boldsymbol{\emptyset}$ and for $A \in (D^{\Theta} \setminus D^{\Theta}_{\mathcal{M}})$,
where $U_{X\cup Y}$ is disjunctive normal form of $X \cup Y$ with all $\cap$s substituted with $\cup$s.

In the case (i) there are positive belief masses assigned only to the $X_i \in D^{\Theta}$ such that $X \notin \boldsymbol{\emptyset}$, hence the simple generalized Dubois-Prade rule, which ignores all the belief masses assigned to $Y \in \boldsymbol{\emptyset}$, may be used. The rule is defined for any couple of BF's which satisfy the constraints.

[11] We present here 3 variants of the generalized Dubois-Prade rule, formulas for all of them include several summations over $X, Y \in D^{\Theta}$, where X, Y are more specified with other conditions. To simplify the formulas in order to increase their readability, we do not repeat the common condition $X, Y \in D^{\Theta}$ in sums in all the following formulas for the generalized Dubois-Prade rule.

In the case (ii) there are no $U_{X\cup Y} \in \boldsymbol{\emptyset}$, hence the generalized Dubois-Prade rule, which ignores multiples of belief masses $m_1(X)m_2(Y)$, where $U_{X\cup Y} \in \boldsymbol{\emptyset}$, may be used.

In the case (iii) the extended generalized Dubois-Prade rule must be used, this rule can handle all the belief masses in any DSm model, see 1a) in Appendix 3.11.3.

It is easy to verify that the generalized Dubois-Prade rule coincides with the classic one in Shafer's model $\mathcal{M}^0$, see 2) in Appendix 3.11.3.

The classic Dubois-Prade rule is not associative, neither the generalized one is. Similarly to the DSm approach we can easily rewrite the definitions of the (generalized) Dubois-Prade rule for a combination of k sources.

Analogically to the generalized Yager's rule, the formulas for the generalized Dubois-Prade's rule work also on the free DSm model $\mathcal{M}^f$ and their results coincide with those of DSmC rules there, see 1b) in Appendix 3.11.3. If we admit also the degenerated (vacuous) DSm model $\mathcal{M}_\emptyset$, i.e., $\Theta_{\mathcal{M}_\emptyset} = \emptyset$, it is enough again to modify conditions for $(m_1 \textcircled{\tiny DP} m_2)(A) = 0$, so that it holds for $\Theta_{\mathcal{M}} \neq A \in \boldsymbol{\emptyset}$ and for $A \in (D^\Theta \setminus D^\Theta_{\mathcal{M}})$. Then the extended generalized Dubois-Prade's rule works also on $\mathcal{M}_\emptyset$ and it trivially coincides with DSmH rule there.

To be easily comparable with the DSm rule, we can rewrite the definitions of the generalized Dubois-Prade rules to an equivalent form similar to that of DSm:

the generalized Dubois-Prade rule:

$$(m_1 \textcircled{\tiny DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_2^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where

$$S_1^{\textcircled{\tiny DP}}(A) = S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\equiv A} m_1(X)m_2(Y),$$

$$S_2^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\, U_{X\cup Y}\equiv A} m_1(X)m_2(Y),$$

$$S_3^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\, (X\cup Y)\equiv A} m_1(X)m_2(Y).$$

the simple generalized Dubois-Prade rule:

$$(m_1 \textcircled{\tiny DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where $S_1^{\textcircled{\tiny DP}}(A), S_3^{\textcircled{\tiny DP}}(A)$ as above;
the extended generalized Dubois-Prade rule:

$$(m_1 \textcircled{\tiny DP} m_2)(A) = \phi(A)[S_1^{\textcircled{\tiny DP}}(A) + S_2^{\textcircled{\tiny DP}}(A) + S_3^{\textcircled{\tiny DP}}(A)]$$

where $S_1^{\textcircled{\tiny DP}}(A), S_3^{\textcircled{\tiny DP}}(A)$ as above, and

$$S_2^{\textcircled{\tiny DP}}(A) = \sum_{X,Y\in \boldsymbol{\emptyset}_{\mathcal{M}},\, [U_{X\cup Y}\equiv A]\vee[U_{X\cup Y}\in\boldsymbol{\emptyset}\wedge A=\Theta_{\mathcal{M}}]} m_1(X)m_2(Y).$$

For a proof of equivalence see 3) in Appendix 3.11.3.

Functions $S_1^{\textcircled{D}}, S_2^{\textcircled{D}}, S_3^{\textcircled{D}}$ have interpretations analogical to $S_i^{\oplus}$ and $S_i^{\textcircled{Y}}$ for $\oplus$ and $\textcircled{Y}$. $S_2^{\textcircled{D}}$ is principal for distinguishing of the variants of the Dubois-Prade rule. In the case (i) no positive belief masses are assigned to $X \in \boldsymbol{\emptyset}$ thus $S_2^{\textcircled{D}}(A) \equiv 0$, in the case (ii) $S_2^{\textcircled{D}}(A)$ sums $m_1(X)m_2(Y)$ only for $U_{X\cup Y} \equiv A$, whereas in the case (iii) also $U_{X\cup Y} \in \boldsymbol{\emptyset}_{\mathcal{M}}$ must be included.

In general, some θ_is can repeat several times in $U_{X\cup Y}$, they are eliminated in DNF. Hence we obtain a union of elements of Θ which are contained in X, Y. Let us note that this union $U_{X\cup Y}$ of elements of Θ coincides with $\mathcal{U} = u(X) \cup u(Y)$, more precisely $U_{X\cup Y} \cap I_{\mathcal{M}}$ coincides with $\mathcal{U} \cap I_{\mathcal{M}} = u(X) \cup u(Y) \cap I_{\mathcal{M}}$. Thus the generalized Dubois-Prade rule gives the same results as the hybrid DSmH rule does. Let us further note that the extension of Dubois-Prade's rule, i.e. addition of $m_1(X)m_2(Y)$ to $m(\Theta_{\mathcal{M}})$ for $X, Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ also coincides with the computation with the DSmH rule in the case, where $\mathcal{U} \in \boldsymbol{\emptyset}_{\mathcal{M}}$. Hence, the extended generalized Dubois-Prade rule is fully equivalent to the DSmH rule.

3.7 A comparison of the rules

As there are no conflicts in the free DSm model $\mathcal{M}^f(\Theta)$ all the presented rules coincide in the free DSm model $\mathcal{M}^f(\Theta)$. Thus the following statement holds:

Statement 1. *Dempster's rule, the non-normalized conjunctive rule, Yager's rule, Dubois-Prade's rule, the hybrid DSmH rule, and the classic DSmC rule are all mutually equivalent in the free DSm model $\mathcal{M}^f(\Theta)$.*

Similarly the classic Dubois-Prade rule is equivalent to the DSm rule for Shafer's model. But in general all the generalized rules $\oplus, \textcircled{Y}, \textcircled{D}$, and DSm rule are different. A very slight difference comes in the case of Dubois-Prade's rule and the DSm rule. A difference appears only in the case of a dynamic fusion where some belief masses of both (of all in an n-ary case) source generalized basic belief assignments are equivalent to the empty set (i.e. $m_1(X), m_2(Y) \in \boldsymbol{\emptyset}_{\mathcal{M}}$ or $m_i(X_i) \in \boldsymbol{\emptyset}_{\mathcal{M}}$). The generalized Dubois-Prade rule is not defined and it must be extended by adding $m_1(X)m_2(Y)$ or $\Pi_i\ m_i(X_i)$ to $m(\Theta_{\mathcal{M}})$ in this case. The generalized Dubois-Prade rule coincides with the DSm rule in all other situations, i.e., whenever all input beliefs fit the DSm model, which is used, and whenever we work with a DSm model without non-existential constraints, see the previous section. We can summarize it as it follows:

Statement 2. *(i) If a hybrid DSm model $\mathcal{M}(\Theta)$ does not include any non-existential constraint or if all the input belief functions satisfy all the constraints of $\mathcal{M}(\Theta)$, then the generalized Dubois-Prade rule is equivalent to the DSm rule in the model $\mathcal{M}(\Theta)$. (ii) The generalized Dubois-Prade rule extended with addition of $m_1(X)m_2(Y)$ (or $\Pi_i\ m_i(X_i)$ in an n-ary case) to $m(\Theta_{\mathcal{M}})$ for $X, Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ (or for $X_i \in \boldsymbol{\emptyset}_{\mathcal{M}}$ in an n-ary case) is fully equivalent to the hybrid DSmH rule on any hybrid DSm model.*

3.7.1 Examples

Let us present examples from Chapter 1 from DSm book 1 [14] for an illustration of the comparison of the generalized rules with the hybrid DSm rule.

Example 1. The first example is defined on $\Theta = \{\theta_1, \theta_2, \theta_3\}$ as Shafer's DSm model $\mathcal{M}^0$ with the additional constraint $\theta_3 \equiv \emptyset$, i.e. $\theta_1 \cap \theta_2 \equiv \theta_3 \equiv \emptyset$ in DSm model $\mathcal{M}_1$, and subsequently $X \equiv Y \equiv \emptyset$ for all $X \subseteq \theta_1 \cap \theta_2$ $Y \subseteq \theta_3$. We assume two independent source belief assignments m_1, m_2, see Table 3.1.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}_1$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{\cup}$	$\textcircled{\cap\cup}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}_1}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{\cup}}$	$m_{\textcircled{\cap\cup}}$
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0	$\emptyset$					
$\theta_1 \cap \theta_2$	0	0	0.21	$\emptyset$					
$\theta_1 \cap \theta_3$	0	0	0.13	$\emptyset$					
$\theta_2 \cap \theta_3$	0	0	0.14	$\emptyset$					
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0	0	0	$\emptyset$					
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0	0	0	$\emptyset$					
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0	0	0.11	$\emptyset$					
$\Box$	0	0	0	$\emptyset$					
θ_1	0.10	0.50	0.21	θ_1	0.34	0.600	0.21	0.21	0.34
θ_2	0.40	0.10	0.11	θ_2	0.25	0.314	0.11	0.11	0.25
θ_3	0.20	0.30	0.06	$\emptyset$					
$\Box\theta_1$	0	0	0	θ_1	0	0	0	0	0
$\Box\theta_2$	0	0	0	θ_2	0	0	0	0	0
$\Box\theta_3$	0	0	0	$\emptyset$					
$\theta_1 \cup \theta_2$	0.30	0.10	0.03	$\theta_1 \cup \theta_2$	0.41	0.086	0.03	0.68	0.41
$\theta_1 \cup \theta_3$	0	0	0	θ_1	0	0	0	0	0
$\theta_2 \cup \theta_3$	0	0	0	θ_2	0	0	0	0	0
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	$\theta_1 \cup \theta_2$	0	0	0	0	0
$\emptyset$				$\emptyset$			0.65		

Table 3.1: Example 1 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{\cup}$, $\textcircled{\cap\cup}$ on hybrid DSm model $\mathcal{M}_1$.

A description of Table 3.1. As DSm theory admits general source basic belief assignments defined on the free DSm model $\mathcal{M}^f$, all elements of D^Θ are presented in the first column of the table. We use the following abbreviations for 4 elements of D^Θ: $\Box$ for $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap (\theta_2 \cup \theta_3)$, $\Box\theta_1$ for $\theta_1 \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3)$, $\Box\theta_2$ for $\theta_2 \cup (\theta_1 \cap \theta_3)$, and $\Box\theta_3$ for $\theta_3 \cup (\theta_1 \cap \theta_2)$. Thus $\Box$ is not any operator here, but just a symbol for abbreviation; it has its origin in the papers about minC combination [3, 5, 6], see also Chapter 10 in DSm book Vol. 1 [14].

Source gbba's m_1, m_2 follow in the second and the third column. The central part of the table contains results of DSm combination of the beliefs: the result obtained with DSmC rule, i.e. resulting gbba m_{DSmC}, is in the 4th column and the result obtained with DSmH is in the 6th column. Column 5 shows equivalence of elements of the free DSm model $\mathcal{M}^f$ to those of the assumed hybrid DSm model $\mathcal{M}_1$. Finally, the right part of the table displays the results of combination of the source gbba's with the generalized combination rules (with the generalized Dempster's rule $\oplus$ in the 7-th column, with the generalized non-normalized Dempster's rule $\textcircled{\cap}$ in column 8, etc.). The resulting values are always cumulated, thus the value for $m(\theta_1)$ is only in the row corresponding to θ_1, whereas all the other rows corresponding to sets equivalent to θ_1 contain 0s. Similarly, all the fields corresponding to empty set are blank with the exception that for $m_{\textcircled{\cap}}(\emptyset)$, i.e. the only one where positive $m(\emptyset)$ is allowed. The same structure of the table is used also in the following examples.

Example 2. Le us assume, now, two independent sources m_1, m_2 over 4-element frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, where Shafer's model $\mathcal{M}^0$ holds, see Table 3.2.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{D}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{D}}$
$\theta_1 \cap \theta_2$	0	0	0.9604	$\emptyset$					
$\theta_1 \cap \theta_4$	0	0	0.0196	$\emptyset$					
$\theta_2 \cap \theta_3$	0	0	0.0098	$\emptyset$					
$\theta_2 \cap \theta_4$	0	0	0.0098	$\emptyset$					
$\theta_3 \cap \theta_4$	0	0	0.0002	$\emptyset$					
θ_1	0.98	0	0	θ_1	0	0	0	0	0
θ_2	0	0.98	0	θ_2	0	0	0	0	0
θ_3	0.01	0	0	θ_3	0	0	0	0	0
θ_4	0.01	0.02	0.0002	θ_4	0.0002	1	0.0002	0.0002	0.0002
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.9604	0	0	0	0.9604
$\theta_1 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_4$	0.0196	0	0	0	0.0196
$\theta_2 \cup \theta_3$	0	0	0	$\theta_2 \cup \theta_3$	0.0098	0	0	0	0.0098
$\theta_2 \cup \theta_4$	0	0	0	$\theta_2 \cup \theta_4$	0.0098	0	0	0	0.0098
$\theta_3 \cup \theta_4$	0	0	0	$\theta_3 \cup \theta_4$	0.0002	0	0	0	0.0002
$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0.9998	0
$\emptyset$				$\emptyset$			0.9998		

Table 3.2: Example 2 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{D}$ on Shafer's DSm model $\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

The structure of Table 3.2 is the same as in the case of Table 3.1. Because of the size of the full table for DSm combination on a 4-element frame of discernment, rows which contain only 0s and blank fields are dropped.

Note, that input values are shortened by one digit here (i.e. 0.98, 0.02, and 0.01 instead of 0.998, 0.002, and 0.001) in comparison with the original version of the example in [14]. Nevertheless the structure and features of both the versions of the example are just the same.

Example 3. This is an example for Smet's case, for the non-normalized Dempster's rule. We assume Shafer's model $\mathcal{M}^0$ on a simple 2-element frame $\Theta = \{\theta_1, \theta_2\}$. We assume $m(\emptyset) \geq 0$, in this example, even if it is not usual in DSm theory, see Table 3.3.

Example 4. Let us assume Shafer's model $\mathcal{M}^0$ on $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ in this example, see Table 3.4.

Example 5. Let us assume again Shafer's model $\mathcal{M}^0$ on a simple 2-element frame $\Theta = \{\theta_1, \theta_2\}$, see Table 3.5.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2$	0	0	0.28	$\emptyset$					
θ_1	0.40	0.60	0.24	θ_1	0.48	0.143	0.24	0.24	0.48
θ_2	0.40	0.10	0.04	θ_2	0.18	0.857	0.04	0.04	0.18
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.34	0	0	0.72	0.34
$\emptyset$	0.20	0.30	0.44	$\emptyset$			0.72		

Table 3.3: Example 3 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on Shafer's DSm model $\mathcal{M}^0$.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2$	0	0	0.9702	$\emptyset$					
$\theta_1 \cap (\theta_3 \cup \theta_4)$	0	0	0.0198	$\emptyset$					
$\theta_2 \cap (\theta_3 \cup \theta_4)$	0	0	0.0098	$\emptyset$					
θ_1	0.99	0	0	θ_1	0	0	0	0	0
θ_2	0	0.98	0	θ_2	0	0	0	0	0
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.9702	0	0	0	0.9702
$\theta_3 \cup \theta_4$	0.01	0.02	0.0002	$\theta_3 \cup \theta_4$	0.0002	1	0.0002	0.0002	0.0002
$\theta_1 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_3 \cup \theta_4$	0.0198	0	0	0	0.0198
$\theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_2 \cup \theta_3 \cup \theta_4$	0.0098	0	0	0	0.0098
$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0.9998	0
$\emptyset$				$\emptyset$			0.9998		

Table 3.4: Example 4 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on Shafer's DSm model $\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

$\mathcal{M}^0$ (rows which contain only 0s and blank fields are dropped).

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}^0$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}^0}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2$	0.40	0.30	0.89	$\emptyset$					
θ_1	0.50	0.10	0.05	θ_1	0.24	$0.\overline{45}$	0.05	0.05	0.24
θ_2	0.10	0.60	0.06	θ_2	0.33	$0.\overline{54}$	0.06	0.06	0.33
$\theta_1 \cup \theta_2$	0	0	0	$\theta_1 \cup \theta_2$	0.43	0	0	0.89	0.43
$\emptyset$				$\emptyset$			0.89		

Table 3.5: Example 5 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on Shafer's DSm model $\mathcal{M}^0$.

Example 6. As all the above examples are quite simple, usually somehow related to Shafer's model, we present also one of the more general examples (Example 3) from Chapter 4 DSm book Vol. 1; it is defined on the DSm model $\mathcal{M}_{4.3}$ based on 3-element frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with constraints $\theta_1 \cap \theta_2 \equiv \theta_2 \cap \theta_3 \equiv \emptyset$; and subsequently $\theta_1 \cap \theta_2 \cap \theta_3 \equiv \theta_2 \cap (\theta_1 \cup \theta_3) \equiv \emptyset$, see Table 3.6.

$\mathcal{M}^f$			$DSmC$	$\mathcal{M}_{4.3}$	$DSmH$	$\oplus$	$\textcircled{\cap}$	$\textcircled{Y}$	$\textcircled{DP}$
D^Θ	m_1	m_2	m_{DSmC}	$D^\Theta_{\mathcal{M}_{4.3}}$	m_{DSmH}	$m_\oplus$	$m_{\textcircled{\cap}}$	$m_{\textcircled{Y}}$	$m_{\textcircled{DP}}$
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0.16	$\emptyset$					
$\theta_1 \cap \theta_2$	0.10	0.20	0.22	$\emptyset$					
$\theta_1 \cap \theta_3$	0.10	0	0.12	$\theta_1 \cap \theta_3$	0.17	0.342	0.12	0.12	0.17
$\theta_2 \cap \theta_3$	0	0.20	0.19	$\emptyset$					
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0	0	0	$\theta_1 \cap \theta_3$	0	0	0	0	0
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0	0	0.05	$\emptyset$					
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0	0	0.01	$\theta_1 \cap \theta_3$	0	0	0.01	0.01	0
$\Box$	0	0	0	$\theta_1 \cap \theta_3$	0	0	0	0	0
θ_1	0.10	0.20	0.08	θ_1	0.16	0.263	0.08	0.08	0.16
θ_2	0.20	0.10	0.03	θ_2	0.12	0.079	0.03	0.03	0.12
θ_3	0.30	0.10	0.10	θ_3	0.23	0.263	0.10	0.10	0.23
$\Box\theta_1$	0	0	0.02	θ_1	0	0	0.02	0.02	0
$\Box\theta_2$	0	0	0	$\Box\theta_2$	0.01	0	0	0	0.01
$\Box\theta_3$	0	0	0	θ_3	0	0	0	0	0
$\theta_1 \cup \theta_2$	0.10	0	0	$\theta_1 \cup \theta_2$	0.11	0	0	0	0.11
$\theta_1 \cup \theta_3$	0.10	0.20	0.02	$\theta_1 \cup \theta_3$	0.08	0.053	0.02	0.02	0.08
$\theta_2 \cup \theta_3$	0	0	0	$\theta_2 \cup \theta_3$	0.05	0	0	0	0.05
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	$\theta_1 \cup \theta_2 \cup \theta_3$	0.07	0	0	0.62	0.07
$\emptyset$				$\emptyset$			0.62		

Table 3.6: Example 6 — combination of gbba's m_1, m_2 with the DSm rules DSmC, DSmH, and with the generalized rules $\oplus$, $\textcircled{\cap}$, $\textcircled{Y}$, $\textcircled{DP}$ on hybrid DSm model $\mathcal{M}_{4.3}$.

3.7.2 A summary of the examples

We can mention that all the rules are defined for all the presented source generalized basic belief assignments. In the case of the generalized Dempster's rule it is based on the fact that no couple of source gbba's is in full contradiction. In the case of the generalized Dubois-Prade's rule we need its extended version in Examples 1, 3, 5, and 6.

In Example 1, it is caused by constraint $\theta_3 \equiv \emptyset$ and positive values $m_1(\theta_3) = 0.20$ and $m_2(\theta_3) = 0.30$, see Table 3.1, hence we have $m_1(\theta_3)m_2(\theta_3) = 0.06 > 0$ and $\theta_3 \cap \theta_3 = \theta_3 \cup \theta_3 = \theta_3 \equiv \emptyset$ in DSm model $\mathcal{M}_1$ in question. In Example 3, it is caused by admission of positive input values for $\emptyset$: $m_1(\emptyset) = 0.20$, $m_2(\emptyset) = 0.30$. In Example 5, it is because both m_1 and m_2 have positive input values for $\theta_1 \cap \theta_2$ which is constrained. We have $m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) = 0.12$ and $(\theta_1 \cap \theta_2) \cap (\theta_1 \cap \theta_2) = \theta_1 \cap \theta_2 \equiv \emptyset \equiv (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_2)$, hence 0.12 should be added to Θ by the extended Dubois-Prade's rule. We have to distinguish this case from different cases such as e.g. $m_1(\theta_1)m_2(\theta_2)$ or $m_1(\theta_1 \cap \theta_2)m_2(\theta_2)$, where values are normally assigned to union of arguments $(\theta_1) \cup (\theta_2)$ or $(\theta_1 \cap \theta_2) \cup \theta_2 = \theta_2$ respectively. In Example 6, it is analogically caused by couples of positive inputs $m_1(\theta_1 \cap \theta_2),\ m_2(\theta_1 \cap \theta_2)$ and $m_1(\theta_1 \cap \theta_2),\ m_2(\theta_2 \cap \theta_3)$.

In Examples 2 and 4, the generalized Dubois-Prade's rule without extension can be used because all the elements of D^Θ which are constrained (prohibited by the constraints) have 0 values of gbbm's.

We can observe that, $m(\emptyset) > 0$ only when using the generalized conjunctive rule $\copyright$, where $m_{\copyright}(\emptyset) = \sum_{Z \equiv \emptyset} m(Z)$ and $m_{\copyright}(X) = m_{DSmC}(X)$ for $X \not\equiv \emptyset$. If we distribute $m_{\copyright}(\emptyset)$ with normalization, we obtain the result $m_\oplus$ of the generalized Dempster's rule $\oplus$; if we relocate (add) $m_{\copyright}(\emptyset)$ to $m_{\copyright}(\Theta)$ we obtain $m_{\textcircled{Y}}$, i.e. the result of the generalized Yager's rule.

The other exception of $m(\emptyset) > 0$ is in Example 3, where $m_{DSmC}(\emptyset) = 0.44 > 0$ because there is $m_1(\emptyset) > 0$ and $m_2(\emptyset) > 0$ what is usually not allowed in DSmT. This example was included into [14] for comparison of DSmH with the classic non-normalized conjunctive rule used in TBM.

In accordance with theoretical results we can verify, that the DSmH rule always gives the same resulting values as the generalized Dubois-Prade rule produces in all 6 examples.

Looking at the tables we can observe, that DSmH and Dubois-Prade's generate more specified results (i.e. higher gbbm's are assigned to smaller elements of D^Θ) than both the generalized non-normalized conjunctive rule $\copyright$ and the generalized Yager's rule $\textcircled{Y}$ produce. There is some lost of information when the generalized $\oplus$ or $\textcircled{Y}$ are applied. Nevertheless, there is some lost of information also within the application of the DSmH rule. Considering the rules investigated and compared in this text we obtain the most specific results when the generalized Dempster's rule $\oplus$ is used. Another rules, which produce more specified results than the DSmH rule and the generalized Dubois-Prade's rule do, are the generalized minC combination rule [5] and PCR combination rules [15], which are out of scope of this chapter.

3.8 Open problems

As an open question remains commutativity of a transformation of generalized belief functions to those which satisfy all the constraints of a used hybrid DSm model with the particular combination rules. Such a commutation may significantly simplify functions S_2 and hence the entire definitions of the corresponding combination rules. If such a commutation holds for some combination rule, we can simply transform all input belief functions to those which satisfy constraints of the DSm model in question at first; and perform static fusion after. No dynamic fusion is necessary in such a case.

A generalization of minC combination rule, whose computing mechanism (not a motivation nor an interpretation) has a relation to the conjunctive rules on the free DSm model $\mathcal{M}^f(\Theta)$ already in its classic case [3], is under recent development. And it will also appear as a chapter of this volume.

We have to mention also the question of a possible generalization of conditionalization, related to particular combination rules to the domain of DSm hyper-power sets.

And we cannot forget for a new family of PCR rules [15], see also a chapter in this volume. Comparison of these rules, rules presented in this chapter, generalized minC combination and possibly some other belief combination rules on hyper-power sets can summarize the presented topic.

3.9 Conclusion

The classic rules for combination of belief functions have been generalized to be applicable to hyper-power sets, which are used in DSm theory. The generalization forms a solid theoretical background for full and objective comparison of the nature of the classic rules with the nature of the DSm rule of combination. It also enables us to place the DSmT better among the other approaches to belief functions.

3.10 References

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3.11 Appendix - proofs

3.11.1 Generalized Dempster's rule

1) *Correctness of the definition:*
1a) $\sum_{X,Y \in D^\Theta} m_1(X)m_2(Y) = 1$ for any gbba's m_1, m_2; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y \equiv A, \emptyset \neq A \in D^\Theta_\mathcal{M}$, all the other multiples (i.e., for $X \cap Y = \emptyset$ and for $X \cap Y = A \notin D^\Theta_\mathcal{M}$) are normalized among $\emptyset \neq A \in D^\Theta_\mathcal{M}$. Hence the formula for the generalized Dempster's rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2.
1b) It holds $\kappa = \sum_{X,Y \in D^\Theta, X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(X) = 0$ and $K = \frac{1}{1-\kappa} = 1$ in the free DSm model $\mathcal{M}^f$. Hence we obtain the formula for the free model $\mathcal{M}^f$ as a special case of the general formula.

2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \emptyset_{\mathcal{M}^0}$ iff $\{\theta_i | \theta_i \subseteq X\} \cap \{\theta_j | \theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$ in both the classic and the generalized Dempster's rule on Shafer's DSm model, and the same multiples are normalized by both of the rules. Thus, the results are the same for any m_1, m_2 on $\mathcal{M}^0$ and for any $A \subseteq \Theta$ and other $A \in D^\Theta$. Hence the generalized Dempster's rule is really a generalization of the classic Dempster's rule.

3) *Equivalence of expressions:* $(m_1 \oplus m_2)(A) \stackrel{?}{=} \phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)]$

$$\begin{aligned}
\phi(A)[S_1^\oplus(A) + S_2^\oplus(A) + S_3^\oplus(A)] = \phi(A) \sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + \\
\phi(A)[\frac{S_1(A)}{\sum_{Z \in D^\Theta \; Z \notin \boldsymbol{\emptyset}} S_1(Z)} \sum_{X,Y \in \; \boldsymbol{\emptyset}_\mathcal{M}} m_1(X)m_2(Y) + \\
\frac{S_1(A)}{\sum_{Z \in D^\Theta \; Z \notin \boldsymbol{\emptyset}} S_1(Z)} \sum_{X \cup Y \notin \; \boldsymbol{\emptyset}, \; X \cap Y \in \; \boldsymbol{\emptyset}_\mathcal{M}} m_1(X)m_2(Y)]
\end{aligned}$$

For $A \notin \boldsymbol{\emptyset}$ we obtain the following (as $m_i(\emptyset) = 0$):

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y) + [\frac{S_1(A)}{\sum_{Z\in D^\Theta\ Z\notin\boldsymbol{\emptyset}} S_1(Z)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)] =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y) + \frac{\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)}{\sum_{X\cap Y\notin\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(1 + \frac{1}{\sum_{X\cap Y\notin\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} \sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(\frac{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)} + \frac{\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)(\frac{1}{1-\sum_{X\cap Y\in\ \boldsymbol{\emptyset}} m_1(X)m_2(Y)}) =$$

$$\sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y)\frac{1}{1-\kappa} = \sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} Km_1(X)m_2(Y) = (m_1 \oplus m_2)(A).$$

For $A \in \boldsymbol{\emptyset}$ we obtain:

$$\phi(A)[S_1^{\oplus}(A) + S_2^{\oplus}(A) + S_3^{\oplus}(A)] = 0 \cdot [S_1^{\oplus}(A) + S_2^{\oplus}(A) + S_3^{\oplus}(A)] = 0 = (m_1 \oplus m_2)(A).$$

Hence the expression in DSm form is equivalent to the definition of the generalized Dempster's rule.

3.11.2 Generalized Yager's rule

1) *Correctness of the definition:*
1a) $\sum_{X,Y\in D^\Theta, X\cap Y=A} m_1(X)m_2(Y) = 1$ for any gbba's m_1, m_2; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cap Y = A \in \boldsymbol{\emptyset}$) are summed to $\Theta_{\mathcal{M}}$. Hence the formula for the generalized Yager's rule produces correct gbba $m_1 \circledY m_2$ for any input gbba's m_1, m_2.
1b) It holds $\sum_{X,Y\in D^\Theta, X\cap Y\in\emptyset} m_1(X)m_2(X) = 0$ in the free DSm model $\mathcal{M}^f$. Thus $(m_1 \circledY m_2)(\Theta) = m_1(\Theta)m_2(\Theta)$. Hence we obtain the formula for the free model $\mathcal{M}^f$ as a special case of the general formula.

2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}}$ iff $\{\theta_i|\theta_i \subseteq X\} \cap \{\theta_j|\theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$, $A \neq \Theta$ in both the classic and the generalized Yager's rule on Shafer's DSm model, and the same multiples are summed to Θ by both of the rules. Thus, the results are the same for any m_1, m_2 on $\mathcal{M}^0$ and any $A \subseteq \Theta$ ($A \in D^\Theta$). Hence the generalized Yager's rule is a correct generalization of the classic Yager's rule.

3) *Equivalence of expressions:* $(m_1 \circledY m_2)(A) \stackrel{?}{=} \phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)]$

For $\Theta_{\mathcal{M}} \neq A \notin \boldsymbol{\emptyset}$ we obtain the following:

$$\phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)] = \phi(A)[\sum_{X\cap Y\equiv A} m_1(X)m_2(Y) + 0 + 0]$$
$$= \sum_{X\cap Y\equiv A\notin\boldsymbol{\emptyset}} m_1(X)m_2(Y) = (m_1 \circledY m_2)(A).$$

For $A = \Theta_{\mathcal{M}}$ we obtain the following:

$$\phi(\Theta_{\mathcal{M}}) \sum_{X\cap Y\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \phi(\Theta_{\mathcal{M}})[\sum_{X,Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$$
$$+ \sum_{X\cup Y\notin\ \boldsymbol{\emptyset},\ X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)]$$
$$= \sum_{X\cap Y\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + [\sum_{X\cap Y\in\ \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)] = (m_1 \circledY m_2)(\Theta_{\mathcal{M}}).$$

For $A \in\ \boldsymbol{\emptyset}$ we obtain $\phi(A)[S_1^{\circledY}(A) + S_2^{\circledY}(A) + S_3^{\circledY}(A)] = 0[S_1^{\circledY}(A) + 0 + 0] = 0 = (m_1 \oplus m_2)(A)$. Hence the expression in DSm form is equivalent to the definition of the generalized Yager's rule.

3.11.3 Generalized Dubois-Prade rule

1) *Correctness of the definition:*
1a) $\sum_{X,Y\in D^\Theta m_1(X)m_2(Y)} = 1$ for any gbba's m_1, m_2; Let us assume that m_1, m_2 satisfy all the constraints of DSm model $\mathcal{M}$, thus $m_1(X) \cup m_2(Y) \notin\ \boldsymbol{\emptyset}$ for any $X, Y \in D^\Theta_{\mathcal{M}}$; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin\ \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cap Y = A \in\ \boldsymbol{\emptyset}$) are summed and added to $m(A)$, where $A = X \cup Y$, with the simple generalized Dubois-Prade rule. Hence the simple generalized Dubois-Prade rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2 which satisfy all the constraints of the used DSm model $\mathcal{M}$.
Let us assume a DSm model $\mathcal{M}$ without non-existential constraints, now, thus $U_{X\cup Y} \notin\ \boldsymbol{\emptyset}$ for any $\emptyset \neq X, Y \in D^\Theta_{\mathcal{M}}$; multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed and added to $m(A)$ for $X \cap Y = A \notin\ \boldsymbol{\emptyset}$, other multiples are summed to $m(A)$ for $X \cup Y = A \notin\ \boldsymbol{\emptyset}$, $X \cap Y = A \in\ \boldsymbol{\emptyset}$, all the other multiples (i.e., for $X \cup Y = A \in\ \boldsymbol{\emptyset}$) are summed and added to $m(A)$ where $A = U_{X\cup Y}$, with the generalized Dubois-Prade rule. Hence the generalized Dubois-Prade rule produces correct gbba $m_1 \oplus m_2$ for any input gbba's m_1, m_2 on DSm model $\mathcal{M}$ without non-existential constraints.
For a fully general dynamic belief fusion on any DSm model the following holds:
multiples $0 \leq m_1(X)m_2(Y) \leq 1$ are summed to $m(A)$ for $X \cap Y = A \notin\ \boldsymbol{\emptyset}$, other multiples are summed and added to $m(A)$ for $X \cup Y = A \notin\ \boldsymbol{\emptyset}$, $X \cap Y = A \in\ \boldsymbol{\emptyset}$, other multiples are summed and added to $m(A)$ for $U_{X\cup Y} = A \notin\ \boldsymbol{\emptyset}$, $X \cup Y = A \in\ \boldsymbol{\emptyset}$, all the other multiples (i.e., for $U_{X\cup Y} = A \in\ \boldsymbol{\emptyset}$) are summed and added to $\Theta_{\mathcal{M}}$. Hence the extended generalized Dubois-Prade rule produces correct gbba $m_1 \circledD m_2$ for any input gbba's m_1, m_2 on any hybrid DSm model.
1b) It holds since $\sum_{X,Y\in D^\Theta, X\cap Y\in\boldsymbol{\emptyset}} m_1(X)m_2(X) = 0 = \sum_{X\cup Y\in\boldsymbol{\emptyset}} m_1(X)m_2(X)$ and one has also $\sum_{X\cup Y\in\boldsymbol{\emptyset}} m_1(X)m_2(X) = \sum_{U_{X\cup Y}\in\boldsymbol{\emptyset}} m_1(X)m_2(X)$ in the free DSm model $\mathcal{M}^f$. Hence, the Dubois-Prade rule for the free model $\mathcal{M}^f$ is a special case of all the simple generalized Dubois-Prade rule, the generalized Dubois-Prade rule, and the extended generalized Dubois-Prade rule.

2) *Correctness of the generalization:*
Let us suppose Shafer's DSm model $\mathcal{M}^0$ and input BF's on $\mathcal{M}^0$, i.e., $\theta_i \cap \theta_j \equiv \emptyset$ for $i \neq j$. There are no non-existential constraints in $\mathcal{M}^0$. $X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}^0}$ iff $\{\theta_i | \theta_i \subseteq X\} \cap \{\theta_j | \theta_j \subseteq Y\} = \emptyset$, hence the same multiples $m_1(X)m_2(Y)$ are assigned to $X \cap Y = A \notin \boldsymbol{\emptyset}$, $A \neq \Theta$ in both the classic and the generalized Dubois-Prade rule on Shafer's DSm model, and the same multiples are summed and added to $X \cup Y = A \notin \boldsymbol{\emptyset}$ by both of the rules. $X \cup Y \notin \boldsymbol{\emptyset}$ for any couple $X, Y \in D^\Theta$ in Shafer's model, thus the 3rd sum in the generalized Dubois-Prade rule and the 4th sum in the extended rule for $\Theta_{\mathcal{M}}$ are always equal to 0 in Shafer's DSm model. Thus, the results are always the same for any m_1, m_2 on $\mathcal{M}^0$ and any $A \subseteq \Theta$ (and $A \in D^\Theta$). Hence all the simple generalized Dubois-Prade rule, the generalized Dubois-Prade rule, and the extended generalized Dubois-Prade rule are correct generalizations of the classic Dubois-Prade rule.

3) *Equivalence of expressions:* $(m_1 \circledR m_2)(A) \stackrel{?}{=} \phi(A)[S_1^{\circledR}(A) + S_2^{\circledR}(A) + S_3^{\circledR}(A)]$

$$\begin{aligned}
\phi(A)[S_1^{\circledR}(A) + S_2^{\circledR}(A) + S_3^{\circledR}(A)] = \\
\phi(A)[\sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + \sum_{X \cup Y \in \boldsymbol{\emptyset}_{\mathcal{M}},\ U_{X \cup Y} \equiv A} m_1(X)m_2(Y) + \\
\sum_{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}},\ (X \cup Y) \equiv A} m_1(X)m_2(Y)]
\end{aligned}$$

For $A \notin \boldsymbol{\emptyset}$ we simply obtain the following:

$$\begin{aligned}
1 \cdot [\sum_{X \cap Y \equiv A} m_1(X)m_2(Y) + \sum_{X \cup Y \in \boldsymbol{\emptyset}_{\mathcal{M}},\ U_{X \cup Y} \equiv A} m_1(X)m_2(Y) + \\
\sum_{X \cap Y \in \boldsymbol{\emptyset}_{\mathcal{M}},\ (X \cup Y) \equiv A} m_1(X)m_2(Y)] = (m_1 \circledR m_2)(A),
\end{aligned}$$

and for $A \in \boldsymbol{\emptyset}$, one gets

$$0 \cdot [S_1^{\circledR}(A) + S_2^{\circledR}(A) + S_3^{\circledR}(A)] = 0 = (m_1 \circledR m_2)(\emptyset).$$

The proof for the simple generalized Dubois-Prade rule is a special case of this proof with $S_2^{\circledR}(A) \equiv 0$.

The same holds for the extended generalized Dubois-Prade rule for $A \in \boldsymbol{\emptyset}$ and for $\Theta_{\mathcal{M}} \neq A \notin \boldsymbol{\emptyset}$.

For $A = \Theta_{\mathcal{M}}$ we obtain the following:

$$
\begin{aligned}
1\cdot[\sum_{X\cap Y\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \sum_{X\cup Y\in\emptyset_{\mathcal{M}},\,[U_{X\cup Y}\equiv\Theta_{\mathcal{M}}]\vee[U_{X\cup Y}\in\emptyset_{\mathcal{M}}]} m_1(X)m_2(Y)\\
+ \sum_{X\cap Y\in\emptyset_{\mathcal{M}},\,(X\cup Y)\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y)] =\\
[\sum_{X\cap Y\equiv A} m_1(X)m_2(Y) + \sum_{X\cup Y\in\emptyset_{\mathcal{M}},\,U_{X\cup Y}\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y) + \sum_{U_{X\cup Y}\in\emptyset_{\mathcal{M}}} m_1(X)m_2(Y)\\
+ \sum_{X\cap Y\in\emptyset_{\mathcal{M}},\,(X\cup Y)\equiv\Theta_{\mathcal{M}}} m_1(X)m_2(Y)] = (m_1 \textcircled{\scriptsize DP} m_2)(\Theta_{\mathcal{M}})
\end{aligned}
$$

Hence all three versions of the expression in DSm form are equivalent to the corresponding versions of the definition of the generalized Dubois-Prade rule.

3.11.4 Comparison statements

Statement 1: trivial.
Statement 2(ii): Let us compare definitions of DSmH rule and the generalized Dubois-Prade rule in DSm form. We have $S_1^{\textcircled{\scriptsize DP}}(A) = S_1(A)$, we can simply observe that $S_3^{\textcircled{\scriptsize DP}}(A) = S_3(A)$. We have already mentioned that $U_{X\cup Y} = \mathcal{U} = u(X)u(Y)$, thus also $S_2^{\textcircled{\scriptsize DP}}(A) = S_2(A)$. Hence $(m_1 \textcircled{\scriptsize DP} m_2)(A) = (m_1 \circledast m_2)(A)$ for any A and any m_1, m_2 in any hybrid DSm model.
Statement 2(i): If all constraints are satisfied by all input beliefs, we have $m_1(X) = m_2(Y) = 0$ for any $X, Y \in \emptyset_{\mathcal{M}}$ and $S_2(A) = 0 = S_2^{\textcircled{\scriptsize DP}}(A)$. If some constraints are not satisfied, but there is no non-existential constraint in model $\mathcal{M}$, then $\mathcal{U} = U_{X\cup Y} \notin \emptyset_{\mathcal{M}}$, and $S_2(A) = \sum_{X,Y\in\emptyset_{\mathcal{M}},\,\mathcal{U}_{\mathcal{M}}=A} m_1(X)m_2(Y) = \sum_{X,Y\in\emptyset_{\mathcal{M}},\,U_{X\cup Y}\cap I_{\mathcal{M}}=A} m_1(X)m_2(Y) = S_2^{\textcircled{\scriptsize DP}}(A)$ again.

Chapter 4

A Comparison of the Generalized minC Combination and the Hybrid DSm Combination Rules

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Abstract: *A generalization of the minC combination to DSm hyper-power sets is presented. Both the special formulas for static fusion or dynamic fusion without non-existential constraints and the quite general formulas for dynamic fusion with non-existential constraints are included. Examples of the minC combination on several different hybrid DSm models are presented. A comparison of the generalized minC combination with the hybrid DSm rule is discussed and explained on examples.*

4.1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Originally belief functions were introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [19].

For combination of beliefs Dempster's rule of combinations is used in DST. Under strict probabilistic assumptions, its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. There are not rare examples where the assumptions are not fulfilled and where results of Dempster's rule are counter intuitive, e.g. see [2, 3, 20], thus a rule with more intuitive results is required in such situations.

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Hence series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are Dubois-Prade's rule [13] and Yager's belief combination rule [23]. Among the others a wide class of operators [17] and an analogous idea proposed in [15], Smets' Transferable Belief Model (TBM) using so-called non-normalized Dempster's rule [22], disjunctive (or dual Dempster's) rule of combination [12], combination 'per elements' with its special case — minC combination, see [4, 8], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [14].

A brand new approach performs the Dezert-Smarandache (or Dempster-Shafer modified) theory (DSmT) with its DSm rule of combination. There are two main differences: 1) mutual exclusivity of elements of a frame of discernment is not assumed in general; mathematically it means that belief functions are not defined on the power set of the frame, but on a so-called hyper-power set, i.e. on the Dedekind lattice defined by the frame; 2) a new combination mechanism which overcomes problems with conflict among the combined beliefs and which also enables a dynamic fusion of beliefs.

As the classical Shafer's frame of discernment may be considered the special case of a so-called hybrid DSm model, the DSm rule of combination is compared with the classic rules of combination in the publications about DSmT [11, 20]. For better and objective comparison with the DSm rule the classic Dempster's, Yager's, and Dubois-Prade's rules were generalized to DSm hyper-power sets [7].

In despite of completely different motivations, ideas and assumptions of minC combination and DSm rule, there is an analogy in computation mechanisms of these approaches described in the author's Chapter 10 in [20]. Unfortunately the minC combination had been designed for classic belief functions defined only on the power set of a frame of discernment in that time. Recently, formulas for computation of minC on general n-element frame discernment has been published [8], and the ideas of minC combination have been generalized to DSm hyper-power sets in [10].

A goal of this contribution is to continue [5] using the recent results from [10], and complete a comparison of minC combination and hybrid DSm rules.

4.2 MinC combination on classic frames of discernment

4.2.1 Basic Definitions

All the classic definitions suppose an exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0, 1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$, the values of bba are called *basic belief masses (bbm)*.[1] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^Θ. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$.

Dempster's (conjunctive) rule of combination $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = K \sum_{X\cap Y=A} m_1(X)m_2(Y)$$

[1] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [19]. A classical counter example is Smets' Transferable Belief Model (TBM) which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.

for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X)m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [19]; putting $K = 1$ and $(m_1 \oplus m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e. g. [22].

An algebra $\mathcal{L} = (L, \wedge, \vee)$ is called a *lattice* if $L \neq \emptyset$ and $\wedge$, $\vee$ are two binary operations *meet* and *join* on L with the following properties: $x \wedge x = x$, $x \vee x = x$ (idempotency), $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (commutativity), $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$ (associativity), and $x \wedge (y \vee x) = x$, $x \vee (y \wedge x) = x$ (absorption). If the operations $\wedge$, $\vee$ satisfy also distributivity, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ we speak about a *distributive lattice*.

We can equivalently write any element of $X \in L$ in *conjunctive normal form* (CNF):
$X = \bigwedge_{i=1,...,m}(\bigvee_{j=1,...,k_i} X_{ij})$ for some m, $k_1, ..., k_m$, $X_{ij} \in L$, i.e. meet of joins.

4.2.2 Ideas of the minC combination

The minC combination (the minimal conflict/contradiction combination) is a generalization of the non-normalized Dempster's rule $\circledcirc$. $m(\emptyset)$ from $\circledcirc$ is considered as a *conflict (or contradiction)* arising by the conjunctive combination. To handle it, a system of different types of conflicts is considered according to the basic belief masses producing it.

We distinguish contradictions (conflicts) according to the sets to which the original bbms were assigned by m_i. There is only one type of contradiction (conflict) $\times$ on the belief functions defined on a binary frame of discernment, $\times$ corresponds to $m(\emptyset)$; hence the generalized level of minC combination fully coincides with the (non-normalized) conjunctive rule there. In the case of an n-element frame of discernment we distinguish different types of conflicts, e.g. $A \times B$, $A \times BC$, $A \times B \times C$, if $m_i(\{A\}), m_j(\{B\}) > 0$, $m_i(\{A\}), m_j(\{B, C\}) > 0$, $m_i(\{A\})$, $m_j(\{B\})$, $m_k(\{C\}) > 0$ etc. A very important role is played by so-called *potential conflicts (contradictions)*, e.g. $AB \times BC$, which is not a conflict in the case of combination of two beliefs ($\{A, B\} \cap \{B, C\} = \{B\} \neq \emptyset$), but it can cause a conflict in a later combination with another belief, e.g. real conflict $AB \times BC \times AC$ because there is $\{A, B\} \cap \{B, C\} \cap \{A, C\} = \emptyset$ which is different from $B \times AC$. Not to have (theoretically) an infinite number of different conflicts, the conflicts are divided into classes of equivalence which are called *types of conflicts*, e.g. $A \times B \sim B \times A \sim A \times B \times B \times B \times A \times A \times A$, etc. For more detail see [4].

In full version of [8], it is shown that the structure of pure and potential conflicts forms a distributive lattice $\mathcal{L}(\Omega) = (L(\Omega), \wedge, \vee)$, where $X \in L(\Omega)$ iff either $X = \{\omega_i\}$, where $\omega_i \in \Omega$, or $X = \{\omega_{i1} \times \omega_{i2} \times .... \times \omega_{ik_i}\}$, where $\omega_{ij} \in \Omega$ for $1 \leq i \leq n$, $1 \leq j \leq k_i$, or $X = U \wedge V$ or $X = U \vee V$ for some couple $U, V \in \mathcal{L}(\Omega)$; $\wedge$ $\vee$ are defined as it follows:
$X \vee Y = \{w \mid w \in X \text{ or } w \in Y \text{ and } (\neg\exists w')(w' \in X \cup Y,\ w' \leq w)\}$,
$X \wedge Y = \{w \mid w \in X \cap Y \text{ or } [w = \omega_{w1} \times \omega_{w2} \times ... \times \omega_{wk_w},$ where $(\exists x \in X)(x \leq w), (\exists y \in Y)(y \leq w)$ and $(\neg\exists w' \leq w)((\exists x \in X)(x \leq w'), (\exists y \in Y)(y \leq w'))]\}$.
Where it is further defined: $x \times x = x$, $y \times x = x \times y$, and $x_{11} \times x_{12} \times ... \times x_{1k_1} \leq x_{21} \times x_{22} \times ... \times x_{2k_2}$ iff $(\forall x_{1k})(\exists x_{2m})(x_{1k} = x_{2m})$. Note that $X \wedge Y = X \cap Y$ if $X \subseteq Y$ or $Y \subseteq X$.

We can extend $\mathcal{L}(\Omega)$ with $\emptyset$ to $\mathcal{L}_\emptyset(\Omega) = (L(\Omega) \cup \{\emptyset\}, \wedge, \vee)$, where $x \wedge \emptyset = \emptyset$ and $x \vee \emptyset = x$ for all $x \in L(\Omega)$. But we do no need it in classical case as no positive gbbm's are assigned to $\emptyset$ in input BF's and $x \wedge y \neq \emptyset$ and $x \vee y \neq \emptyset$ for any $x, y \in \mathcal{L}(\Omega)$.

The generalized level of minC combination gives non-negative weights to all elements of $\mathcal{L}(\Theta)$, i.e. also to the conflicts/contradictions and potential conflicts, i.e. it produces and combines so-called *generalized bba's* and *generalized* belief functions defined on the so-called *generalized frame of discernment* $\mathcal{L}(\Theta)$, which includes also all corresponding types of conflicts.

The generalized level of minC combination is associative and commutative operation and it commutes also with coarsening of frame of discernment. After performance of the generalized level of the minC, all bbms of both pure and potential conflicts should be reallocated / proportionalized among all corresponding non-conflicting elements of $\mathcal{P}(\Theta)$.

Unfortunately such proportionalizations break associativity of the minC combination. Hence all the input bba's must be combined on the generalized level at first, and the proportionalization may not be performed before finishing of the generalized level combination. So it is useful to keep also generalized level results because of to be prepared for possible additional source of belief, which we possibly want to combine together with the present input beliefs.

4.2.3 Formulas for the minC combination

Let $\bigcap X = X_1 \cap X_2 \cap ... \cap X_k$ and $c(X) = \{X_1, ..., X_k\}$, where $CNF(X) = X_1 \wedge X_2 \wedge ... \wedge X_k$, similarly let $\bigcup X = X_1 \cup X_2 \cup ... \cup X_k$, where $CNF(X) = X_1 \wedge X_2 \wedge ... \wedge X_k$, it holds that $X_i = X_{i1} \vee X_{i2} \vee ... \vee X_{ik_i}$ for any of these X_is thus it corresponds to $\{X_{i1}, X_{i2}, ..., X_{ik_i}\}$, and $\bigcup X \in \mathcal{P}(\Theta)$, let further $p(X) = \{Y_1 \cup ... \cup Y_m \mid 1 \leq m \leq k, Y_i \in c(X) \text{ for } i = 1, ..., m\}$. Let all X from $\mathcal{L}(\Theta)$ be in CNF in the following formulas, unless another form of X is explicitly specified.

The generalized level of the minC combination is computed for all $A \in \mathcal{L}(\Theta)$ as

$$m^0(A) = \sum_{X \wedge Y = A} m_1(X)\, m_2(Y).$$

Reallocation of gbbm's of potential conflicts: for all $\emptyset \neq A \in \mathcal{P}(\Theta)$,

$$m^1(A) = m^0(A) + \sum_{\substack{X \in \mathcal{L}(\Theta) \\ X \neq A,\ \bigcap X = A}} m^0(X) = \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \bigcap X = A}} m^0(X).$$

Final classic bba m we obtain after proportionalization of gbbm's of pure conflicts.

$$m(A) = \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \cap X = A}} m^0(X) \ + \sum_{\substack{X \in \mathcal{L}(\Theta) \\ \cap X = \emptyset,\, A \subseteq \cup X}} prop(A, X)\, m^0(X),$$

where
$prop_{11}(A, X) = prop_{12}(A, X) = \frac{m^1(A)}{\sum_{Y \in p(X)} m^1(Y)}$ for $A \in p(X)$, $\sum_{Y \in p(X)} m^1(Y) > 0$,
$prop_{11}(A, X) = prop_{12}(A, X) = 0$ for $A \notin p(X)$,
$prop_{11}(A, X) = \frac{1}{|p(X)|-1}$ for $A \in p(X)$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{12}(A, X) = 1$ for $A = \bigcup X$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{12}(A, X) = 0$ for $A \subset \bigcup X$, $\sum_{Y \in p(X)} m^1(Y) = 0$,
$prop_{21}(A, X) = prop_{22}(A, X) = \frac{m^1(A)}{cbel^1(X)}$ for $cbel^1(X) > 0$,
$prop_{21}(A, X) = \frac{1}{2^{|\bigcup X|} - 1}$ for $cbel^1 = 0$,
$prop_{22}(A, X) = \frac{m^1(A)}{cbel^1(X)}$ for $cbel^1(X) > 0$,
$prop_{22}(A, X) = 1$ for $cbel^1(X) = 0$ and $A = \bigcup X$,
$prop_{22}(A, X) = 0$ for $cbel^1(X) = 0$ and $A \subset \bigcup X$,
where $cbel^1(X) = \sum_{\emptyset \neq Y \in \mathcal{P}(\Theta),\ Y \subseteq \bigcup X} m^1(Y)$, $m(\emptyset) = 0$ $(= m^0(\emptyset) = m^1(\emptyset))$.

Proportionalization coefficient function $prop_{ij}(\_,\_)$ determines the proportionalization ratio for distribution of conflicting gbbm's. The first index i indicates whether 1) $m^0(X)$ is proportionalized only among elements of $p(X)$, i.e, among all conjuncts from $CNF(X)$ and among all disjunctions of these conjuncts for $i = 1$, or 2) $m^0(X)$ is proportionalized among all subsets of $\bigcup X$ for $i = 2$. The second index indicates the way of proportionalization when the proportionalization ratio is "$\frac{0}{0}$": 1) division of $m^0(X)$ to the same parts and distribution of these parts among all conjuncts in question (for $i = 1, j = 1$) or among all subsets of $\bigcup X$ (for $i = 2, j = 1$) is used, or 2) whole conflicting gbbm $m^0(X)$ is relocated to $\bigcup X$ for $j = 2$. $prop_{1j}$ corresponds to proportionalization a) from [4, 5] and $prop_{2j}$ corresponds to proportionalization b) from [5] (resp. to c) from [4]). For another proportionalizations see the full version of [8].

Let us present the proportionalization on a small example $m^0(X)$, where $X = \theta_1 \wedge (\theta_2 \vee \theta_3)$: X is already in CNF, i.e. $CNF(X) = X$, it has two conjuncts singleton θ_1 and disjunction $\theta_2 \vee \theta_3$, we can construct the only nontrivial disjunction $\theta_1 \vee \theta_2 \vee \theta_3$ from these conjuncts, $\bigcup X = \theta_1 \vee \theta_2 \vee \theta_3$.

$prop_{1j}$ proportionalizes conflicting $m^0(X)$ among conjuncts θ_1, $\theta_2 \vee \theta_3$, and their disjunction $\theta_1 \vee \theta_2 \vee \theta_3$:
if $m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) > 0$ we have:

$$prop_{1j}(\theta_1, X) = \frac{m^1(\theta_1)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

$$prop_{1j}(\theta_2 \vee \theta_3, X) = \frac{m^1(\theta_2 \vee \theta_3)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

$$prop_{1j}(\theta_1 \vee \theta_2 \vee \theta_3, X) = \frac{m^1(\theta_1 \vee \theta_2 \vee \theta_3)}{m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3)}$$

if $m^1(\theta_1) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) = 0$ we have:

$$prop_{11}(\theta_1, X) = prop_{11}(\theta_2 \vee \theta_3, X) = prop_{11}(\theta_1 \vee \theta_2 \vee \theta_3) = 1/3$$

$$prop_{12}(\theta_1, X) = prop_{11}(\theta_2 \vee \theta_3, X) = 0, prop_{12}(\theta_1 \vee \theta_2 \vee \theta_3) = 1.$$

$prop_{2j}$ proportionalizes conflicting $m^0(X)$ among all subsets of $\bigcup X = \theta_1 \vee \theta_2 \vee \theta_3$, i.e. among $\theta_1, \theta_2, \theta_3, \theta_1 \vee \theta_2, \theta_1 \vee \theta_3, \theta_2 \vee \theta_3, \theta_1 \vee \theta_2 \vee \theta_3$:
if $S = m^1(\theta_1) + m^1(\theta_2) + m^1(\theta_3) + m^1(\theta_1 \vee \theta_2) + m^1(\theta_1 \vee \theta_3) + m^1(\theta_2 \vee \theta_3) + m^1(\theta_1 \vee \theta_2 \vee \theta_3) > 0$ we have, $prop_{2j}(A, X) = \frac{m^1(A)}{S}$ for all $A \subseteq \bigcup X$;
if $S = 0$ we have, $prop_{21}(A, X) = 1/7$ for all $A \subseteq \bigcup X$. $prop_{22}(A, X) = 0$ for all $A \subset \bigcup X$, $prop_{22}(\bigcup X) = 1$.

4.3 Introduction to DSm theory

Because DSmT is a new theory which is in permanent dynamic evolution, we have to note that this text is related to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [20]. Rapid development of the theory is demonstrated by appearing of the current second volume of the book. For new advances of DSmT see other chapters of this volume.

4.3.1 Dedekind lattice and other basic DSm notions

Dempster-Shafer modified Theory or Dezert-Smarandache Theory (DSmT) by J. Dezert and F. Smarandache [11, 20] allows mutually overlapping elements of a frame of discernment. Thus a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[2]. DSmT allows that an object can have 2 or 3 colours in the same time: e.g. it can be both red and blue, or red and green and blue in the same time, it corresponds to a composition of general colours from the 3 basic ones.

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), but they are defined on so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions they are called generalized basic belief assignments and generalized belief functions[3].

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ with exclusive elements is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset,\ \theta_1 \cap \theta_2,\ \theta_1,\ \theta_2,\ \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. For $\Theta = \{\theta_1, \theta_2, \theta_3\}$ we have $|\Theta| = 3$, $|D^\Theta| = 19$.

A *DSm generalized basic belief assignment (DSm gbba)* m is a mapping $m : D^\Theta \longrightarrow [0, 1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *DSm generalized basic belief mass (DSm gbbm) of A*. A *DSm generalized belief function (DSm gBF) Bel* is a mapping $Bel : D^\Theta \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$.

4.3.2 DSm models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i. e. all elements of an exhaustive frame of discernment can mutually overlap themselves, we speak about the *free DSm model* $\mathcal{M}^f(\Theta)$, i. e. about DSm model free of constraints.

In general it is possible to add exclusivity or non-existential constraints into DSm models, we speak about *hybrid DSm models* in such cases.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows them: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case all the elements of the 3-element frame of discernment

[2] www.gallup.unm.edu/~smarandache/DSmT.htm

[3] If we want to distinguish these generalized notions from the generalized level of minC combination we use DSm generalized basic belief assignment, DSm generalized belief mass and function, and analogically minC generalized basic belief assignment and minC gbbm further in this text, on the other hand no minC generalized BF has been defined.

$\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model as *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings an additional information about a frame of discernment saying that θ_3 is impossible, it forces all the gbbm of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination, which is used in a dynamic fusion.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only gbbm on $\mathcal{M}_\emptyset$, and it is the only case, where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$, where $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set[4]. Because we will not work with $\mathcal{M}_\emptyset$ in the present contribution, we will work only $\emptyset \neq X \in D^\Theta$, thus $X \in \boldsymbol{\emptyset}$ is the same as $X \in \boldsymbol{\emptyset}_\mathcal{M}$ in this text.

For a given DSm model we can define (in addition to [20]) $\Theta_\mathcal{M} = \{\theta_i | \theta_i \in \Theta, \theta_i \notin \boldsymbol{\emptyset}_\mathcal{M}\}$, $\Theta_\mathcal{M} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_\mathcal{M} = \bigcup_{\theta_i \in \Theta_\mathcal{M}} \theta_i$, i.e. $I_\mathcal{M} \overset{\mathcal{M}}{\equiv} I_t$, $I_\mathcal{M} = I_t \cap \Theta_\mathcal{M}$, $I_{\mathcal{M}_\emptyset} = \emptyset$. $D^{\Theta_\mathcal{M}}$ is a hyper-power set on the DSm frame of discernment $\Theta_\mathcal{M}$, i.e. on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. It holds $\Theta_\mathcal{M} = \Theta$, $D^{\Theta_\mathcal{M}} = D^\Theta$and $I_\mathcal{M} = I_t$ for any DSm model without non-existential constraint. Whereas *reduced hyper-power set* $D^\Theta_\mathcal{M}$ from Chapter 4 in [20] arises from D^Θ by identifying of all $\mathcal{M}$-equivalent elements. $D^\Theta_{\mathcal{M}^0}$ corresponds to classic power set 2^Θ.

4.3.3 The DSm rule of combination

The *classic DSm rule* (DSmC) is defined for belief combination on the free DSm model as it follows[5]:

$$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledplus m_2)(A) = \sum_{X \cap Y = A,\ X,Y \in D^\Theta} m_1(X)\, m_2(Y).$$

Since D^Θ is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classic DSm rule guarantees that $(m_1 \circledplus m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [20].

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the nature of the problem under consideration, which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_\emptyset(\Theta)$, for $k \geq 2$ independent sources of information is defined as: $m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledplus m_2 \circledplus ... \circledplus m_k)(A) = \phi(A)[S_1(A)+S_2(A)+S_3(A)]$, in full generality, see [20]. For a comparison with minC combination we use binary version of the rule, thus we have:

$$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledplus m_2)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)],$$

where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined by

$$S_1(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X)\ , m_2(Y)$$

[4] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_\mathcal{M}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_\mathcal{M} \cup \{\emptyset\}$.

[5] To distinguish the DSm rule from Dempster's rule, we use $\circledplus$ instead of $\oplus$ for the DSm rule in this text.

$$S_2(A) = \sum_{X,Y\in\ \emptyset,\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\ \emptyset)\wedge(A=I_t)]} m_1(X)\,m_2(Y)$$

$$S_3(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ X\cap Y\in\ \emptyset} m_1(X)\,m_2(Y)$$

with $\mathcal{U} = u(X) \cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are supposed to be in some canonical form, e.g. CNF. Unfortunately no mention about the canonical form is included in [20].

As size of hyper-power set D^Θ rapidly increase with cardinality of the frame of discernment Θ some readers may be interested in Chapter 2 of [20] on the generation of hyper-power sets, including subsection about memory size and complexity. For applications of DSmT see contributions in second parts of both the volumes of DSmT book.

In [20], is was shown that DSm hyper-power set corresponds to minC generalized frame of discernment extended with $\emptyset$, where overlappings of elements in DSm hyper-power set correspond to elementary conflicts in minC generalized frame of discernment and that the classic DSm rule numerically coincides with the generalized level of minC combination.

4.4 MinC combination on hyper-power sets

4.4.1 Generalized level of minC combination on hyper-power set

From the correspondence of hyper-power set (Dedekind Lattice) D^Θ with distributive lattice $\mathcal{L}_\emptyset(\Theta)$ representing extended minC generalized frame of discernment and from numerical coincidence of the classic DSm rule with generalized level of minC combination, we obtain coincidence of generalized level of minC on the hyper-power set with the generalized level of the classic minC combination and with the classic DSm rule (DSmC). Hence the generalized level of the minC combination on the hyper-power set is given by the following formula:

$$(m_1 \circledcirc m_2)^0(A) = m^0(A) = \sum_{X\wedge Y=A} m_1(X)\,m_2(Y) \ = \sum_{X\cap Y=A} m_1(X)\,m_2(Y).$$

4.4.2 MinC combination on the free DSm model $\mathcal{M}^f$

There are no constraints on the free DSm model, all elements of hyper-power set are allowed to have a positive (DSm generalized) bbm. It means that there are no conflicting bbms in minC combination generalized to the free DSm model. Thus no reallocation of bbms is necessary in minC combination generalized to the free DSm model. Thus minC combination generalized to the free DSm model coincides with its generalized level from the previous subsection:

$$m(A) = m^0(A) = \sum_{X\cap Y=A} m_1(X)\,m_2(Y).$$

Hence the generalized level of the minC combination and the minC combination on the free DSm model is associative and commutative operation on DSm generalized belief functions. The combination also commutes with coarsening of the frame of discernment.

Let us note that $m(\emptyset) = 0 = m^0(\emptyset)$ always holds as $X \cap Y \neq \emptyset$ for any $X, Y \in D^\Theta$, and $m_i(\emptyset) = 0$ for any DSm gbba on D^Θ.

4.4.3 Static minC combination on hybrid DSm models

Let us continue our generalization with a static combination, where DSm model is not changed within the combination process, i.e. all input belief functions are defined on a hybrid model in question. Let us suppose a fixed DSm model $\mathcal{M}$, thus we can use $\equiv$ instead of $\stackrel{\mathcal{M}}{\equiv}$ for simplification of generalized minC formulas.

As some of the elements of D^Θ are equal to other ones in hybrid DSm model $\mathcal{M}$, we have to reallocate their m^0 gbbm's to a corresponding elements $D^\Theta_\mathcal{M}$ as it follows:

$$m^1(A) = m^0(A) + \sum_{X \neq A,\ X \in D^\Theta,\ X \equiv A} m^0(X) \ = \sum_{X \in D^\Theta,\ X \equiv A} m^0(X),$$

for all $\emptyset \not\equiv A \in D^\Theta$, (i.e. for all $A \notin \emptyset_\mathcal{M}$). This step corresponds to relocation of potential conflicts in classic minC combination.

The rest is reallocation of m^0 bbms of sets which are equivalent to $\emptyset$; such sets correspond to pure conflicts in the classic case. Analogically to the proportionalization of gbbm of pure conflict X to its power set $\mathcal{P}(\bigcup X)$ in the classic minC combination, we proportionalize[6] conflicting gbbm $m^0(X)$ to substructure of the DSm model $\mathcal{M}$ defined by $\bigcup X$, i.e. to $D^{\bigcup X}_\mathcal{M}$, we do not care about $Y \equiv \emptyset_\mathcal{M}$ because they are not allowed by model $\mathcal{M}$.

$$m(A) = m^1(A) + \ reallocated\ gbbm's\ of\ conflicts.$$

$$m(A) = \sum_{\substack{X \in D^\Theta \\ X \equiv A}} m^0(X) \ + \sum_{\substack{X \in D^\Theta \\ X \equiv \emptyset,\, A \subseteq \cup X}} prop(A, X)\, m^0(X),$$

where proportionalization coefficient function *prop* is analogous to the *prop* in the classic version; there are only the following differences in notation: we use $X \in D^\Theta$ instead of $X \in \mathcal{L}(\Theta)$, $X \in D^\Theta_\mathcal{M}$ instead of $X \in \mathcal{P}(\Theta)$, $bel^1_\mathcal{M}$ instead of $cbel^1$, $|D^\Theta_\mathcal{M}|$ .... $\mathcal{P}(\Theta) = 2^{|\Theta|}$, $|D^{\bigcup X}_\mathcal{M}|$ .... $2^{|\bigcup X|} = |p(X)|$, $A \in D^{c(X)}_\mathcal{M}$ .... $A \in p(X)$, $Z \in \boldsymbol{\emptyset}_\mathcal{M}$ .... $\bigcup Z = \emptyset$, and similarly. Where $D^{c(X)}_\mathcal{M} = \{Y \in D^\Theta_\mathcal{M} \mid c(Y) \subseteq c(X)\}$, i.e. elements of $D^{c(X)}_\mathcal{M}$ are all unions and intersections constructed from conjuncts from $CNF(X)$ (from X_i such that $CNF(X) = X_1 \cap ... \cap X_k$). Let X be such that $CNF(X) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap \theta_4$ for example, thus $c(X) = \{\theta_1 \cup \theta_2,\ \theta_1 \cup \theta_3,\ \theta_4\}$, and $D^{c(X)}_{\mathcal{M}^f}$ contains e.g. $\theta_1 \cup \theta_2 \cup \theta_4$ and $(\theta_1 \cup \theta_2) \cap \theta_4$, but neither $\theta_1 \cup \theta_4$ or $\theta_1 \cap \theta_4$ nor $\theta_2 \cup \theta_3 \cup \theta_4$ as $\theta_1, \theta_2, \theta_3, \theta_2 \cup \theta_3, \theta_2 \cup \theta_4, \theta_3 \cup \theta_4$ are not elements of $c(X)$.

For $m^0(X) > 0$ we have that $(\bigcup X) \notin \emptyset_\mathcal{M}$ in static combination, because $X \subseteq \bigcup X$ and similarly for all input focal elements X_i from which $m^0(X)$ is computed $X_i \subseteq \bigcup X$. Thus we have no problem with cardinality $|D^{\cup X}_\mathcal{M}|$ which is always ≥ 2.
It is possible to show that $\sum_{X \in D^\Theta_\mathcal{M}} m(X) = 1$, i.e. $m(A)$ correctly defines static combination of gbba's on hybrid DSm model $\mathcal{M}$. We can also show that the above definition coincides with the classic minC combination on the Shafer's DSm model $\mathcal{M}_0$. Hence the above definition really generalizes the classic minC combination.

4.4.4 Dynamic minC combination

To make a full generalization of minC combination in the DSm nature. We have to allow also a change of a DSm model during combination, i.e. to allow input belief functions which

[6] If a proportionalization ratio is not defined, i.e. if it should be "$\frac{0}{0}$" then either 1) division to the same parts or 2) reallocation to $\bigcup X$ is used, analogically to the classic case.

are defined on more general model that is the resulting one, i.e. we have to be prepared to prohibition of some input focal elements. In such a case we have no immediate suggestion how reallocate $m^0(X)$ for $X \equiv \emptyset$ such that also $\bigcup X \equiv \emptyset$. In correspondence to non-defined proportionalization ratios we can distribute it among all non-empty elements of DSm model $\mathcal{M}$ or to relocate it to whole $I_{\mathcal{M}}$. We can represent both these proportionalizations with coefficient functions $prop(A, X)$ for computation of proportion of conflicting gbbm $m^0(X)$ which to be reallocated to $\emptyset \not\equiv A \in D^{\cup X}_{\mathcal{M}}$ and analogical $dyn(A, X)$ for dynamic fusion proportionalization of $m^0(X)$ where $\bigcup X \equiv \emptyset$. With respect to two types of proportionalization and two variants of non-defined proportionalizaton ratios managing we obtain four variants of coefficient function *prop* and two variants coefficient function *dyn*: of $prop_{11}(A, X)$, $prop_{12}(A, X)$, $prop_{21}(A, X)$, $prop_{22}(A, X)$, $dyn_1(A, X)$, and $dyn_2(A, X)$. We can summarize the dynamic minC combination as it follows:

$$m^0(A) = \sum_{\substack{X,Y \in D^\Theta \\ X \cap Y = A}} m_1(X)\, m_2(Y)$$

$$m_{ij}(A) = \sum_{\substack{X \in D^\Theta \\ X \equiv A}} m^0(X) + \sum_{\substack{\emptyset \equiv X \in D^\Theta \\ A \subseteq \bigcup X}} prop_{ij}(A, X)\, m^0(X) + \sum_{\substack{X \in D^\Theta \\ \bigcup X \equiv \emptyset}} dyn_j(A, X)\, m^0(X)$$

for all $\emptyset \not\equiv A \in D^\Theta_{\mathcal{M}}$, where $|D^\Theta_{\mathcal{M}}| > 1$ and where $prop_{ij}(A, X), dyn_j(A, X)$ are defined as it follows:

$prop_{11}(A, X) = prop_{12}(A, X) = \frac{m^1(A)}{\sum_{A,Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y)}$ for $A \in D^{c(X)}_{\mathcal{M}}$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) > 0$,
$prop_{11}(A, X) = prop_{12}(A, X) = 0$ for $A \notin D^{c(X)}_{\mathcal{M}}$,
$prop_{11}(A, X) = \frac{1}{|D^{c(X)}_{\mathcal{M}}| - 1}$ for $A \in D^{c(X)}_{\mathcal{M}}$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,
$prop_{12}(A, X) = 1$ for $A = \bigcup X$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,
$prop_{12}(A, X) = 0$ for $A \subset \bigcup X$, $\sum_{Y \in D^{c(X)}_{\mathcal{M}}} m^1(Y) = 0$,

$prop_{21}(A, X) = \frac{m^1(A)}{bel^1_{\mathcal{M}}(X)}$ for $bel^1_{\mathcal{M}}(X) > 0$,
$prop_{21}(A, X) = \frac{1}{|D^{\cup X}_{\mathcal{M}}| - 1}$ for $bel^1_{\mathcal{M}}(X) = 0$,
$prop_{22}(A, X) = \frac{m^1(A)}{bel^1_{\mathcal{M}}(X)}$ for $bel^1_{\mathcal{M}}(X) > 0$,
$prop_{22}(A, X) = 1$ for $bel^1_{\mathcal{M}}(X) = 0$ and $A = \bigcup X$,
$prop_{22}(A, X) = 0$ for $bel^1_{\mathcal{M}}(X) = 0$ and $A \subset \bigcup X$,

$dyn_1(A, \_) = \frac{m^1(A)}{\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z)}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) > 0$,
$dyn_1(A, \_) = \frac{1}{|D^\Theta_{\mathcal{M}}| - 1}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$,
$dyn_2(A, \_) = \frac{m^1(A)}{\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z)}$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) > 0$,
$dyn_2(I_{\mathcal{M}}, \_) = 1$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$,
$dyn_2(A, \_) = 0$, if $\sum_{Z \in D^\Theta_{\mathcal{M}}} m^1(Z) = 0$, $A \neq I_{\mathcal{M}}$,

$m_{ij}(A) = 0$ for $A \equiv \emptyset$.

Similarly to the classic case we can show that $\sum_{X \in D^\Theta_{\mathcal{M}}} m(X) = 1$ hence the above formulas produce a correct gbba also for dynamic combination.

If we want to combine 3 or more (k) gBF's, we apply twice or more times (k times) the binary combination on the generalized level (in the classic minC terminology), i.e. on the free

			$\mathcal{M}^f$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_7$
	m_1	m_2	m^0	m_{ij}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{1j}	m_{2j}	m_{ij}	m_{1j}	m_{2j}
$\theta_1\cap\theta_2\cap\theta_3$	0	0	0.16												
$\theta_1\cap\theta_2$	0.10	0.20	0.22	0.26											
$\theta_1\cap\theta_3$	0.10	0	0.12	0.14	0.15	0.20	0.18	0.41							
$\theta_2\cap\theta_3$	0	0.20	0.19	0.23	0.30	0.41					0.42	0.70			
$\theta_1\cap(\theta_2\cup\theta_3)$	0	0	0	0.00											
$\theta_2\cap(\theta_1\cup\theta_3)$	0	0	0.05	0.06											
$\theta_3\cap(\theta_1\cup\theta_2)$	0	0	0.01	0.01	0.01	0.02									
$\square$	0	0	0	0											
θ_1	0.10	0.20	0.08	0.10	0.26	0.14	0.31	0.24	0.39	0.41				0.62	0.67
θ_2	0.20	0.10	0.03	0.04	0.10	0.05	0.17	0.09	0.17	0.15	0.27	0.07		0.37	0.33
θ_3	0.30	0.10	0.10	0.12	0.13	0.13	0.29	0.23	0.37	0.39	0.31	0.23	1.00		
$\square\theta_1$	0	0	0.02	0.02	0.025	0.03									
$\square\theta_2$	0	0	0	0	0	0									
$\square\theta_3$	0	0	0	0											
$\theta_1\cup\theta_2$	0.10	0	0	0	0	0	0	0	0	0				0.01	0
$\theta_1\cup\theta_3$	0.10	0.20	0.02	0.02	0.025	0.03	0.05	0.03	0.06	0.05					
$\theta_2\cup\theta_3$	0	0	0	0	0	0	0	0	0	0	0	0			
$\theta_1\cup\theta_2\cup\theta_3$	0	0	0	0	0	0	0	0	0	0					

Table 4.1: MinC combination of gbba's m_1 and m_2 on hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$.

DSm model, (or equivalently k-ary combination on the free DSm model), and after it we use some proportionalization in the same way as in the case of the minC combination of two gBF's. Hence we can see that the minC combination is defined on any DSm model for any k generalized belief functions.

4.5 Examples of minC combination

Three simple examples for both the static and dynamic fusion on Shafer's DSm model $\mathcal{M}^0$ have been presented in [10]. Nevertheless, for an illustration of all main properties of the generalized minC rule it is necessary to see, how the rule works on general hybrid DSm models. Therefore we present examples of fusion on seven different hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$ in this text, see Table 4.1.

For easier comparison of the generalized minC combination with the hybrid DSm rule we use the models from Examples 1 — 7, see DSm book Vol. 1 [20], Chapter 4. All the combinations are applied to two generalized belief functions on a 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The hybrid DSm models from the examples are given as it follows:

$\mathcal{M}_1$: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$,

$\mathcal{M}_2$: $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, thus also $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$,

$\mathcal{M}_3$: $\theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_3}{\equiv} \emptyset$ and hence also $\theta_1 \cap \theta_2 \overset{\mathcal{M}_3}{\equiv} \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$,

$\mathcal{M}_4 = \mathcal{M}^0$: $\theta_1 \cap \theta_2 \overset{\mathcal{M}_4}{\equiv} \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_1 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \emptyset$ and hence also $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_1 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv}$ $\theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv} \theta_3 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_4}{\equiv} \square \overset{\mathcal{M}_4}{\equiv} \emptyset$, and further $\square\theta_1 \overset{\mathcal{M}_4}{\equiv} \theta_1$, $\square\theta_2 \overset{\mathcal{M}_4}{\equiv} \theta_2$, $\square\theta_3 \overset{\mathcal{M}_4}{\equiv} \theta_3$,

$\mathcal{M}_5$: $\theta_1 \overset{\mathcal{M}_5}{\equiv} \emptyset$ (θ_1 is removed from $\Theta = \{\theta_1, \theta_2, \theta_2\}$ in fact) thus all $X \in D^\Theta$ which include intersection with θ_1 are forced to be empty (i.e. $X \overset{\mathcal{M}_5}{\equiv} \emptyset$), and all $Y \in D^\Theta$ which include union with θ_1 are forced to be equivavent to some element of $D^\Theta_{\mathcal{M}_5}$,

$\mathcal{M}_6$: $\theta_1 \overset{\mathcal{M}_6}{\equiv} \theta_2 \overset{\mathcal{M}_6}{\equiv} \emptyset$, thus $\theta_1 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_1 \cup \theta_2 \cup \theta_3 \overset{\mathcal{M}_6}{\equiv} \theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_6}{\equiv} \theta_3$, and all

other $X \in D^\Omega$ are forced to be empty (i.e. $X \overset{\mathcal{M}_6}{\equiv} \emptyset$),
$\mathcal{M}_7$: $\theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_7}{\equiv} \emptyset$, i.e. also $\theta_3 \overset{\mathcal{M}_7}{\equiv} \emptyset$ and $\theta_1 \cap \theta_2 \overset{\mathcal{M}_7}{\equiv} \emptyset$, thus only $\theta_1 \cup (\theta_2 \cap \theta_3) \overset{\mathcal{M}_7}{\equiv} \theta_1 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, $\theta_2 \cup (\theta_1 \cap \theta_3) \overset{\mathcal{M}_7}{\equiv} \theta_2 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, $\theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_7}{\equiv} \theta_3 \overset{\mathcal{M}_7}{\not\equiv} \emptyset$, and all the other $X \in D^\Theta$ are constrained, for more details see [20].

We use the following abbreviations for 4 elements of D^Θ: $\square$ for $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3) \cap (\theta_2 \cup \theta_3)$, $\square\theta_1$ for $\theta_1 \cup (\theta_2 \cap \theta_3) = (\theta_1 \cup \theta_2) \cap (\theta_1 \cup \theta_3)$, $\square\theta_2$ for $\theta_2 \cup (\theta_1 \cap \theta_3)$, and $\square\theta_3$ for $\theta_3 \cup (\theta_1 \cap \theta_2)$. Thus $\square$ is not any operator here, but just a symbol for abbreviation; it has its origin in the papers about minC combination [4, 10], see also Chapter 10 in DSm book Vol. 1 [20].

The generalized BF's Bel_1 and Bel_2 are represented by generalized bba's m_1 and m_2 from the referred Examples 1—7 again. For the values of gbbm's $m_i(A)$ see the 2nd and 3rd column of Table 4.1. All elements of the hyper-power set D^Θ, which correspond to the given frame of the discernment Θ, are placed in first column of the table.

For better comparison of different results of the generalized minC combination on different DSm models we put all the results into one table. Every row of the table body contain an element A of D^Θ, corresponding values of source gbba's $m_i(A)$, value $m^0(A)$, which corresponds to the free DSm model $\mathcal{M}^f$, and gbbm's $m_{ij}(A)$ corresponding to hybrid DSm models $\mathcal{M}_1$ — $\mathcal{M}_7$ referred in the first row of the table head. The fourth column of Table 4.1 present values $m^0(A)$ of the generalized level of the generalized minC combination. These values coincide with the resulting values $m(A)$ on the free DSm model $\mathcal{M}^f$, where values for all elements $A \in D^\Theta$ are defined and printed.

To space economizing, we present the DSm models $\mathcal{M}_i$ together with the resulting gbbm values $m_{ij}(A)$ in the corresponding columns of Table 4.1: only values for $A \in D^\Theta_{\mathcal{M}_i}$ are printed. The 0 values for $A \in D^\Theta$ which are constrained (forced by constraints to be empty) are not printed, similarly the 0 values for $X \in D^\Theta$ which are $\mathcal{M}_i$-equivalent to some $A \in D^\Theta_{\mathcal{M}_i}$ ($A \overset{\mathcal{M}_i}{\equiv} X \neq A$) are also not printed. Thus for example $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_1}{\equiv} \emptyset$, $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$ consequently $m_{ij}(\theta_1 \cap \theta_2 \cap \theta_3) = 0$ in both models $\mathcal{M}_1$ and $\mathcal{M}_2$ and $m_{ij}(\theta_1 \cap \theta_2) = 0$ in model $\mathcal{M}_2$, hence the corresponding cells in the table are blank. Similarly $\theta_1 \cap (\theta_2 \cup \theta_3) \overset{\mathcal{M}_2}{\equiv} \theta_1 \cap \theta_3$, $\theta_2 \cap (\theta_1 \cup \theta_3) \overset{\mathcal{M}_2}{\equiv} \theta_2 \cap \theta_3$, $\square\theta_3 = \theta_3 \cup (\theta_1 \cap \theta_2) \overset{\mathcal{M}_2}{\equiv} \theta_3$, and $\square \overset{\mathcal{M}_2}{\equiv} \theta_3 \cap (\theta_1 \cup \theta_2)$, thus values $m^0(X)$ are added to values $m^0(A)$ and $m^1(X) = m_{ij}(X) = 0$ for all such Xs and corresponding As ($A \overset{\mathcal{M}_2}{\equiv} X \neq A$), i.e. $m_{ij}(\theta_1 \cap (\theta_2 \cup \theta_3)), m_{ij}(\theta_2 \cap (\theta_1 \cup \theta_3)), m_{ij}(\square\theta_3) m_{ij}(\square)$ are forced to be 0 in DSm model $\mathcal{M}_2$, hence the corresponding cells in the 6th and 7th columns of the table are also blank. On the other hand there are printed 0 values for $m_{ij}(\theta_1 \cup \theta_2) = m_{ij}(\theta_2 \cup \theta_3) = m_{ij}(\theta_1 \cup \theta_2 \cup \theta_3) = m_{ij}(\square\theta_2) = 0$ because these 0 values are not forced by constraints of the model $\mathcal{M}_2$ but they follow values of input gbba's m_1 and m_2. $\mathcal{M}_4 \equiv \mathcal{M}^0$ is Shafer's DSm model thus the values are printed just for $A \in 2^\Theta$ in the 10-th and 11-th columns. For details on equivalence of $A \in D^\Theta$ on hybrid DSm models $\mathcal{M}_3, \mathcal{M}_5, \mathcal{M}_6, \mathcal{M}_7$, see Chapter 4 in DSm book Vol. 1 [20]; for the model $\mathcal{M}_3$ see also Example 6 in Chapter 3 of this volume, specially the 5-th column of Table 3.6 as the model $\mathcal{M}_3$ coincides with DSm model $\mathcal{M}_{4.3}$ there. There is no row for $\emptyset$ in Table 4.1 as all the cells should be blank there.

Because of the values $m_i(A)$ of the used gbba's m_1 and m_2, there is no difference between m_{i1} and m_{i2} on all the models $\mathcal{M}_1, ..., \mathcal{M}_7$, moreover, there is also no difference between m_{1j} and m_{2j} on model $\mathcal{M}_1$. Trivially, there is no difference on trivial DSm model $\mathcal{M}_6$ which have the only element θ_3 not equivalent to empty set ($D^\Theta_{\mathcal{M}_6} = \{\theta_3, \emptyset\}$) thus there is the only possible gbbm $m(\theta_3) = 1$ on the model $\mathcal{M}_6$. Trivially, there is also no difference among m_{ij} on the free DSm model $\mathcal{M}^f$ because there is no constraint and consequently no proportionalization there.

To economize space in the table again, only columns with different values are printed. Results m^1 of the combination step which groups together values $m^0(X)$ of $\mathcal{M}_i$-equivalent elements of D^Θ are not presented from the same reason.

4.6 Comparison of the generalized minC combination and hybrid DSm combination rules

There is presented minC combination of generalized BF's Bel_1 and Bel_2 on the free DSm model and on 7 hybrid DSm models in the previous section. For a comparison of the generalized minC combination rule with the hybrid DSm rule (DSmH rule), we compute or recall the DSm rule results on the same DSm models from the examples in DSm book 1 [20], Chapter 4. We present the results in the same way as there were presented the results of the generalized minC combination in the previous section, see Table 4.2. From the definitions of the both the rules

			$\mathcal{M}^f$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$
	m_1	m_2	$m_{\mathcal{M}^f}$	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}	m_{DSmH}
$\theta_1 \cap \theta_2 \cap \theta_3$	0	0	0.16							
$\theta_1 \cap \theta_2$	0.10	0.20	0.22	0.22						
$\theta_1 \cap \theta_3$	0.10	0	0.12	0.12	0.14	0.17				
$\theta_2 \cap \theta_3$	0	0.20	0.19	0.19	0.26			0.33		
$\theta_1 \cap (\theta_2 \cup \theta_3)$	0	0	0	0.02						
$\theta_2 \cap (\theta_1 \cup \theta_3)$	0	0	0.05	0.07						
$\theta_3 \cap (\theta_1 \cup \theta_2)$	0	0	0.01	0.03	0.03					
$\square$	0	0	0	0						
θ_1	0.10	0.20	0.08	0.08	0.12	0.16	0.18			0.43
θ_2	0.20	0.10	0.03	0.03	0.08	0.12	0.13	0.24		0.24
θ_3	0.30	0.10	0.10	0.10	0.17	0.23	0.24	0.39	1.00	
$\square\theta_1$	0	0	0.02	0.04	0.04					
$\square\theta_2$	0	0	0	0.01	0.01	0.01				
$\square\theta_3$	0	0	0	0.07						
$\theta_1 \cup \theta_2$	0.10	0	0	0	0.09	0.11	0.11			0.33
$\theta_1 \cup \theta_3$	0.10	0.20	0.02	0.02	0.06	0.08	0.17			
$\theta_2 \cup \theta_3$	0	0	0	0	0	0.05	0.05	0.04		
$\theta_1 \cup \theta_2 \cup \theta_3$	0	0	0	0	0	0.07	0.12			

Table 4.2: DSmH combination of gbba's m_1 and m_2 on hybrid DSm models $\mathcal{M}_1, ..., \mathcal{M}_7$.

it is obvious that the minC and DSmH rules coincide themselves on the free DSm model and that they coincide also with the classic DSm (DSmC) rule and with the conjunctive rule of combination of gBF's on DSm hyper-power sets. In the examples we can compare the fourth columns in both the tables.

Trivially, both the rules coincide also on trivial DSm models with the only non-empty element, see e.g. $\mathcal{M}_6$ and the corresponding columns in the tables.

The presented examples are not enough conflicting to present differences between proportionalizations $prop_{i1}$ and $prop_{i2}$. Therefore we add another example for presentation of their differences and for better presentation of their relation to DSmH rule. For this reason we use a modified Zadeh's example on Shafer's model on 4-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$: $\mathcal{M}_8 = \mathcal{M}^f(\Theta)$. The small non-conflicting element is split to two parts θ_3 and θ_4 and similarly its bbms. In the same time, it is a modification of the example from subsection 5.4.1 from Chapter 5 in DSm book Vol. 1, where small parts of $m(\theta_3 \cup \theta_4)$ are more specified to θ_3 and θ_4 in inputs bba's, see Table 4.3. When coarsening $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ to $\{\theta_1, \theta_2, \theta_3 \equiv \theta_4\}$ in our present example, we obtain an instance of the classic Zadeh's example. Hence our example in Table 4.3 is just one of many possible refinements of Zadeh's example.

The structure of the table is analogous to that of previous tables. As the whole table representing D^Θ has 167 rows, all the rows which include only 0s and blank cells are skipped. Different results of minC using 4 proportionalizations are presented in 5-8th columns of the table. DSmH results are presented in 9-th column. As it is already mentioned in the introduction, we cannot forget that Dempster's rule produces correct results for combination of any 2 belief functions which correctly represent mutually probabilistically independent evidences, which are not in full contradiction, on Shafer's model. Therefore we present also the result of application of Dempster's rule in the last column of Table 4.3.

			$\mathcal{M}^f$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$	$\mathcal{M}_8$
	m_1	m_2	m_0	m_{11}	m_{12}	m_{21}	m_{22}	m_{DSmH}	$m_\oplus$
$\theta_1 \cap \theta_2$	0	0	0.9506						
$\theta_1 \cap \theta_3$	0	0	0.0098						
$\theta_2 \cap \theta_4$	0	0	0.0097						
$\theta_3 \cap \theta_4$	0	0	0.0001						
$\theta_1 \cap (\theta_3 \cup \theta_4)$	0	0	0.0196						
$\theta_2 \cap (\theta_3 \cup \theta_4)$	0	0	0.0097						
θ_1	0.98	0	0	$0.3168\overline{6}$	0	$0.3168\overline{6}$	0	0	0
θ_2	0	0.97	0	$0.3168\overline{6}$	0	$0.3168\overline{6}$	0	0	0
θ_3	0	0.01	0.0001	0.00992	0.00992	0.01578	0.01578	0.0001	0.20
θ_4	0.01	0	0.0002	0.00994	0.00994	0.02166	0.02166	0.0002	0.40
$\theta_1 \cup \theta_2$	0	0	0	$0.3168\overline{6}$	0.95060	$0.3168\overline{6}$	0.95060	0.9506	0
$\theta_1 \cup \theta_3$	0	0	0	0	0	0	0	0.0098	0
$\theta_2 \cup \theta_4$	0	0	0	0	0	0	0	0.0097	0
$\theta_3 \cup \theta_4$	0.01	0.02	0.0002	0.02954	0.02954	0.01196	0.01196	0.0003	0.40
$\theta_1 \cup \theta_3 \cup \theta_4$	0	0	0	0	0	0	0	0.0196	0
$\theta_2 \cup \theta_3 \cup \theta_4$	0	0	0	0	0	0	0	0.0097	0

Table 4.3: Comparison of minC combination, hybrid DSm and Dempster's rules on a modified Zadeh's example on Shafer's model $\mathcal{M}_8 \equiv \mathcal{M}^0(\Theta)$ for a 4-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. (Only non-empty non-zero rows of the table are printed.)

Results of the minC combination are usually more specified (i.e. gbbm's are located to less focal elements) in general cases, compare the columns corresponding to the same DSm models in Tables 4.1 and 4.2, see also comparison in Table 4.3. It holds more when using proportionalizations $prop_{i1}$, which produce more specified results than proportionalizations $prop_{i2}$ do. There are also examples, where it is not possible to say which rule produces more of less specified results. It is in cases of totally conflicting focal elements, where all input gbbm's corresponding to these elements are assigned to $X \equiv \emptyset$ by $m^0 \equiv m_{\mathcal{M}^f}$.

Moreover the counter examples arise in a special cases of input gBF's with focal elements which are all totally conflicting and some of them assign(s) gbba to overlapping element(s) of frame of discernment. For example, let us assume hybrid DSm model $\mathcal{M}_2$ and gBF's Bel_3, Bel_4, Bel_5 given by gbba's $m_3(\theta_1) = 1$, $m_4(\theta_2 \cap \theta_3) = 1$ and $m_5(\theta_1 \cap \theta_2 \cap \theta_3) = 1$.

When combining Bel_3 and Bel_4 using $prop_{22}$ we obtain a counter example for static fusion: $m_{11}(\theta_1) = m_{11}(\theta_2 \cap \theta_3) = m_{11}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1/3$, $m_{12}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1$, $m_{21}(X) = 1/12$, $m_{22}(\theta_1 \cup \theta_2 \cup \theta_3) = 1$, whereas for DSmH we obtain $m_{DSmH}(\theta_1 \cup (\theta_2 \cap \theta_3)) = 1$, i.e. $m_{DSmH}(\Box\theta_1) = 1$. We can immediately see that $\theta_1 \cup \theta_2 \cup \theta_3 \supset \theta_1 \cup (\theta_2 \cap \theta_3)$. When using $prop_{21}$ it not possible to say which of the rules produces more specified results as m_{21} assigns 1/12 to every element of model $\mathcal{M}$: one of them is equal to $\Box\theta_1 = \theta_1 \cup (\theta_2 \cap \theta_3)$ (to what DSmH assigns 1), 4 of them are subset of $\Box\theta_1$, 3 of them are supersets of $\Box\theta_1$ and 4 of them are incomparable.

When combining Bel_3 and Bel_5 using $prop_{21}$ we obtain a similar case for dynamic fusion: $m_{11}(\theta_1) = m_{12}(\theta_1) = m_{22}(\theta_1) = m_{DSmH}(\theta_1) = 1$ and $m_{21}(X) = 1/12$ for all $\emptyset \not\equiv X \in D^{\Theta}_{\mathcal{M}_2}$. m_{21} assigns 1/12 to every element of model $\mathcal{M}$ again: one of them is equal to θ_1 (to what DSmH assigns 1, 1 of them $\theta_1 \cap \theta_3$ is subset of θ_1, 4 of them $(\Box\theta_1, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3)$ are supersets of θ_1 and other 6 of them are incomparable.

A detail study of situations where it is not possible say whether minC combination produces more specified results and situations where DSmH rule produces more specified results is an open problem for future.

The principal difference between the minC combination and the hybrid DSm rule is the following: DSmH rule handles separately individual multiples of gbbm's $m_1(X)m_2(Y)$ and assign them to intersection (if non-empty) or to union (if non-empty) of focal elements X and Y. Whereas the minC combination groups together all the multiples, where $X \cap Y$ are mutually $\mathcal{M}$-equivalent and assigns the result to $X \cap Y$ (if non-empty) or proportionalizes it to focal elements derived from $\bigcup(X \cap Y)$. Hence multiples $m_i(\theta_1)m_j(\theta_2 \cap \theta_3), m_i(\theta_1)m_j(\theta_1 \cap \theta_2 \cap \theta_3), m_i(\theta_1 \cap \theta_2)m_j(\theta_2 \cap \theta_3), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cap \theta_2 \cap \theta_3)$ and other $\mathcal{M}$-equivalent are reallocated all together in the minC combination. Similarly multiples $m_i(\theta_1)m_j(\theta_2), m_i(\theta_1)m_j(\theta_1 \cap \theta_2), m_i(\Box\theta_1)m_j(\theta_1 \cap \theta_2), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cup \theta_2), m_i(\theta_1 \cap \theta_2)m_j(\theta_1 \cup \theta_2 \cup \theta_3)$ and other $\mathcal{M}$-equivalent are reallocated also all together in the minC combination. This is also the reason of minC results in the special cases, where $X \cup Y \subset \bigcup(X \cap Y)$ and $m^1(Z) = 0$ for all $Z \in D^{\Theta}_{\mathcal{M}}$, as in the previous paragraph.

The other principal difference is necessity of n-ary version of the rule for DSmH. Whereas we can apply (n-1) times computation of binary m^0 and some proportionalization after, in the case of the binary minC combination.

4.7 Related works.

We have to remember again the comparison of classic minC with DSmH on Shafer's DSm model at first, see Chapter 10 in [20].

To have a solid theoretical background for comparison of DSm rules with the classic ones, a generalization of Dempster's rule, Yager's rule [23], and Dubois-Prade rule [13] has been presented in [6, 7], see also Chapter 3 in this volume, and the generalized minC combination in [8].

We cannot forget for new types of DSm rules, especially Proportional Conflict Redistribution Rules [21], which are "between" DSmC and DSmH rules on one side and minC approach on the other side. Comparison of these rules with the generalized minC approach is a very interesting task for forthcoming research.

We have to mention also works by Besnard [1] and his collaborators Jaouen [16] and Perin [18], who propose to replace the classical Boolean algebras with a distributive lattice, hoping it might solve Smets' bomb issue. Their distributed lattice generated on a frame of discernment is the free DSm model in fact, it also coincides with a lattice $\mathcal{L}(\Theta)$ in minC combination. Moreover these authors use a conflicting relation for a construction of their evidential structure. There is no concept of negation similarly to DSm approach. Comparison of the conflicting relation with DSm constraints and of the evidential structures with hybrid DSm models is still an open problem for future research to formulate a relation between the two independently developed approaches to belief combination on distributive lattices. Nevertheless neither this issue really new as it has been started and unfortunately unfinished by Philippe Smets in 2004/2005.

4.8 Conclusion

The minC combination rule generalized to DSm hyper-power sets and general hybrid DSm models has been presented both for static and dynamic fusion of generalized belief functions.

Examples of the generalized minC combination on several hybrid DSm models have been presented and discussed. After it, a comparison of the generalized minC combination and the hybrid DSm rule has been performed and several open problems for a future research has been defined.

A step for inclusion of minC combination into family of DSm combination rules has been done.

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Chapter 5

Pre-Boolean algebra, ordered DSmT and DSm continuous models

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Abstract: *When implementing the DSmT, a difficulty may arise from the possible huge dimension of hyper-power sets, which are indeed free structures. However, it is possible to reduce the dimension of these structures by involving logical constraints. In this chapter, the logical constraints will be related to a predefined order over the logical propositions. The use of such orders and their resulting logical constraints will ensure a great reduction of the model complexity. Such results will be applied to the definition of continuous DSm models. In particular, a simplified description of the continuous impreciseness is considered, based on impreciseness intervals of the sensors. From this viewpoint, it is possible to manage the contradictions between continuous sensors in a DSmT manner, while the complexity of the model stays handleable.*

5.1 Introduction

Recent advances [6] in the Dezert Smarandache Theory have shown that this theory was able to handle the contradiction between propositions in a quite flexible way. This new theory has been already applied in different domains; *e.g.*:

- Data association in target tracking [9] ,
- Environmental prediction [2] .

Although free DSm models are defined over hyper-power sets, which sizes evolve exponentially with the number of *atomic* propositions, it appears that the manipulation of the fusion rule is still manageable for practical problems reasonably well shaped. Moreover, the hybrid DSm models are of lesser complexity.

If DSmT works well for discrete spaces, the manipulation of continuous DSm models is still an unknown. Nevertheless, the management of continuous data is an issue of main interest. It is necessary for implementing a true fusion engine for localization informations; and associated with a principle of conditioning, it will be a main ingredient for implementing filters for the localization. But a question first arises: *what could be an hyper-power set for a continuous DSm model?* Such first issue does not arises so dramatically in Dempster Shafer Theory or for Transfer Belief Models [7]. In DST, a continuous proposition could just be a measurable subset. On the other hand, a free DSm model, defined over an hyper-power set, will imply that any pair of propositions will have a non empty intersection. This is disappointing, since the notion of *point* (a minimal non empty proposition) does not exist anymore in an hyper-power set.

But even if it is possible to define a continuous propositional model in DST/TBM, the manipulation of continuous basic belief assignment is still an issue [4, 8]. In [4] , Ristic and Smets proposed a restriction of the bba to intervals of $\mathbb{R}$. It was then possible to derive a mathematical relation between a continuous bba density and its Bel function.

In this chapter, the construction of continuous DSm models is proposed. This construction is based on a constrained model, where the logical constraints are implied by the definition of an order relation over the propositions.

A one-dimension DSm model will be implemented, where the definition of the basic belief assignment relies on a *generalized notion of intervals*. Although this construction has been fulfilled on a different ground, it shares some surprising similarities with Ristic and Smets viewpoint. As in [4], the bba will be seen as density defined over a 2-dimension measurable space. We will be able to derive the Belief function from the basic belief assignment, by applying an integral computation. At last, the conjunctive fusion operator, $\oplus$, is derived by a rather simple integral computation.

Section 5.2 makes a quick introduction of the Dezert Smarandache Theory. Section 5.3 is about ordered DSm models. In section 5.4, a continuous DSm model is defined. This method is restricted to only one dimension. The related computation methods are detailed. In section 5.5, our algorithmic implementation is described and an example of computation is given. The paper is then concluded.

5.2 A short introduction to the DSmT

The theory and its meaning are widely explained in [6]. However, we will particularly focus on the notion of hyper-power sets, since this notion is fundamental subsequently.

The *Dezert Smarandache Theory* belongs to the family of *Evidence Theories*. As the *Dempster Shafer Theory* [3] [5] or the *Transferable Belief Models* [7], the DSmT is a framework for fusing belief informations, originating from independent sensors. However, free DSm models are defined over Hyper-power sets, which are *fully open-world extensions* of sets. It is possible to restrict this full open-world hypothesis by adding propositional constraints, resulting in the definition of an *hybrid Dezert Smarandache model*.

The notion of hyper-power set is thus a fundamental ingredient of the DSmT. Hyper-power sets could be considered as a free pre-Boolean algebra. As these structures will be of main importance subsequently, the next sections are devoted to introduce them in details. As a prerequisite, the notion of Boolean algebra is quickly introduced now.

5.2.1 Boolean algebra

Definition. A Boolean algebra is a sextuple $(\Phi, \wedge, \vee, \neg, \bot, \top)$ such that:

- Φ is a set, called set of propositions,
- $\bot, \top$ are specific propositions of Φ, respectively called *false* and *true*,
- $\neg : \Phi \to \Phi$ is a unary operator,
- $\wedge : \Phi \times \Phi \to \Phi$ and $\vee : \Phi \times \Phi \to \Phi$ are binary operators,

and verifying the following properties:

A1. $\wedge$ and $\vee$ are commutative:
$$\forall \phi, \psi \in \Phi,\ \phi \wedge \psi = \psi \wedge \phi \text{ and } \phi \vee \psi = \psi \vee \phi\ ,$$

A2. $\wedge$ and $\vee$ are associative:
$$\forall \phi, \psi, \eta \in \Phi,\ (\phi \wedge \psi) \wedge \eta = \phi \wedge (\psi \wedge \eta) \text{ and } (\phi \vee \psi) \vee \eta = \phi \vee (\psi \vee \eta)\ ,$$

A3. $\top$ is neutral for $\wedge$ and $\bot$ is neutral for $\vee$:
$$\forall \phi \in \Phi,\ \phi \wedge \top = \phi \text{ and } \phi \vee \bot = \phi\ ,$$

A4. $\wedge$ and $\vee$ are distributive for each other:
$$\forall \phi, \psi, \eta \in \Phi,\ \phi \wedge (\psi \vee \eta) = (\phi \wedge \psi) \vee (\phi \wedge \eta) \text{ and } \phi \vee (\psi \wedge \eta) = (\phi \vee \psi) \wedge (\phi \vee \eta)\ ,$$

A5. $\neg$ defines the complement of any proposition:
$$\forall \phi \in \Phi,\ \phi \wedge \neg\phi = \bot \text{ and } \phi \vee \neg\phi = \top\ .$$

The Boolean algebra $(\Phi, \wedge, \vee, \neg, \bot, \top)$ will be also referred to as the *Boolean algebra* Φ, the structure being thus implied. An order relation $\subset$ is defined over Φ by:
$$\forall \phi, \psi \in \Phi\,, \quad \phi \subset \psi \overset{\Delta}{\Longleftrightarrow} \phi \wedge \psi = \phi\ .$$

Fundamental examples. The following examples are two main conceptions of Boolean algebra.

Example 1. *Let Ω be a set and $\mathcal{P}(\Omega)$ be the set of its subsets. For any $A \subset \Omega$, denote $\sim A = \Omega \setminus A$ its complement. Then $\big(\mathcal{P}(\Omega), \cap, \cup, \sim, \emptyset, \Omega\big)$ is a Boolean algebra.*

The proof is immediate by verifying the properties A1 to A5.

Example 2. *For any $i \in \{1, \ldots, n\}$, let $\theta_i = \{0,1\}^{i-1} \times \{0\} \times \{0,1\}^{n-i}$. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ and denote $\bot = \emptyset$, $\top = \{0,1\}^n$ and $\mathcal{B}(\Theta) = \mathcal{P}\big(\{0,1\}^n\big)$. Define the operators $\wedge$, $\vee$ and $\neg$ by $\phi \wedge \psi = \phi \cap \psi$, $\phi \vee \psi = \phi \cup \psi$ and $\neg\phi = \top \setminus \phi$ for any $\phi, \psi \in \mathcal{B}(\Theta)$. Then $\big(\mathcal{B}(\Theta), \wedge, \vee, \neg, \bot, \top\big)$ is a Boolean algebra.*

The second example seems just like a rewriting of the first one, but it is of the most importance. It is called the *free Boolean algebra generated by the set of atomic propositions* Θ. Figure 5.1 shows the structure of such algebra, when $n = 2$. The free Boolean algebra $\mathcal{B}(\Theta)$ is deeply related to the classical propositional logic: it gives the (logical) equivalence classes of the propositions generated from the atomic propositions of Θ. Although we give here an explicit definition of $\mathcal{B}(\Theta)$ by means of its binary coding $\mathcal{P}\big(\{0,1\}^n\big)$, the truly rigorous definition of $\mathcal{B}(\Theta)$ is made by means of the logical equivalence (which is out of the scope of this presentation). Thus, the binary coding of the atomic propositions $\theta_i \in \Theta$ is only implied.

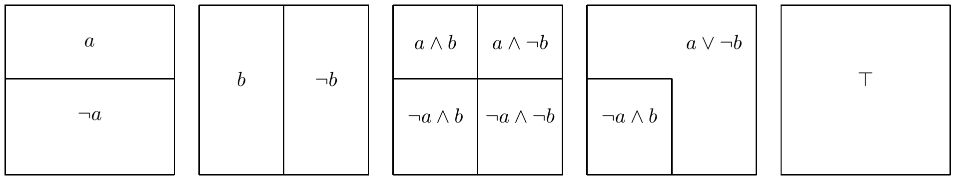

Figure 5.1: Boolean algebra $\mathcal{B}\big(\{a, b\}\big)$; (partial)

Fundamental proposition.

Proposition 3. *Any Boolean algebra is isomorph to a Boolean algebra derived from a set,* i.e. $\big(\mathcal{P}(\Omega), \cap, \cup, \sim, \emptyset, \Omega\big)$.

Proofs should be found in any good reference; see also [1].

5.2.2 Hyper-power sets

Definition of hyper-power set. Let's consider a finite set Θ of atomic propositions, and denote $\big(\mathcal{B}(\Theta), \wedge, \vee, \neg, \perp, \top\big)$ the free Boolean algebra generated by Θ. For any $\Sigma \subset \mathcal{P}(\Theta)$, define $\varphi(\Sigma)$, element of $\mathcal{B}(\Theta)$, by[1] $\varphi(\Sigma) = \bigvee_{\sigma\in\Sigma} \bigwedge_{\theta\in\sigma} \theta$. The set $< \Theta >= \big\{\varphi(\Sigma) \;/\; \Sigma \subset \mathcal{P}(\Theta)\big\}$ is called hyper-power set generated by Θ.

It is noticed that both $\perp = \varphi(\emptyset)$ and $\top = \varphi\big(\mathcal{P}(\Theta)\big)$ are elements of $< \Theta >$. Figure 5.2 shows the structure of the hyper-power set, when $n = 2$. Typically, it appears that the elements of the hyper-power set are built only from $\neg$-free components.

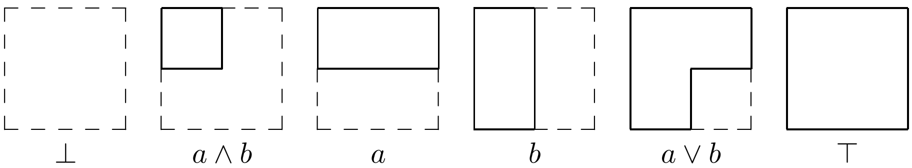

Figure 5.2: Hyper-power set $< a, b >= \{\perp, a \wedge b, a, b, a \vee b, \top\}$

***Example* 3.** *Hyper-power set generated by* $\Theta = \{a, b, c\}$.

$$
\begin{aligned}
< a, b, c >= \big\{ & \perp, a, b, c, a \wedge b \wedge c, a \wedge b, b \wedge c, c \wedge a, a \vee b \vee c, \top \\
& a \vee b, b \vee c, c \vee a, (a \wedge b) \vee c, (b \wedge c) \vee a, (c \wedge a) \vee b, \\
& (a \vee b) \wedge c, (b \vee c) \wedge a, (c \vee a) \wedge b, (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\big\}
\end{aligned}
$$

The following table associates some $\Sigma \subset \mathcal{P}(\Theta)$ *to their related hyper-power element* $\varphi(\Sigma)$.

[1] It is assumed $\bigvee_{\phi\in\emptyset} = \perp$ and $\bigwedge_{\phi\in\emptyset} = \top$.

This table is partial; there is indeed 256 possible choices for Σ. It appears that φ is not one-to-one:

Σ	$\varphi(\Sigma)$	*reduced form in* $<\Theta>$
$\emptyset$	$\bot$	$\bot$
$\{\emptyset\}$	$\top$	$\top$
$\big\{\{a\};\{b\};\{c\}\big\}$	$a\vee b\vee c$	$a\vee b\vee c$
$\big\{\{a,b\};\{b,c\};\{c,a\}\big\}$	$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$	$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$
$\big\{\{a,c\};\{b,c\};\{a,b,c\}\big\}$	$(a\wedge c)\vee(b\wedge c)\vee(a\wedge b\wedge c)$	$(a\vee b)\wedge c$
$\big\{\{a,c\};\{b,c\}\big\}$	$(a\wedge c)\vee(b\wedge c)$	$(a\vee b)\wedge c$

Remark. In the DSmT book 1 [6], the hyper-power sets have been defined by means of the Smarandache encoding. Our definition is quite related to this encoding. In fact this encoding is just implied in the definition of φ.

Hyper-power set as a free pre-Boolean algebra. It is easy to verify on example 3 that $<\Theta>$ is left unchanged by any application of the operators $\wedge$ and $\vee$. For example:

$$(a\wedge b)\wedge\big((b\wedge c)\vee a\big)=(a\wedge b\wedge b\wedge c)\vee(a\wedge b\wedge a)=a\wedge b\;.$$

This result is formalized by the following proposition.

Proposition 4. *Let $\phi,\psi\in<\Theta>$. Then $\phi\wedge\psi\in<\Theta>$ and $\phi\vee\psi\in<\Theta>$.*

Proof. Let $\phi,\psi\in<\Theta>$.

There are $\Sigma\subset\mathcal{P}(\Theta)$ and $\Gamma\subset\mathcal{P}(\Theta)$ such that $\phi=\varphi(\Sigma)$ and $\psi=\varphi(\Gamma)$.

By applying the definition of φ, it comes immediately:

$$\varphi(\Sigma)\vee\varphi(\Gamma)=\bigvee_{\sigma\in\Sigma\cup\Gamma}\bigwedge_{\theta\in\sigma}\theta\;.$$

It is also deduced:

$$\varphi(\Sigma)\wedge\varphi(\Gamma)=\left(\bigvee_{\sigma\in\Sigma}\bigwedge_{\theta\in\sigma}\theta\right)\wedge\left(\bigvee_{\gamma\in\Gamma}\bigwedge_{\theta\in\gamma}\theta\right)\;.$$

By applying the distributivity, it comes:

$$\varphi(\Sigma)\wedge\varphi(\Gamma)=\bigvee_{\sigma\in\Sigma}\bigvee_{\gamma\in\Gamma}\left(\left(\bigwedge_{\theta\in\sigma}\theta\right)\wedge\left(\bigwedge_{\theta\in\gamma}\theta\right)\right)=\bigvee_{(\sigma,\gamma)\in\Sigma\times\Gamma}\;\bigwedge_{\theta\in\sigma\cup\gamma}\theta\;.$$

Then $\varphi(\Sigma)\wedge\varphi(\Gamma)=\varphi(\Lambda)$, with $\Lambda=\big\{\sigma\cup\gamma\;/\;(\sigma,\gamma)\in\Sigma\times\Gamma\big\}$.

□□□

Corollary and definition. Proposition 4 implies that $\wedge$ and $\vee$ infer inner operations within $<\Theta>$. As a consequence, $\big(<\Theta>,\wedge,\vee,\bot,\top\big)$ is an algebraic structure by itself. Since it does not contains the negation $\neg$, this structure is called the *free pre-Boolean algebra generated by* Θ.

5.2.3 Pre-Boolean algebra

Generality. Typically, a free algebra is an algebra where the only constraints are the intrinsic constraints which characterize its fundamental structures. For example in a free Boolean algebra, the only constraints are A1 to A5, and there are no other constraints put on the propositions. But conversely, it is indeed possible to derive any algebra by constraining its free counterpart. This will be our approach for defining pre-Boolean algebra in general: a pre-Boolean algebra will be a *constrained* free pre-Boolean algebra. Constraining a structure is a quite intuitive notion. However, a precise mathematical definition needs the abstract notion of equivalence relations and classes. Let us start with the intuition by introducing an example.

***Example* 4.** *Pre-Boolean algebra generated by* $\Theta = \{a, b, c\}$ *and constrained by* $a \wedge b = a \wedge c$ *and* $a \wedge c = b \wedge c$.
For coherence with forthcoming notations, these constraints will be designated by using the set of propositional pairs $\Gamma = \big\{(a \wedge b, a \wedge c), (a \wedge c, b \wedge c)\big\}$.
The idea is to start from the free pre-Boolean algebra $< a, b, c >$, *propagate the constraints, and then reduce the propositions identified by the constraints.*

It is first deduced $a \wedge b = a \wedge c = b \wedge c = a \wedge b \wedge c$.
It follows $(a \wedge b) \vee c = c$, $(b \wedge c) \vee a = a$ *and* $(c \wedge a) \vee b = b$.
Also holds $(a \vee b) \wedge c = (b \vee c) \wedge a = (c \vee a) \wedge b = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = a \wedge b \wedge c$.
By discarding these cases from the free structure $< a, b, c >$, *it comes the following constrained pre-Boolean algebra:*

$$< a, b, c >_\Gamma = \big\{\bot, a \wedge b \wedge c, a, b, c, a \vee b, b \vee c, c \vee a, a \vee b \vee c, \top\big\}$$

Of course, it is necessary to show that there is actually no further reduction in $< a, b, c >_\Gamma$. *This is done by explicating a model; for example the structure of figure 5.3.*

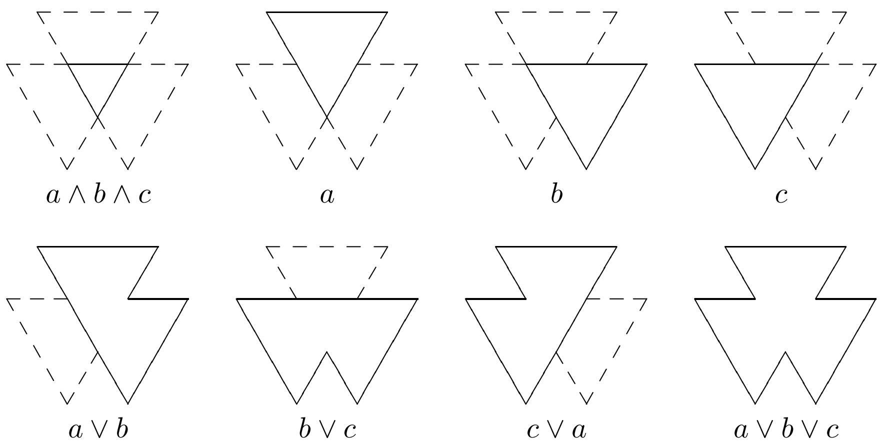

Figure 5.3: Pre-Boolean algebra $< a, b, c >_\Gamma$; ($\bot$ and $\top$ are omitted)

For the reader not familiar with the notion of equivalence classes, the following construction is just a mathematical formalization of the constraint propagation which has been described in example 4. Now, it is first introduced the notion of morphism between structures.

Magma. A $(\wedge, \vee, \perp, \top)$-*magma*, also called *magma* for short, is a quintuple $(\Phi, \wedge, \vee, \perp, \top)$ where Φ is a set of propositions, $\wedge$ and $\vee$ are binary operators on Φ, and $\perp$ and $\top$ are two elements of Φ.

The magma $(\Phi, \wedge, \vee, \perp, \top)$ may also be referred to as the magma Φ, the structure being thus implied. Notice that an hyper-power set is a magma.

Morphism. Let $(\Phi, \wedge, \vee, \perp, \top)$ and $(\Psi, \wedge, \vee, \perp, \top)$ be two magma. A morphism μ from the magma Φ to the magma Ψ is a mapping from Φ to Ψ such that:

- $\mu(\phi \wedge \psi) = \mu(\phi) \wedge \mu(\psi)$ and $\mu(\phi \vee \psi) = \mu(\phi) \vee \mu(\psi)$,
- $\mu(\perp) = \perp$ and $\mu(\top) = \top$.

A morphism is an isomorphism if it is a bijective mapping. In such case, the magma Φ and the magma Ψ are said to be isomorph, which means that they share the same structure.

The notions of $(\wedge, \vee)$*-magma and of* $(\wedge, \vee)$*-morphism are defined similarly by discarding* $\perp$ *and* $\top$.

Propagation relation. Let $< \Theta >$ be a free pre-Boolean algebra. Let $\Gamma \subset < \Theta > \times < \Theta >$ be a set of propositional pairs; for any pair $(\phi, \psi) \in \Gamma$ is defined the constraint $\phi = \psi$. The propagation relation associated to the constraints, and also denoted Γ, is defined recursively by:

- $\phi \Gamma \phi$, for any $\phi \in < \Theta >$,
- If $(\phi, \psi) \in \Gamma$, then $\phi \Gamma \psi$ and $\psi \Gamma \phi$,
- If $\phi \Gamma \psi$ and $\psi \Gamma \eta$, then $\phi \Gamma \eta$,
- If $\phi \Gamma \eta$ and $\psi \Gamma \zeta$, then $(\phi \wedge \psi) \Gamma (\eta \wedge \zeta)$ and $(\phi \vee \psi) \Gamma (\eta \vee \zeta)$

The relation Γ is thus obtained by propagating the constraint over $< \Theta >$. It is obviously reflexive, symmetric and transitive; it is an equivalence relation. An equivalence class for Γ contains propositions which are identical in regards to the constraints.

It is now time to define the pre-Boolean algebra.

Pre-Boolean algebra.

Proposition 5. *Let be given a free pre-Boolean algebra* $< \Theta >$ *and a set of propositional pairs* $\Gamma \subset < \Theta > \times < \Theta >$. *Then, there is a magma* $< \Theta >_\Gamma$ *and a morphism* $\mu : < \Theta > \rightarrow < \Theta >_\Gamma$ *such that:*

$$\begin{cases} \mu\big(< \Theta >\big) = < \Theta >_\Gamma , \\ \forall \phi, \psi \in < \Theta >,\ \mu(\phi) = \mu(\psi) \iff \phi \Gamma \psi . \end{cases}$$

The magma $< \Theta >_\Gamma$ *is called the pre-Boolean algebra generated by* Θ *and constrained by the constraints* $\phi = \psi$ *where* $(\phi, \psi) \in \Gamma$.

Proof. For any $\phi \in < \Theta >$, define $\phi_\Gamma = \big\{ \psi \in < \Theta > \ / \ \psi \Gamma \phi \big\}$; this set is called the class of ϕ for Γ .

It is a well known fact, and the proof is immediate, that $\phi_\Gamma = \psi_\Gamma$ or $\phi_\Gamma \cap \psi_\Gamma = \emptyset$ for any

$\phi, \psi \in <\Theta>$; in particular, $\phi_\Gamma = \psi_\Gamma \iff \phi\Gamma\psi$.
Now, assume $\eta_\Gamma = \phi_\Gamma$ and $\zeta_\Gamma = \psi_\Gamma$, that is $\eta\Gamma\phi$ and $\zeta\Gamma\psi$.
It comes $(\eta\wedge\zeta)\Gamma(\phi\wedge\psi)$ and $(\eta\vee\zeta)\Gamma(\phi\vee\psi)$.
As a consequence, $(\eta\wedge\zeta)_\Gamma = (\phi\wedge\psi)_\Gamma$ and $(\eta\vee\zeta)_\Gamma = (\phi\vee\psi)_\Gamma$.
At last:

$$\Big(\eta_\Gamma = \phi_\Gamma \text{ and } \zeta_\Gamma = \psi_\Gamma\Big) \Rightarrow \Big((\eta\wedge\zeta)_\Gamma = (\phi\wedge\psi)_\Gamma \text{ and } (\eta\vee\zeta)_\Gamma = (\phi\vee\psi)_\Gamma\Big)$$

The proof is then concluded easily, by setting:

$$\begin{cases} <\Theta>_\Gamma = \left\{\phi_\Gamma \,/\, \phi \in <\Theta>\right\}, \\ \forall\phi,\psi \in <\Theta>,\ \phi_\Gamma\wedge\psi_\Gamma = (\phi\wedge\psi)_\Gamma \text{ and } \phi_\Gamma\vee\psi_\Gamma = (\phi\vee\psi)_\Gamma, \\ \forall\phi \in <\Theta>,\ \mu(\phi) = \phi_\Gamma. \end{cases}$$

□□□

From now on, the element $\mu(\phi)$, where $\phi \in <\Theta>$, will be denoted ϕ as if ϕ were an element of $<\Theta>_\Gamma$. In particular, $\mu(\phi) = \mu(\psi)$ will imply $\phi = \psi$ in $<\Theta>_\Gamma$ (but not in $<\Theta>$).

Proposition 6. *Let be given a free pre-Boolean algebra $<\Theta>$ and a set of propositional pairs $\Gamma \subset <\Theta> \times <\Theta>$. Let $<\Theta>_\Gamma$ and $<\Theta>'_\Gamma$ be pre-Boolean algebras generated by Θ and constrained by the family Γ. Then $<\Theta>_\Gamma$ and $<\Theta>'_\Gamma$ are isomorph.*

Proof. Let $\mu :<\Theta>\rightarrow<\Theta>_\Gamma$ and $\mu' :<\Theta>\rightarrow<\Theta>'_\Gamma$ be as defined in proposition 5.
For any $\phi \in <\Theta>$, define $\nu\big(\mu(\phi)\big) = \mu'(\phi)$.
Then, $\nu\big(\mu(\phi)\big) = \nu\big(\mu(\psi)\big)$ implies $\mu'(\phi) = \mu'(\psi)$.
By definition of μ', it is derived $\phi\Gamma\psi$ and then $\mu(\phi) = \mu(\psi)$.
Thus, ν is one-to-one.
By definition, it is also implied that ν is onto.

□□□

This property thus says that there is a structural uniqueness of $<\Theta>_\Gamma$.

***Example* 5.** *Let us consider again the pre-Boolean algebra generated by $\Theta = \{a,b,c\}$ and constrained by $a\wedge b = a\wedge c$ and $a\wedge c = b\wedge c$. In this case, the mapping $\mu :<\Theta>\rightarrow<\Theta>_\Gamma$ is defined by:*

- $\mu\big(\{\bot\}\big) = \{\bot\}$, $\mu\Big(\big\{a,(b\wedge c)\vee a\big\}\Big) = \{a\}$, $\mu\Big(\big\{b,(c\wedge a)\vee b\big\}\Big) = \{b\}$,
  $\mu\Big(\big\{c,(a\wedge b)\vee c\big\}\Big) = \{c\}$, $\mu\big(\{a\vee b\vee c\}\big) = \{a\vee b\vee c\}$, $\mu\big(\{\top\}\big) = \{\top\}$,
- $\mu\big(\{a\vee b\}\big) = \{a\vee b\}$, $\mu\big(\{b\vee c\}\big) = \{b\vee c\}$, $\mu\big(\{c\vee a\}\big) = \{c\vee a\}$,
- $\mu\Big(\big\{a\wedge b\wedge c, a\wedge b, b\wedge c, c\wedge a, (a\vee b)\wedge c, (b\vee c)\wedge a, (c\vee a)\wedge b,$
  $(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)\big\}\Big) = \{a\wedge b\wedge c\}$.

Between sets and hyper-power sets.

Proposition 7. *The Boolean algebra* $\big(\mathcal{P}(\Theta),\cap,\cup,\sim,\emptyset,\Theta\big)$*, considered as a* $(\wedge,\vee,\bot,\top)$*-magma, is isomorph to the pre-Boolean algebra* $<\Theta>_\Gamma$*, where* Γ *is defined by:*

$$\Gamma=\big\{(\theta\wedge\vartheta,\bot)\;/\;\theta,\vartheta\in\Theta\ \textit{and}\ \theta\neq\vartheta\big\}\cup\left\{\left(\bigvee_{\theta\in\Theta}\theta,\top\right)\right\}\,.$$

Proof. Recall the notation $\varphi(\Sigma)=\bigvee_{\sigma\in\Sigma}\bigwedge_{\theta\in\sigma}\theta$ for any $\Sigma\subset\mathcal{P}(\Theta)$.

Define $\mu:<\Theta>\rightarrow\mathcal{P}(\Theta)$ by setting[2] $\mu\big(\varphi(\Sigma)\big)=\bigcup_{\sigma\in\Sigma}\bigcap_{\theta\in\sigma}\{\theta\}$ for any $\Sigma\subset\mathcal{P}(\Theta)$.

It is immediate that μ is a morphism.

Now, by definition of Γ, $\mu\big(\varphi(\Sigma)\big)=\mu\big(\varphi(\Lambda)\big)$ is equivalent to $\varphi(\Sigma)\Gamma\varphi(\Lambda)$.

The proof is then concluded by proposition 6.

□□□

Thus, sets, considered as Boolean algebra, and hyper-power sets are both extremal cases of the notion of pre-Boolean algebra. But while hyper-power sets extend the structure of sets, hyper-power sets are more complex in structure and size than sets. A practical use of hyper-power sets becomes quickly impossible. Pre-Boolean algebra however allows intermediate structures between sets and hyper-power sets.

A specific kind of pre-Boolean algebra will be particularly interesting when defining the DSmT. Such pre-Boolean algebra will forbid any interaction between the trivial propositions $\bot,\top$ and the other propositions. These algebra, called insulated pre-Boolean algebra, are characterized now.

Insulated pre-Boolean algebra. A pre-Boolean algebra $<\Theta>_\Gamma$ verifies the *insulation* property if $\Gamma\subset\big(<\Theta>\setminus\{\bot,\top\}\big)\times\big(<\Theta>\setminus\{\bot,\top\}\big)$.

Proposition 8. *Let* $<\Theta>_\Gamma$ *a pre-Boolean algebra verifying the insulation property. Then holds for any* $\phi,\psi\in<\Theta>_\Gamma$ *:*

$$\left\{\begin{array}{l}\phi\wedge\psi=\bot\Rightarrow(\phi=\bot\ \textit{or}\ \psi=\bot)\,,\\ \phi\vee\psi=\top\Rightarrow(\phi=\top\ \textit{or}\ \psi=\top)\,.\end{array}\right.$$

In other words, all propositions are independent *with each other in a pre-Boolean algebra with insulation property.*

The proof is immediate, since it is impossible to obtain $\phi\wedge\psi\Gamma\bot$ or $\phi\vee\psi\Gamma\top$ without involving $\bot$ or $\top$ in the constraints of Γ. Examples 3 and example 4 verify the insulation property. On the contrary, a non empty set does not.

Corollary and definition. Let $<\Theta>_\Gamma$ be a pre-Boolean algebra, verifying the insulation property. Define $\ll\Theta\gg_\Gamma=<\Theta>_\Gamma\setminus\{\bot,\top\}$. The operators $\wedge$ and $\vee$ restrict to $\ll\Theta\gg_\Gamma$, and $\big(\ll\Theta\gg_\Gamma,\wedge,\vee\big)$ is an algebraic structure by itself, called *insulated* pre-Boolean algebra. This structure is also referred to as the insulated pre-Boolean algebra $\ll\Theta\gg_\Gamma$.

[2] It is defined $\bigcap_{\theta\in\emptyset}\theta=\Theta$.

Proposition 9. *Let $< \Theta >_\Gamma$ and $< \Theta >'_\Gamma$ be pre-Boolean algebras with insulation properties. Assume that the insulated pre-Boolean algebra $\ll \Theta \gg_\Gamma$ and $\ll \Theta \gg'_\Gamma$ are $(\wedge, \vee)$-isomorph. Then $< \Theta >_\Gamma$ and $< \Theta >'_\Gamma$ are isomorph.*

Deduced from the insulation property.

All ingredients are now gathered for the definition of Dezert Smarandache models.

5.2.4 The *free* Dezert Smarandache Theory

Dezert Smarandache Model. Assume that Θ is a finite set. A Dezert Smarandache model (DSmm) is a pair (Θ, m), where Θ is a set of propositions and the *basic belief assignment* m is a non negatively valued function defined over $< \Theta >$ such that:

$$\sum_{\phi \in <\Theta>} m(\phi) = \sum_{\phi \in \ll\Theta\gg} m(\phi) = 1 \ .$$

The property $\sum_{\phi \in \ll\Theta\gg} m(\phi) = 1$ implies that the propositions of Θ are exhaustive.

Belief Function. Assume that Θ is a finite set. The belief function Bel related to a bba m is defined by:

$$\forall \phi \in < \Theta >, \ \mathrm{Bel}(\phi) = \sum_{\psi \in <\Theta>: \psi \subset \phi} m(\psi) \ . \tag{5.1}$$

The equation (6.1) is invertible:

$$\forall \phi \in < \Theta >, \ m(\phi) = \mathrm{Bel}(\phi) - \sum_{\psi \in <\Theta>: \psi \subsetneq \phi} m(\psi) \ .$$

Fusion rule. Assume that Θ is a finite set. For a given universe Θ, and two basic belief assignments m_1 and m_2, associated to independent sensors, the fused basic belief assignment is $m_1 \oplus m_2$, defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1, \psi_2 \in <\Theta>: \psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2) \ . \tag{5.2}$$

Remarks. It appears obviously that the previous definitions could be equivalently restricted to $\ll \Theta \gg$, owing to the insulation properties.

From the insulation property $(\phi \neq \perp \text{ and } \psi \neq \perp) \Rightarrow (\phi \wedge \psi) \neq \perp$ and the definition of the fusion rule, it appears also that these definitions could be generalized to any algebra $< \Theta >_\Gamma$ with the insulation property.

5.2.5 Extensions to any insulated pre-Boolean algebra

Let $\ll \Theta \gg_\Gamma$ be an insulated pre-Boolean algebra. The definition of bba m, belief Bel and fusion $\oplus$ is thus kept unchanged.

- A *basic belief assignment* m is a non negatively valued function defined over $\ll \Theta \gg_\Gamma$ such that:

$$\sum_{\phi \in \ll\Theta\gg_\Gamma} m(\phi) = 1 \ .$$

- The belief function Bel related to a bba m is defined by:

$$\forall\phi \in \ll \Theta \gg_\Gamma,\ \mathrm{Bel}(\phi) = \sum_{\psi \in \ll \Theta \gg_\Gamma : \psi \subset \phi} m(\psi)\ .$$

- Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1, \psi_2 \in \ll \Theta \gg_\Gamma : \psi_1 \wedge \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2)\ .$$

These extended definitions will be applied subsequently.

5.3 Ordered DSm model

From now on, we are working only with insulated pre-Boolean structures.

In order to reduce the complexity of the free DSm model, it is necessary to introduce logical constraints which will lower the size of the pre-Boolean algebra. Such constraints may appear clearly in the hypotheses of the problem. In this case, constraints come naturally and approximations may not be required. However, when the model is too complex and there are no explicit constraints for reducing this complexity, it is necessary to approximate the model by introducing some new constraints. Two rules should be applied then:

- Only weaken informations[3]; do not produce information from nothing,
- minimize the information weakening.

First point guarantees that the approximation does not introduce false information. But some significant informations (*e.g.* contradictions) are possibly missed. This drawback should be avoided by second point.

In order to build a good approximation policy, some external knowledge, like distance or order relation among the propositions could be used. Behind these relations will be assumed some kind of distance between the informations: *more are the informations distant, more are their conjunctive combination valuable.*

5.3.1 Ordered atomic propositions

Let $(\Theta, \leq)$ be an ordered set of atomic propositions. This order relation is assumed to describe the relative distance between the information. For example, the relation $\phi \leq \psi \leq \eta$ implies that ϕ and ψ are closer informations than ϕ and η. Thus, the information contained in $\phi \wedge \eta$ is stronger than the information contained in $\phi \wedge \psi$. Of course, this comparison does not matter when all the information is kept, but when approximations are necessary, it will be useful to be able to choose the best information.

[3]Typically, a constraint like $\phi \wedge \psi \wedge \eta = \phi \wedge \psi$ will weaken the information, by erasing η from $\phi \wedge \psi \wedge \eta$.

Sketchy example. Assume that 3 independent sensors are giving 3 measures about a continuous parameter, that is x, y and z. The parameters x, y, z are assumed to be real values, not of the *set* $\mathbb{R}$ but of its pre-Boolean extension (theoretical issues will be clarified later[4]). The fused information could be formalized by the proposition $x \wedge y \wedge z$ (in a DSmT viewpoint). What happen if we want to reduce the information by removing a proposition. Do we keep $x \wedge y$, $y \wedge z$ or $x \wedge z$? This is of course an information weakening. But it is possible that one information is better than an other. At this stage, the order between the values x, y, z will be involved. Assume for example that $x \leq y < z$. It is clear that the proposition $x \wedge z$ indicates a greater contradiction than $x \wedge y$ or $y \wedge z$. Thus, the proposition $x \wedge z$ is the one which should be kept! The discarding constraint $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$ is implied then.

5.3.2 Associated pre-Boolean algebra and complexity.

In regard to the previous example, the insulated pre-Boolean algebra associated to the ordered propositions $(\Theta, \leq)$ is $\ll \Theta \gg_\Gamma$, where Γ is defined by:

$$\Gamma = \big\{(\phi \wedge \psi \wedge \eta, \phi \wedge \eta) / \phi, \psi, \eta \in \Theta \text{ and } \phi \leq \psi \leq \eta\big\} .$$

The following property give an approximative bound of the size of $\ll \Theta \gg_\Gamma$ in the case of a total order.

Proposition 10. *Assume that* $(\Theta, \leq)$ *is totally ordered. Then,* $\ll \Theta \gg_\Gamma$ *is a substructure of the set* Θ^2.

proof. Since the order is total, first notice that the added constraints are:

$$\forall \phi, \psi, \eta \in \Theta\,,\ \phi \wedge \psi \wedge \eta = \min\{\phi, \psi, \eta\} \wedge \max\{\phi, \psi, \eta\} .$$

Now, for any $\phi \in \Theta$, define $\breve{\phi}$ by[5]:

$$\breve{\phi} \triangleq \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \phi \leq \varphi_2\big\}$$

It is noteworthy that:

$$\breve{\phi} \cap \breve{\psi} = \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \min\{\phi, \psi\} \text{ and } \max\{\phi, \psi\} \leq \varphi_2\big\}$$

and

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \big\{(\varphi_1, \varphi_2) \in \Theta^2 \,/\, \varphi_1 \leq \min\{\phi, \psi, \eta\} \text{ and } \max\{\phi, \psi, \eta\} \leq \varphi_2\big\} .$$

By defining $m = \min\{\phi, \psi, \eta\}$ and $M = \max\{\phi, \psi, \eta\}$, it is deduced:

$$\breve{\phi} \cap \breve{\psi} \cap \breve{\eta} = \breve{m} \cap \breve{M} . \tag{5.3}$$

Figure 5.4 illustrates the construction of $\breve{\phi}$, $\breve{\phi} \cap \breve{\psi}$ and property (5.3).

---

[4] In particular, as we are working in a pre-Boolean algebra, $x \wedge y$ makes sense and it is possible that $x \wedge y \neq \bot$ even when $x \neq y$.

[5] $where the symbol \triangleq$ means *equals by definition.*

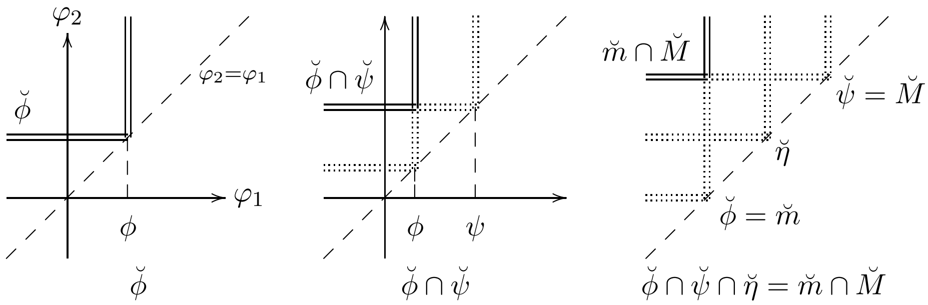

Figure 5.4: Construction of $\breve\phi$

Let $\mathcal{A} \subset \mathcal{P}(\Theta^2)$ be generated by $\breve\phi|_{\phi\in\Theta}$ with $\cap$ and $\cup$, *ie.*:

$$\mathcal{A} = \bigcup_{n\geq 0}\left\{ \bigcup_{k=1}^{n} (\breve\phi_k \cap \breve\psi_k) \Big/ \forall k\,,\ \breve\phi_k, \breve\psi_k \in \Theta \right\}\ .$$

A consequence of (5.3) is that $\mathcal{A}$ is an insulated pre-Boolean algebra which satisfies the constraints of Γ. Then, the mapping:

$$\smile\ : \begin{cases} \ll \Theta \gg_\Gamma \ \longrightarrow\ \mathcal{A} \\ \displaystyle\bigvee_{k=1}^{n}\bigwedge_{l=1}^{n_k} \phi_{k,l} \longmapsto \bigcup_{k=1}^{n}\bigcap_{l=1}^{n_k} \breve\phi_{k,l} \end{cases} ,\quad \text{where } \phi_{k,l}\in\Theta$$

is an onto morphism of pre-Boolean algebra.
Now, let us prove that $\smile$ is a one-to-one morphism.

Lemma 11. *Assume:*

$$\bigcup_{k=1}^{n} (\breve\phi_k^1 \cap \breve\phi_k^2) \subset \bigcup_{l=1}^{m} (\breve\psi_l^1 \cap \breve\psi_l^2) \quad , \quad \textit{where } \phi_k^j\,,\ \psi_l^j \in \Theta\ .$$

Then:

$$\forall k\,,\ \exists l\,,\ \min\{\phi_k^1, \phi_k^2\} \leq \min\{\psi_l^1, \psi_l^2\} \ \textit{and}\ \max\{\phi_k^1, \phi_k^2\} \geq \max\{\psi_l^1, \psi_l^2\}$$

and

$$\forall k\,,\ \exists l\,,\ \breve\phi_k^1 \cap \breve\phi_k^2 \subset \breve\psi_l^1 \cap \breve\psi_l^2\ .$$

Proof of lemma. Let $k \in [\![1, n]\!]$.
Define $m = \min\{\phi_k^1, \phi_k^2\}$ and $M = \max\{\phi_k^1, \phi_k^2\}$.
Then holds $(m, M) \in \breve\phi_k^1 \cap \breve\phi_k^2$, implying $(m, M) \in \bigcup_{l=1}^{m} (\breve\psi_l^1 \cap \breve\psi_l^2)$.
Let l be such that $(m, M) \in \breve\psi_l^1 \cap \breve\psi_l^2$.
Then $m \leq \min\{\psi_l^1, \psi_l^2\}$ and $M \geq \max\{\psi_l^1, \psi_l^2\}$.
At last, $\breve\phi_k^1 \cap \breve\phi_k^2 \subset \breve\psi_l^1 \cap \breve\psi_l^2$.

□□

From inequalities $\min\{\phi_k^1,\phi_k^2\}\leq\min\{\psi_l^1,\psi_l^2\}$ and $\max\{\phi_k^1,\phi_k^2\}\geq\max\{\psi_l^1,\psi_l^2\}$ is also deduced $(\phi_k^1\wedge\phi_k^2)\wedge(\psi_l^1\wedge\psi_l^2)=\phi_k^1\wedge\phi_k^2$ (definition of Γ).
This property just means $\phi_k^1\wedge\phi_k^2\subset\psi_l^1\wedge\psi_l^2$. It is lastly deduced:

Lemma 12. *Assume:*

$$\bigcup_{k=1}^{n}(\breve{\phi}_k^1\cap\breve{\phi}_k^2)\subset\bigcup_{l=1}^{m}(\breve{\psi}_l^1\cap\breve{\psi}_l^2)\quad,\quad\text{where }\phi_k^j\,,\ \psi_l^j\in\Theta\,.$$

Then:

$$\bigvee_{k=1}^{n}(\phi_k^1\wedge\phi_k^2)\subset\bigvee_{l=1}^{m}(\psi_l^1\wedge\psi_l^2)\,.$$

From this lemma, it is deduced that $\smile$ is one to one.
At last $\smile$ is an isomorphism of pre-Boolean algebra, and $\ll\Theta\gg_\Gamma$ is a substructure of Θ^2.

□□□

5.3.3 General properties of the model

In the next section, the previous construction will be extended to the continuous case, *ie.* $(\mathbb{R},\leq)$. However, a strict logical manipulation of the propositions is not sufficient and instead a measurable generalization of the model will be used. It has been seen that a proposition of $\ll\Theta\gg_\Gamma$ could be described as a subset of Θ^2. In this subsection, the proposition model will be characterized precisely. This characterization will be used and extended in the next section to the continuous case.

Proposition 13. *Let* $\phi\in\ll\Theta\gg_\Gamma$.
Then $\smile(\phi)\subset\mathcal{T}$, *where* $\mathcal{T}=\left\{(\phi,\psi)\in\Theta^2/\phi\leq\psi\right\}$.

Proof. Obvious, since $\forall\phi\in\Theta$, $\breve{\phi}\subset\mathcal{T}$.

□□□

Definition 14. *A subset* $\theta\subset\Theta^2$ *is* increasing *if and only if:*

$$\forall\,(\phi,\psi)\in\theta\,,\ \forall\eta\leq\phi\,,\ \forall\zeta\geq\psi\,,\ (\eta,\zeta)\in\theta\,.$$

Let $\mathcal{U}=\left\{\theta\subset\mathcal{T}/\theta\text{ is increasing and }\theta\neq\emptyset\right\}$ be the set of increasing non-empty subsets of $\mathcal{T}$. Notice that the intersection or the union of increasing non-empty subsets are increasing non-empty subsets, so that $(\mathcal{U},\cap,\cup)$ is an insulated pre-Boolean algebra.

Proposition 15. *For any choice of* Θ, $\left\{\smile(\phi)/\phi\in\ll\Theta\gg_\Gamma\right\}\subset\mathcal{U}$.
When Θ *is finite,* $\mathcal{U}=\left\{\smile(\phi)/\phi\in\ll\Theta\gg_\Gamma\right\}$.

Proof of $\supset$. Obvious, since $\breve{\phi}$ is increasing for any $\phi\in\Theta$.

Proof of $\subset$. Let $\theta\in\mathcal{U}$ and let $(a,b)\in\theta$.
Since $\breve{a}\cap\breve{b}=\left\{(\alpha,\beta)\in\Theta^2/\alpha\leq a\text{ and }\beta\geq b\right\}$ and θ is increasing, it follows $\breve{a}\cap\breve{b}\subset\theta$.
At last, $\theta=\bigcup_{(a,b)\in\theta}\breve{a}\cap\breve{b}=\smile\left(\bigvee_{(a,b)\in\theta}a\wedge b\right)$.
Notice that $\bigvee_{(a,b)\in\theta}a\wedge b$ is actually defined, since θ is finite when Θ is finite.

□□□

Figure 5.5 gives an example of increasing subsets, element of $\mathcal{U}$.

When infinite $\vee$-ing are allowed, notice that $\mathcal{U}$ may be considered as a model for $\ll \Theta \gg_\Gamma$ even

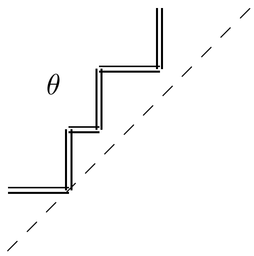

Figure 5.5: Example of increasing subset $\theta \in \mathcal{U}$

if Θ is infinite. In the next section, the *continuous* pre-Boolean algebra related to $(\mathbb{R}, \leq)$ will be modeled by the *measurable increasing subsets* of $\left\{(x,y) \in \mathbb{R}^2 / x \leq y\right\}$.

5.4 Continuous DSm model

In this section, the case $\Theta = \mathbb{R}$ is considered.

Typically, in a continuous model, it will be necessary to manipulate any measurable proposition, and for example intervals. It comes out that most intervals could not be obtained by a finite logical combination of the atomic propositions, but rather by infinite combinations. For example, considering the set formalism, it is obtained $[a,b] = \bigcup_{x\in[a,b]}\{x\}$, which suggests the definition of the infinite disjunction "$\bigvee_{x\in[a,b]} x$". It is known that infinite disjunctions are difficult to handle in a logic. It is better to manipulate the models directly. The pre-Boolean algebra to be constructed should verify the property $x \leq y \leq z \Rightarrow x \wedge y \wedge z = x \wedge z$. As discussed previously and since infinite disjunctions are allowed, a model for such algebra are the measurable increasing subsets.

5.4.1 Measurable increasing subsets

A measurable subset $A \subset \mathbb{R}^2$ is a measurable increasing subset if:

$$\begin{cases} \forall\,(x,y) \in A\,,\ x \leq y\,, \\ \forall\,(x,y) \in A\,,\ \forall a \leq x\,,\ \forall b \geq y\,,\ (a,b) \in A\,. \end{cases}$$

The set of measurable increasing subsets is denoted $\mathcal{U}$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a non decreasing measurable mapping such that $f(x) \geq x$ for any $x \in \mathbb{R}$. The set $\left\{(x,y) \in \mathbb{R}^2 / f(x) \leq y\right\}$ is a measurable increasing subset.

"Points". For any $x \in \mathbb{R}$, the measurable increasing subset $\breve{x}$ is defined by:

$$\breve{x} = \left\{(a,b) \in \mathbb{R}^2 \ / \ a \leq x \leq b\right\}.$$

The set $\breve{x}$ is of course a model for the point $x \in \mathbb{R}$ within the pre-Boolean algebra (refer to section 5.3).

Generalized intervals. A particular class of increasing subsets, the generalized intervals, will be useful in the sequel.

For any $x \in \mathbb{R}$, the measurable sets $\grave{x}$ and $\acute{x}$ are defined by:

$$\begin{cases} \grave{x} = \left\{(a,b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } x \leq b\right\} , \\ \acute{x} = \left\{(a,b) \in \mathbb{R}^2 \;/\; a \leq b \text{ and } a \leq x\right\} . \end{cases}$$

The following properties are derived:

$$\breve{x} = \grave{x} \cap \acute{x} \ , \ \grave{x} = \bigcup_{z \in [x,+\infty[} \breve{z} \quad \text{and} \quad \acute{x} = \bigcup_{z \in]-\infty,x]} \breve{z}$$

Moreover, for any x, y such that $x \leq y$, it comes:

$$\grave{x} \cap \acute{y} = \bigcup_{z \in [x,y]} \breve{z} \ .$$

As a conclusion, the set $\grave{x}$, $\acute{x}$ and $\grave{x} \cap \acute{y}$ (with $x \leq y$) are the respective models for the intervals $[x,+\infty[$, $]-\infty,x]$ and $[x,y]$ within the pre-Boolean algebra. Naturally, the quotation marks $\grave{}$ (opening) and $\acute{}$ (closing) are used respectively for opening and closing the intervals. Figure 5.6 illustrates various cases of interval models.

At last, the set $\grave{x} \cap \acute{y}$, where $x, y \in \mathbb{R}$ are not constrained, constitutes a generalized definition of

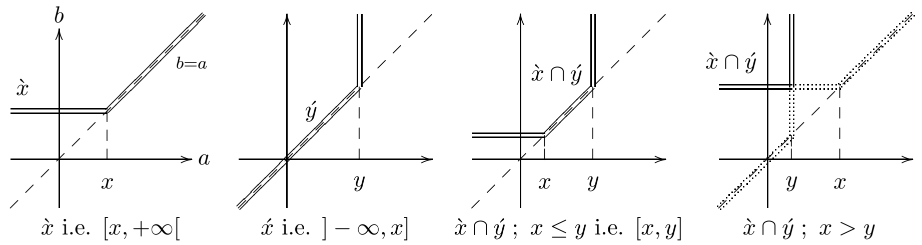

Figure 5.6: Interval models

the notion of interval. In the case $x \leq y$, it works like "classical" interval, but in the case $x > y$, it is obtained a new class of intervals with negative width (last case in figure 5.6). Whatever, $\grave{x} \cap \acute{y}$ comes with a non empty inner, and may have a non zero measure.

The width $\delta = \frac{y-x}{2}$ of the interval $\grave{x} \cap \acute{y}$ could be considered as a measure of contradiction associated with this proposition, while its center $\mu = \frac{x+y}{2}$ should be considered as its median value. The interpretation of the measure of contradiction is left to the human. Typically, a possible interpretation could be:

- $\delta < 0$ means contradictory informations,
- $\delta = 0$ means exact informations,
- $\delta > 0$ means imprecise informations.

It is also noteworthy that the set of generalized intervals

$$\mathcal{I} = \{\grave{x} \cap \acute{y} / x, y \in \mathbb{R}\}$$

is left unchanged by the operator $\cap$, as seen in the following proposition 16:

Proposition 16 (Stability). Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
Define $x = \max\{x_1, x_2\}$ and $y = \min\{y_1, y_2\}$.
Then $(\grave{x}_1 \cap \acute{y}_1) \cap (\grave{x}_2 \cap \acute{y}_2) = \grave{x} \cap \acute{y}$.

Proof is obvious.

This last property make possible the definition of basic belief assignment over generalized intervals only. This assumption is clearly necessary in order to reduce the complexity of the evidence modeling. Behind this assumption is the idea that a continuous measure is described by an imprecision/contradiction around the measured value. Such hypothesis has been made by Smets and Ristic [4]. From now on, all the defined bba will be zeroed outside $\mathcal{I}$. Now, since $\mathcal{I}$ is invariant by $\cap$, it is implied that all the bba which will be manipulated, from sensors or after fusion, will be zeroed outside $\mathcal{I}$. This makes the basic belief assignments equivalent to a density over the 2-dimension space $\mathbb{R}^2$.

5.4.2 Definition and manipulation of the belief

The definitions of bba, belief and fusion result directly from section 5.2, but of course the bba becomes density and the summations are replaced by integrations.

Basic Belief Assignment. As discussed previously, it is hypothesized that the measures are characterized by a precision interval around the measured values. In addition, there is an uncertainty about the measure which is translated into a basic belief assignment over the precision intervals.

According to these hypotheses, a bba will be a non negatively valued function m defined over $\mathcal{U}$, zeroed outside $\mathcal{I}$ (set of generalized intervals), and such that:

$$\int_{x,y\in\mathbb{R}} m(\grave{x} \cap \acute{y})\,dxdy = 1 .$$

Belief function. The function of belief, Bel, is defined for any measurable proposition $\phi \in \mathcal{U}$ by:

$$\mathrm{Bel}\,(\phi) = \int_{\grave{x}\cap\acute{y}\subset\phi} m(\grave{x} \cap \acute{y})\,dxdy .$$

In particular, for a generalized interval $\grave{x} \cap \acute{y}$:

$$\mathrm{Bel}\,(\grave{x} \cap \acute{y}) = \int_{u=x}^{+\infty} \int_{v=-\infty}^{y} m(\grave{u} \cap \acute{v})\,dudv .$$

Fusion rule. Being given two basic belief assignments m_1 and m_2, the fused basic belief assignment $m_1 \oplus m_2$ is defined by the curvilinear integral:

$$m_1 \oplus m_2(\grave{x} \cap \acute{y}) = \int_{\mathcal{C}=\{(\phi,\psi)/\phi\cap\psi=\grave{x}\cap\acute{y}\}} m_1(\phi)m_2(\psi)\, d\mathcal{C} \,.$$

Now, from hypothesis it is assumed that m_i is positive only for intervals of the form $\grave{x}_i \cap \acute{y}_i$. Proposition 16 implies:

$$\grave{x}_1 \cap \acute{y}_1 \cap \grave{x}_2 \cap \acute{y}_2 = \grave{x} \cap \acute{y} \text{ where } \begin{cases} x = \max\{x_1, x_2\} \,, \\ y = \min\{y_1, y_2\} \,. \end{cases}$$

It is then deduced:

$$\begin{aligned} m_1 \oplus m_2(\grave{x} \cap \acute{y}) &= \int_{x_2=-\infty}^{x} \int_{y_2=y}^{+\infty} m_1(\grave{x} \cap \acute{y}) m_2(\grave{x}_2 \cap \acute{y}_2) dx_2 dy_2 \\ &+ \int_{x_1=-\infty}^{x} \int_{y_1=y}^{+\infty} m_1(\grave{x}_1 \cap \acute{y}_1) m_2(\grave{x} \cap \acute{y}) dx_1 dy_1 \\ &+ \int_{x_1=-\infty}^{x} \int_{y_2=y}^{+\infty} m_1(\grave{x}_1 \cap \acute{y}) m_2(\grave{x} \cap \acute{y}_2) dx_1 dy_2 \\ &+ \int_{x_2=-\infty}^{x} \int_{y_1=y}^{+\infty} m_1(\grave{x} \cap \acute{y}_1) m_2(\grave{x}_2 \cap \acute{y}) dx_2 dy_1 \,. \end{aligned}$$

In particular, it is now justified that a bba, from sensors or fused, will always be zeroed outside $\mathcal{I}$.

5.5 Implementation of the continuous model

Setting. In this implementation, the study has been restricted to the interval $[-1, 1]$ instead of $\mathbb{R}$. The previous results still hold by truncating over $[-1, 1]$. In particular, any bba m is zeroed outside $\mathcal{I}_{-1}^{1} = \{\grave{x} \cap \acute{y}/x, y \in [-1, 1]\}$ and its related belief function is defined by:

$$\text{Bel}\,(\grave{x} \cap \acute{y}) = \int_{u=x}^{1} \int_{v=-1}^{y} m(\grave{u} \cap \acute{v}) du dv \,,$$

for any generalized interval of $\mathcal{I}_{-1}^{1}$. The bba resulting of the fusion of two bba's m_1 and m_2 is defined by:

$$\begin{aligned} m_1 \oplus m_2(\grave{x} \cap \acute{y}) &= \int_{x_2=-1}^{x} \int_{y_2=y}^{1} m_1(\grave{x} \cap \acute{y}) m_2(\grave{x}_2 \cap \acute{y}_2) dx_2 dy_2 \\ &+ \int_{x_1=-1}^{x} \int_{y_1=y}^{1} m_1(\grave{x}_1 \cap \acute{y}_1) m_2(\grave{x} \cap \acute{y}) dx_1 dy_1 \\ &+ \int_{x_1=-1}^{x} \int_{y_2=y}^{1} m_1(\grave{x}_1 \cap \acute{y}) m_2(\grave{x} \cap \acute{y}_2) dx_1 dy_2 \\ &+ \int_{x_2=-1}^{x} \int_{y_1=y}^{1} m_1(\grave{x} \cap \acute{y}_1) m_2(\grave{x}_2 \cap \acute{y}) dx_2 dy_1 \,. \end{aligned}$$

Method. A theoretical computation of these integrals seems uneasy. An approximation of the densities and of the integrals has been considered. More precisely, the densities have been approximated by means of 2-dimension *Chebyshev polynomials* , which have several good properties:

- The approximation grows quickly with the degree of the polynomial, without oscillation phenomena,
- The Chebyshev transform is quite related to the Fourier transform, which makes the parameters of the polynomials really quickly computable by means of a Fast Fourier Transform,
- Integration is easy to compute.

In our tests, we have chosen a Chebyshev approximation of degree 128×128, which is more than sufficient for an almost exact computation.

Example. Two bba m_1 and m_2 have been constructed by normalizing the following functions mm_1 and mm_2 defined over $[-1,1]^2$:

$$mm_1\big(\grave{x} \cap \acute{y}\big) = \exp\big(-(x+1)^2 - y^2\big)$$

and

$$mm_2\big(\grave{x} \cap \acute{y}\big) = \exp\big(-x^2 - (y-1)^2\big)\ .$$

The fused bba $m_1 \oplus m_2$ and the respective belief function $b_1, b_2, b_1 \oplus b_2$ have been computed. This computation has been instantaneous. All functions have been represented in the figures 5.7 to 5.14.

Interpretation. The bba m_1 is a density centered around the interval $[-1,0]$, while m_2 is a density centered around $[0,1]$. This explains why the belief b_1 increases faster from the interval $[-1,-1]$ to $[-1,1]$ than from the interval $[1,1]$ to $[-1,1]$. And this property is of course inverted for b_2.

A comparison of the fused bba $m_1 \oplus m_2$ with the initial bba's m_1 and m_2 makes apparent a global forward move of the density. This just means that the fused bba is put on intervals with less imprecision, and possibly on some intervals with negative width (*ie.* associated with a degree of contradiction). Of course there is nothing surprising here, since information fusion will reduce imprecision and produce some contradiction! It is also noticed that the fused bba is centered around the interval $[0,0]$. This result matches perfectly the fact that m_1 and m_2, and their related sensors, put more belief respectively over the interval $[-1,0]$ and the interval $[0,1]$; and of course $[-1,0] \cap [0,1] = [0,0]$.

5.6 Conclusion

A problem of continuous information fusion has been investigated and solved in the DSmT paradigm. The conceived method is based on the generalization of the notion of hyper-power set. It is versatile and is able to specify the typical various degrees of contradiction of a DSm model. It has been implemented efficiently for a bounded continuous information. The work

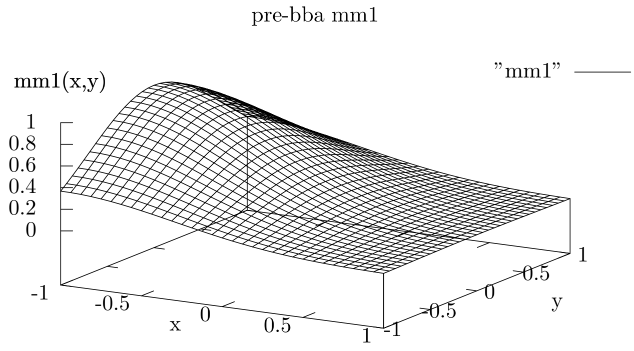

Figure 5.7: Non normalized bba mm_1

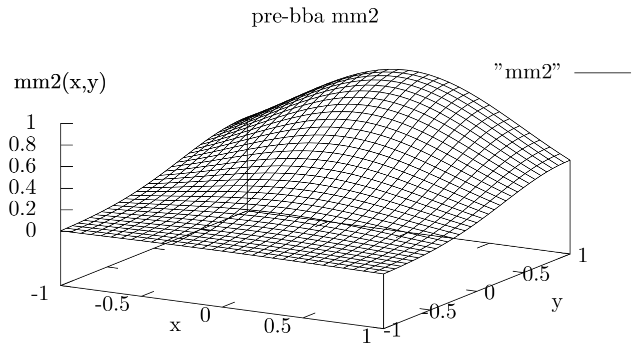

Figure 5.8: Non normalized bba mm_2

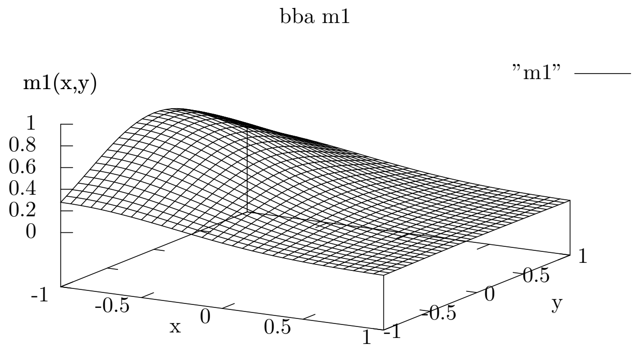

Figure 5.9: Basic belief assignment m_1

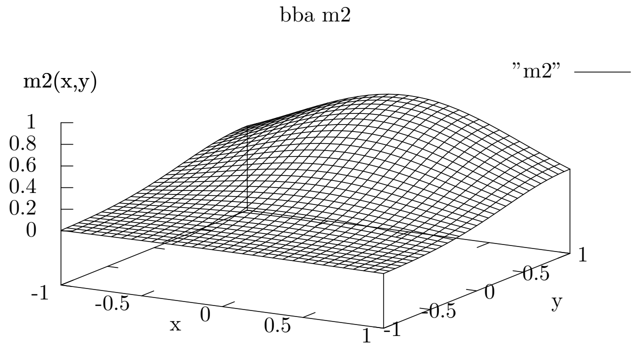

Figure 5.10: Basic belief assignment m_2

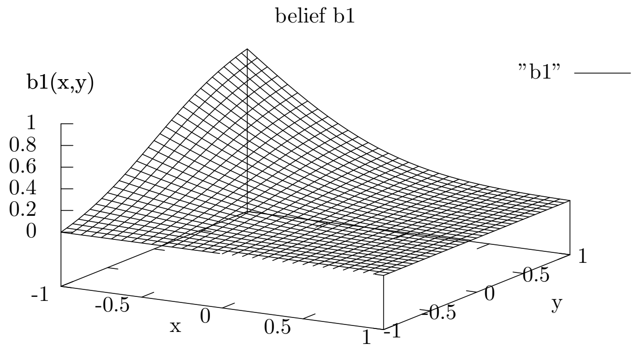

Figure 5.11: Belief function b_1

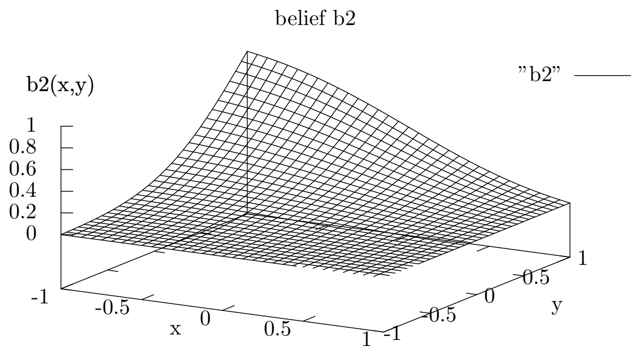

Figure 5.12: Belief function b_2

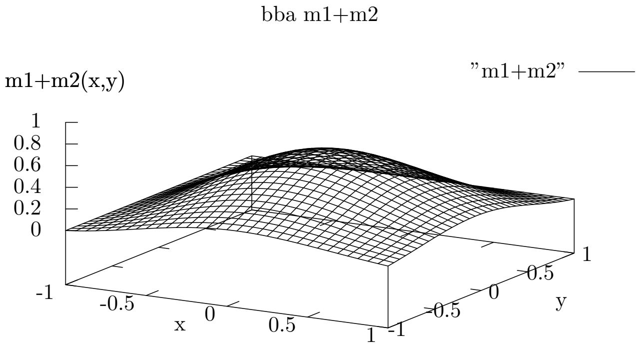

Figure 5.13: Fused bba $m_1 \oplus m_2$

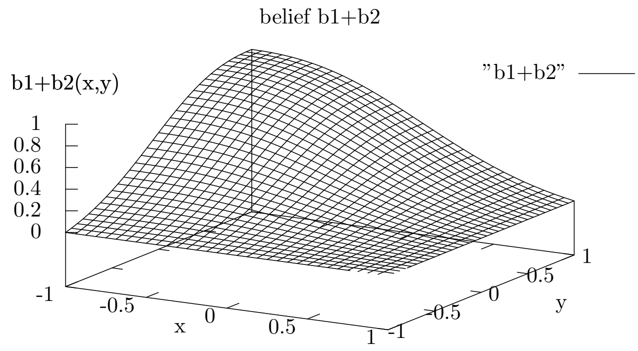

Figure 5.14: Fused bba $b_1 \oplus b_2$

is still prospective, but applications should be done in the future on localization problems. At this time, the concept is restricted to one-dimension informations. However, works are now accomplished in order to extend the method to multiple-dimensions domains.

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Chapter 6

Conflict Free Rule for Combining Evidences

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Abstract: *Recent works have investigated the problem of the conflict redistribution in the fusion rules of evidence theories. As a consequence of these works, many new rules have been proposed. Now, there is not a clear theoretical criterion for a choice of a rule instead another. The present chapter proposes a new theoretically grounded rule, based on a new concept of sensor independence. This new rule avoids the conflict redistribution, by an adaptive combination of the beliefs. Both the logical grounds and the algorithmic implementation are considered.*

6.1 Introduction

Recent works have underlined the limitations of the historical rule of Dempster and Shafer for fusing the information [4, 9]. The difficulty comes essentially from the conflict generation which is inherent to the rule definition. By the way, a sequential use of the rules would result in an accumulation of the conflict, if there were not a process for removing it. Many solutions have been proposed for managing this conflict. The following methods are noteworthy:

- Constraining, relaxing or adapting the models in order to avoid the conflict,
- Weakening the conflicting information with the time,
- Redistributing the conflict within the rules.

Model adaptations are of different natures. Close to Dempster-Shafer theory, Appriou [1] suggests to reduce the possible contradictions by a convenient setting of the problem hypotheses. Smets [11] removes the nullity constraint on the belief of the empty proposition (TBM); this way, the conflict is no more a problem. Dezert and Smarandache [5, 10] defined evidences on models with weakened negations (free DSmT and similar models). By weakening or suppressing the negation, the conflict actually disappears. The TBM of Smets and the DSmT of Dezert and Smarandache are both theoretically grounded. TBM is axiomatically derived, while free DSmT is constructed logically [3]. Moreover, although the DSmT keeps the nullity constraint for the empty proposition, it is possible to interpret the TBM by means of a constrained DSm model.

Avoiding the conflict by adapted models is not satisfactory however. Indeed, there are many cases where such models appear quite artificial and not well suited to represent the real world. Weakening the information is not satisfactory either; in many cases, the choice of a weakening criterion is rather subjective. Experimentations [8] have shown better results by means of rules with conflict redistributions adapted to the problem.[1] Florea, Jousselme and al [7] proposed recently a new family of rules which are adaptive with the conflict level. In this case, there is an important idea: the redistribution policy is now changing automatically as a function of the conflict.

Many new rules have been proposed. However, there is not a clear theoretical criterion for a choice of a rule instead another. Now, these new rules, and particularly the adaptive rule of Florea and Jousselme, have uncovered a new fact: *there is not a simple and permanent definition of the fusion rule for any fusion problem.* More precisely, the structure of the fusion rule may depend on the structure of the problem. In this chapter, we are proposing a methodology for computing fusion rules, being given a problem setting. This methodology is logically grounded and based on a new concept of sensor independence. As a result, the rules are obtained from a constrained convex optimization. These computed rules cannot be derived mathematically in general.

The next section introduces evidence theories and their various viewpoints. As a general framework for these theories, the notions of hyperpower sets and of pre-Boolean algebras are briefly reminded. Section 6.3 settles a new methodology for deriving the fusion rule. This methodology is based on an entropic notion of sensor independence. Then, section 6.4 discusses about the implementations and the properties of the new rule. Typical examples are considered. Section 6.5 is more theoretical and exposes the logical fundaments of our methodology. At last, section 6.6 concludes.

6.2 Viewpoints in evidence theories

In this section, we are discussing about theories for combining evidences expressed by belief functions. Since pre-Boolean algebra is a common framework for all these theories, in particular as a generalization of sets and hyperpower sets, we are now introducing briefly this notion.

6.2.1 Pre-Boolean algebra

The theory of Dezert and Smarandache is based on the fundamental notion of pre-Boolean algebra, or hyperpower sets. These algebras will describe the logical modeling of the knowledge. This chapter is not dedicated to a thorough exposition of the theory of pre-Boolean algebra. The reader should refer to the chapter 5 of this book for a precise theoretical definition. Now, the present section will introduce these notions qualitatively, and some typical examples will be provided.

[1] In fact, Dempster-Shafer rule is also a rule with redistribution of the conflict. But in this case, the redistribution is uniform.

6.2.1.1 General principle

Subsequently, the conjunction and disjunction are denoted $\wedge$ and $\vee$. The negation, when used, is denoted $\neg$. The empty set is denoted $\perp$ while the tautology, or full ignorance, is denoted $\top$. Notice that these notations are not the most classical in the domain of evidence theories. Typically, $\cap, \cup, \Theta \setminus \cdot, \emptyset, \Theta$ are used instead of $\wedge, \vee, \neg, \perp, \top$. However, $\wedge, \vee, \neg, \perp, \top$ are notations widely used in logics and Boolean algebra. Since the connexions are important between these theories, we will use the logical notations in general.

Definition. A pre-Boolean algebra could be seen as a subset of a Boolean algebra which is stable for the conjunction and the disjunction. As a consequence, a pre-Boolean algebra together with the two operators, conjunction and disjunction, is an algebraic structure.

This algebraic structure has the same properties than a Boolean algebra, except that it does not implement explicitly the notion of negation. In particular, the following properties are provided by the pre-Boolean algebra for the binary operators:

Commutativity. $\phi \wedge \psi = \psi \wedge \phi$ and $\phi \vee \psi = \psi \vee \phi$,

Associativity. $\phi \wedge (\psi \wedge \eta) = (\phi \wedge \psi) \wedge \eta$ and $\phi \vee (\psi \vee \eta) = (\phi \vee \psi) \vee \eta$,

Distributivity. $\phi \wedge (\psi \vee \eta) = (\phi \wedge \psi) \vee (\phi \wedge \eta)$ and $\phi \vee (\psi \wedge \eta) = (\phi \vee \psi) \wedge (\phi \vee \eta)$,

Idempotence. $\phi \wedge \phi = \phi$ and $\phi \vee \phi = \phi$,

Neutral sup/sub-elements. $\phi \wedge (\phi \vee \psi) = \phi$ and $\phi \vee (\phi \wedge \psi) = \phi$,

for any ϕ, ψ, η in the pre-Boolean algebra.

6.2.1.2 Example

Free pre-Boolean algebra. Let a, b, c be three atomic propositions. Consider the free Boolean algebra $\mathcal{B}(a, b, c)$ generated by a, b, c:

$$\mathcal{B}(a,b,c) = \left\{ \bigvee_{(\alpha,\beta,\gamma)\in A} (\alpha \wedge \beta \wedge \gamma) \Big/ A \subset \{a, \neg a\} \times \{b, \neg b\} \times \{c, \neg c\} \right\} .$$

It is well known that $\mathcal{B}(a, b, c)$ contains $2^{2^3} = 256$ elements.

The free pre-Boolean algebra generated by the propositions a, b, c is the smaller subset of $\mathcal{B}(a, b, c)$ containing a, b, c and stable for $\wedge$ and $\vee$. This set, denoted $< a, b, c >$, is defined extensionally by:

$$\begin{aligned} < a, b, c >= \{&\perp, a \wedge b \wedge c, a \wedge b, a \wedge c, b \wedge c, a \wedge (b \vee c), b \wedge (a \vee c), c \wedge (a \vee b), a, b, c, \\ &(a \wedge b) \vee (a \wedge c) \vee (b \wedge c), (a \wedge b) \vee c, (a \wedge c) \vee b, (b \wedge c) \vee a, a \vee b, a \vee c, b \vee c, a \vee b \vee c, \top\} \end{aligned}$$

It is easily verified that a conjunctive or disjunctive combination of propositions of $< a, b, c >$ is still a proposition of $< a, b, c >$. For example:

$$\big(a \wedge (b \vee c)\big) \vee b = \big((a \wedge b) \vee (a \wedge c)\big) \vee b = (a \wedge c) \vee b \, .$$

Moreover, $< a, b, c >$ is obviously the smallest set, which is stable for $\wedge$ and $\vee$. In particular, it is noticed that $\perp, \top \in < a, b, c >$ since $\perp = \bigvee_{\alpha \in \emptyset} \alpha$ and $\top = \bigwedge_{\alpha \in \emptyset} \alpha$.

The free pre-Boolean algebra $< a, b, c >$ is also called *hyperpower set* generated by a, b, c. It is also denoted D^Θ, where $\Theta = \{a, b, c\}$. Notice that the tautology $\top$ is often excluded from the definition of the hyperpower set [10]. By the way, Dambreville excluded both $\bot$ and $\top$ from a previous definition [3]. These differences have a quite limited impact, when considering the free DSmT. Whatever, it is generally assumed that $a \vee b \vee c = \top$; *but this is an additional hypothesis.*

A Boolean algebra is a constrained pre-Boolean algebra. A Boolean algebra is a subset of itself and is stable for $\wedge$ and $\vee$. Thus, it is a pre-Boolean algebra. Now, we will see on an example that a set could be seen as an hyperpower set which has been constrained by logical constraints. Since a Boolean algebra could be considered as a set, this result implies more generally that a Boolean algebra could be obtained by constraining a free pre-Boolean algebra.

Denote $\Theta = \{a, b, c\}$. Consider the Boolean algebra $\mathcal{P}(\Theta)$ related to the set operators $\cap, \cup, \Theta \setminus \cdot$ and neutral elements $\emptyset, \Theta$. This Boolean algebra is extensionally defined by:

$$\mathcal{P}(\Theta) = \big\{\emptyset, a, b, c, \{a, b\}, \{a, c\}, \{b, c\}, \Theta\big\} .$$

Now, consider the hyperpower set $< a, b, c >$ and apply to it the constraints:

$$\Gamma = \big\{a \wedge b = b \wedge c = a \wedge c = \bot \ , \ a \vee b \vee c = \top\big\} .$$

It is then derived:

$$\begin{cases} a \wedge b \wedge c = a \wedge (b \vee c) = b \wedge (a \vee c) = c \wedge (a \vee b) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = \bot \ , \\ (a \wedge b) \vee c = c \ , (a \wedge c) \vee b = b \ , (b \wedge c) \vee a = a \ , \\ a \vee b \vee c = \top \ . \end{cases}$$

Denoting $< a, b, c >_\Gamma$ the resulting constrained pre-Boolean algebra, it comes:

$$< a, b, c >_\Gamma = \{\bot, a, b, c, a \vee b, a \vee c, b \vee c, \top\} .$$

Then, $< a, b, c >_\Gamma$ contains exactly the same number of elements than $\mathcal{P}(\Theta)$. More precisely, by the Boolean properties of $\wedge$ and $\vee$, it is clear that $< a, b, c >_\Gamma$ and $\mathcal{P}(\Theta)$ are isomorph as pre-Boolean algebra. While $< a, b, c >_\Gamma$ does not define the negation explicitly, this isomorphism shows that the negation is implicitly defined in $< a, b, c >_\Gamma$. In fact, the negation of $< a, b, c >_\Gamma$ has been built by the constraints. This is an important property of pre-Boolean algebra:

> *The constraints put on a free pre-Boolean algebra partially characterize the negation operator.*

As a consequence, there is a partial definition of the negation in a pre-Boolean algebra. This negation is entirely undefined in an hyperpower set and is entirely defined in a set. But there are many intermediate cases.

Example of constrained pre-Boolean algebra. Let $\Theta = \{a, b, c\}$ be a set of atomic propositions and $\Gamma = \{a \wedge b = a \wedge c\}$ be a set of constraints. By propagating the constraints, it is obtained:

$$a \wedge b = a \wedge c = (a \wedge b) \vee (a \wedge c) = (a \wedge b) \wedge (a \wedge c) = a \wedge b \wedge c .$$

Consequently:

$$a \wedge (b \vee c) = a \wedge b \wedge c\,,\ \ (a \wedge b) \vee c = c\,,\ \ (a \wedge c) \vee b = b\,.$$
$$b \wedge (a \vee c) = c \wedge (a \vee b) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) = b \wedge c\,.$$

At last, the constrained pre-Boolean algebra is extensionally defined by:

$$< a, b, c >_\Gamma = \{\bot, a \wedge b \wedge c, b \wedge c, a, b, c, (b \wedge c) \vee a, a \vee b, a \vee c, b \vee c, a \vee b \vee c, \top\}$$

This configuration is modeled in figure 6.1. This model ensures that the propagation of the constraints is complete in the definition of $< a, b, c >_\Gamma$.

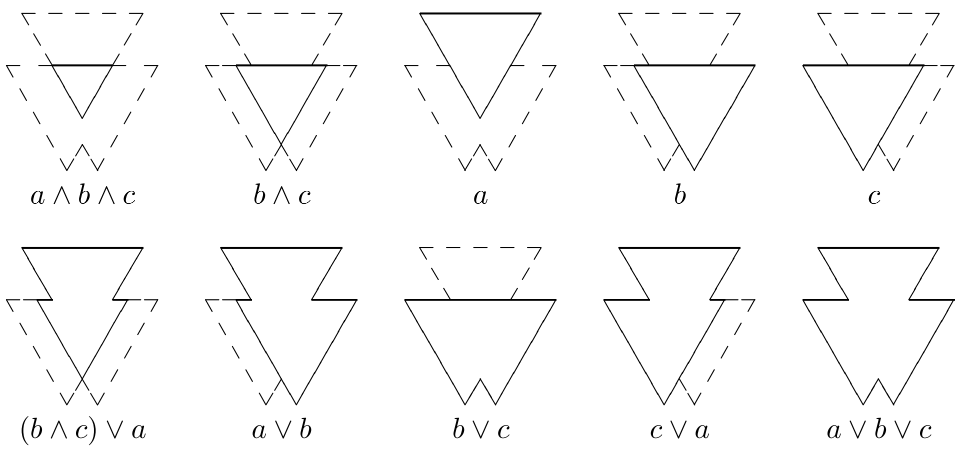

Figure 6.1: Pre-Boolean algebra $< a, b, c >_\Gamma$; ($\bot$ and $\top$ are omitted)

6.2.1.3 Notations

Let Θ be a set of atomic propositions. The free pre-Boolean algebra generated by Θ is denoted $< \Theta >$.

Now, let Γ be a set of constraints over the propositions of $< \Theta >$. The pre-Boolean algebra generated by Θ and constrained by Γ is denoted $< \Theta >_\Gamma$. Of course, it comes $< \Theta >_\emptyset = < \Theta >$ (the pre-Boolean algebra generated by Θ and constrained by an empty Γ is an hyperpower set).

A proposition ϕ is a subproposition of proposition ψ if and only if $\phi \wedge \psi = \phi$; subsequently, the property $\phi \wedge \psi = \phi$ is also denoted $\phi \subset \psi$.

6.2.2 Belief

It is now given a pre-Boolean algebra $< \Theta >_\Gamma$ as a logical framework for the knowledge representation. The theories of evidence also implement a belief on each logical proposition. This belief contains both an imprecision and an uncertainty information. The following sections consider two main styles for implementing the belief. In the DSmT and DST (Dempster Shafer Theory) [9], the belief over the empty proposition is always zero. In the TBM (Transferable Belief Model) [11], the belief over the empty proposition may be non zero. These viewpoints are related to two slightly different logical interpretations, as stated in section 6.5.

6.2.2.1 DSmT and DST

DSmT defines the notion of belief in a same way than DST. The only difference is that DST works on a set, while DSmT works on any pre-Boolean algebra. Fundamental differences will also arise, when defining the fusion of the information (section 6.2.3).

Basic Belief Assignment. A basic belief assignment (bba) to the pre-Boolean algebra $< \Theta >_\Gamma$ is a real valued function defined over $< \Theta >_\Gamma$ such that:

$$\sum_{\phi \in < \Theta >_\Gamma} m(\phi) = 1 \;, \quad m(\perp) = 0 \quad \text{and} \quad m \geq 0 \;.$$

Typically, m represents the knowledge of an expert or of a sensor. By hypothesizing $m(\perp) = 0$, the DSmT assumes the *coherence* of the information.

The bba is a belief density, describing the information intrinsic to the propositions. The *full* belief of a proposition is thus the compilation of the bba of its sub-propositions.

Belief function. The belief function Bel related to a bba m is defined by:

$$\forall \phi \in < \Theta >_\Gamma, \, \mathrm{Bel}(\phi) = \sum_{\psi \in < \Theta >_\Gamma : \psi \subset \phi} m(\psi) \;. \tag{6.1}$$

It is generally considered that $\mathrm{Bel}\left(\bigvee_{\theta \in \Theta} \theta\right) = 1$, which means that Θ matches all possible information.

6.2.2.2 TBM and TBM-like bba

Like the DST, the TBM works on a set. However, in the TBM interpretation the belief put on the empty set is not necessarily zeroed. It is also possible to mix this hypothesis with a pre-Boolean modeling, as follows.

TBM-like Basic Belief Assignment. A basic belief assignment to the pre-Boolean algebra $< \Theta >_\Gamma$ is a real valued function defined over $< \Theta >_\Gamma$ such that:

$$\sum_{\phi \in < \Theta >_\Gamma} m(\phi) = 1 \quad \text{and} \quad m \geq 0 \;.$$

By removing the hypothesis $m(\perp) = 0$, the coherence of the information is canceled. The coherence and non-coherence hypotheses have a logical interpretation, as explained in section 6.5.

In fact, it is also possible to simulate the TBM (and TBM-like models) by means of the DSmT (with the coherence hypothesis). The idea is to simulate the empty set of TBM by the pre-Boolean proposition $\bigwedge_{\theta \in \Theta} \theta$. This result, although simple, is outside the scope of this chapter and will not be developed further. To end with this subsection, it is noticed that Smets proposes a slightly different definition of the belief function by excluding the belief of the empty set. Smets belief function will be denoted and defined by:

$$\mathrm{Bel}_S(\phi) = \sum_{\psi \in < \Theta >_\Gamma : \perp \neq \psi \subset \phi} m(\psi) \;.$$

This truncated belief function is not used subsequently, since we work essentially on bba and on the full belief function Bel as defined in (6.1).

6.2.3 Fusion rules

The main contribution of evidence theories consists in their fusion rules. It is assumed then that two or more sources of information are providing a viewpoint about the universe. These viewpoints are described by specific bbas for each sensor. The question then is to make a unique representation of the information, *i.e.* a unique bba, from these several bbas. Several rules for fusing such information have been elaborated.

There are essentially two kinds of rules. The first kind avoids any conflict redistribution. The theorists generally agree then on a unique fusion rule, the conjunctive rule (without redistribution). Two models avoid the conflict redistribution: the transferable belief model of Smets and the free DSmT. In both theories, a strong constraint is put on the model. TBM puts non zero weights on the empty set, while free DSmT removes the negation from the model. In many cases however, these hypotheses are too restrictive.

When the conflict is taken into account and is redistributed, many possible rules have been proposed. No model restriction is needed anymore, but it is difficult to decide for a definitive fusion rule.

The following sections introduce shortly these various concepts of rules.

6.2.3.1 Fusion rule in free DSmT and similar models.

Free DSmT is defined on an hyperpower set. A fundamental property of an hyperpower set is that the empty proposition cannot be generated from non empty propositions. More generally, a pre-Boolean algebra $<\Theta>_\Gamma$, where the constraints in Γ do not generate $\perp$, will also satisfy such property:

$$\phi,\psi\in<\Theta>_\Gamma\setminus\{\perp\}\Longrightarrow\phi\wedge\psi\in<\Theta>_\Gamma\setminus\{\perp\}\,. \tag{6.2}$$

This property will be called an *insulation* property.

Assume now a pre-Boolean algebra $<\Theta>_\Gamma$ satisfying (6.2). Then, two bbas m_1 and m_2 over $<\Theta>_\Gamma$ will be fused into a bba $m_1\oplus m_2$ as follows:

$$\forall\phi\in<\Theta>_\Gamma\,,\ m_1\oplus m_2(\phi)=\sum_{\psi\wedge\eta=\phi}m_1(\psi)m_2(\eta)\ . \tag{6.3}$$

This definition is compatible with the constraint $m_1\oplus m_2(\perp)=0$ of DSmT, since it comes by the insulation property:

$$m_1(\psi)\neq 0 \text{ and } m_2(\eta)\neq 0\ \text{ imply }\ \psi\wedge\eta\in<\Theta>_\Gamma\setminus\{\perp\}\,.$$

The insulation property is often a too strong hypothesis for many problems. The TBM viewpoint will not request such structure constraints. But as a consequence, the coherence property of the bba will be removed.

6.2.3.2 Fusion rule for TBM-like bbas

In the TBM paradigm, two bbas m_1 and m_2 over $<\Theta>_\Gamma$ will be fused into a bba $m_1\oplus m_2$ as follows:

$$\forall\phi\in<\Theta>_\Gamma\,,\ m_1\oplus m_2(\phi)=\sum_{\psi\wedge\eta=\phi}m_1(\psi)m_2(\eta)\ . \tag{6.4}$$

There is no particular restriction on the choice of $< \Theta >_\Gamma$ in this case. It is for example possible that the model contains two non empty propositions ψ and η such that $\psi \wedge \eta = \perp$. Assuming that the initial bbas m_1 and m_2 are such that $m_1(\psi) > 0$ and $m_2(\eta) > 0$, it comes from the definition that $m_1 \oplus m_2(\perp) > 0$. But the rule is still compatible with the TBM paradigm, since then the *coherence* constraint $m_1 \oplus m_2(\perp) = 0$ is removed. By the way, removing this constraint is not satisfactory in many cases. In particular, it is well known that the weight of the contradiction may increase up to 1 by iterating the fusion stages.

6.2.3.3 General case

While the fusion rule is clearly defined by (6.3) for models avoiding the conflict, there are many possible rules when this conflict has to be redistributed. Typically, the rule could be defined in two steps. First, compute the conjunctive function μ of m_1 and m_2 by:

$$\forall \phi \in < \Theta >_\Gamma , \ \mu(\phi) = \sum_{\psi \wedge \eta = \phi} m_1(\psi) m_2(\eta) \ .$$

The function μ is like the fusion rule in the TBM paradigm. It cannot be used directly, since $\mu(\perp)$ have to be redistributed when $\mu(\perp) > 0$. Redistributing the conflict means:

- Constructing a function ρ on $< \Theta >_\Gamma$ such that:

$$\rho(\perp) = 0 \ , \qquad \sum_{\phi \in < \Theta >_\Gamma} \rho(\phi) = 1 \quad \text{and} \quad \rho \geq 0 \ ,$$

- Derive the bba $m_1 \oplus m_2$ by:

$$m_1 \oplus m_2(\phi) = \mu(\phi) + \rho(\phi)\mu(\perp) \ .$$

There are many possible rules deduced from the redistribution principle. Moreover, the redistribution may be dependent of a local conflict, like the PCR rules [6, 8] also introduced in chapter 2 of this book. It is also noticed that some authors [7] allows negative redistributions by removing the constraint $\rho \geq 0$. These new rules are as well legitimate and interesting, but by allowing negative redistributions, the criterion for defining rules is again weakened. The question now is *how to decided for a rule or another?* This choice is certainly dependent of the structure of the fusion problem. Actually, Florea, Jousselme and al [7] proposed a rule adaptive with the conflict level. More generally, it is foreseeable that a fusion rule should be defined or computed specifically for a given fusion problem.

In the next sections, we will derive logically a new ruling method, which avoids the conflict redistribution by exploiting a new concept of independence of the sensors. The new rules will be essentially computed from an entropic optimization problem. This problem may be unsolvable, which will imply a rejection of the hypotheses (too high conflict between the sources). Otherwise, it will adapt the belief dispatching in a much more flexible way than the usual conjunctive function μ.

6.3 Entropic approach of the rule definition

To begin with this new rule concept, we will directly settle the concrete optimization principles of our method. The logical justifications will come later, in section 6.5.

6.3.1 Independent sources and entropy

Actually, the idea is not completely new, and Dezert used it in order to give a first justification to the free DSmT [5]. More precisely, the free rule could be rewritten:

$$\forall \phi \in < \Theta >_\Gamma\,,\ m_1 \oplus m_2(\phi) = \sum_{\psi \wedge \eta = \phi} f_o(\psi, \eta)\ ,$$

where:

$$f_o(\psi, \eta) = m_1(\psi) m_2(\eta)\ . \tag{6.5}$$

If we are interpreting m_i as a kind of probability, the relation (6.5) is like the probabilistic independence, where $f_o(\psi, \eta)$ is a kind of joint probability. Section 6.5 will clarify this probabilistic viewpoint. Now, there is a link between the notion of probabilistic independence and the notion of entropy, which is often forgotten. The law $f_o(\psi, \eta) = m_1(\psi) m_2(\eta)$ is a maximizer of the entropy, with respect to the constraint of marginalization:

$$\begin{array}{l} f_o \in \arg\max\limits_f - \sum\limits_{\psi,\eta} f(\psi,\eta) \ln f(\psi,\eta) \\ \text{under constraints:} \\ \qquad f \geq 0\ , \quad \sum\limits_{\psi} f(\psi,\eta) = m_2(\eta) \quad \text{and} \quad \sum\limits_{\eta} f(\psi,\eta) = m_1(\psi)\ . \end{array} \tag{6.6}$$

This is actually how Dezert derived the conjunctive rule of free DSmT [5], although he did not make an explicit mention to the probability theory. Now, the equation (6.6) has a particular interpretation in the paradigm of information theory: f_o is the law which contains the maximum of information, owing to the fact that its marginals are m_1 and m_2. By the way, independent sources of information should provide the maximum of information, so that the maximization of entropy appears as the good way to characterize independent sources. When the constraints are just the marginalizations, the solution to this maximization is the independence relation $f_o(\psi, \eta) = m_1(\psi) m_2(\eta)$. In Bayesian theory particularly, the marginalizations are generally the only constraints, and the notion of independent sources of information reduces to the notion of independent propositions. But in the case of evidence theories, *there is the problem of the conflict*, which adds constraints.

6.3.2 Definition of a new rule for the DSmT

Let be defined a pre-Boolean algebra $< \Theta >_\Gamma$, constituting the logical framework of the information. Let be defined two bbas m_1 and m_2 over $< \Theta >_\Gamma$. The bbas are assumed to be *coherent*, so that $m_1(\bot) = m_2(\bot) = 0$. Then the fusion of m_1 and m_2 is the bba $m_1 \oplus m_2$

defined by:

$$\forall \phi \in < \Theta >_\Gamma\,,\ m_1 \oplus m_2(\phi) = \sum_{\psi \wedge \eta = \phi} f_o(\psi,\eta)\ ,$$

where:

$$\begin{array}{l} f_o \in \arg\max_f -\sum_{\psi,\eta} f(\psi,\eta) \ln f(\psi,\eta) \\ \text{under constraints:} \\ \qquad f \geq 0\ , \quad \sum_{\psi} f(\psi,\eta) = m_2(\eta)\ , \quad \sum_{\eta} f(\psi,\eta) = m_1(\psi)\ , \\ \qquad \text{and} \quad \forall \psi,\eta \in < \Theta >_\Gamma\,,\ \psi \wedge \eta = \bot \Longrightarrow f(\psi,\eta) = 0\,. \end{array} \tag{6.7}$$

This rule will be named *Entropy Maximizing Rule* (EMR).

Corollary of the definition. The fused basic belief assignment is compatible with the coherence constraint $m_1 \oplus m_2(\bot) = 0$ of DSmT.

Proof is immediate owing to the constraints $\psi \wedge \eta = \bot \Longrightarrow f(\psi,\eta) = 0$ in the optimization.

6.3.3 Feasibility of the rule

The rule is feasible when there is a solution to the optimization. The feasibility is obtained as soon there is a solution to the constraints.

Definition. The fused bba $m_1 \oplus m_2$ is defined if and only if there exists a function f such that:

$$\begin{array}{c} f \geq 0\ , \quad \sum_{\psi} f(\psi,\eta) = m_2(\eta)\ , \quad \sum_{\eta} f(\psi,\eta) = m_1(\psi)\ , \\ \text{and} \quad \forall \psi,\eta \in < \Theta >_\Gamma\,,\ \psi \wedge \eta = \bot \Longrightarrow f(\psi,\eta) = 0\,. \end{array} \tag{6.8}$$

In next section, it will be shown on examples that the fusion is not always feasible. Actually, the infeasibility of the rule is a consequence of fundamental incompatibilities of the information.

6.3.4 Generalizations

6.3.4.1 Fusion of N bbas

It will be seen that the fusion rule defined previously is not associative. This means that the sources of information do not have the same *weight* in a sequential fusion. However, when it is needed to fuse N sources of information simultaneously, the fusion method has then to be generalized to N bbas. The subsequent definition makes this generalization.

N-ary rule. Let be defined a pre-Boolean algebra $< \Theta >_\Gamma$, constituting the logical framework of the information. Let be defined N coherent bbas $m_i|_{1 \leq i \leq N}$ over $< \Theta >_\Gamma$. Then the fusion of

$m_i|_{1\leq i\leq N}$ is the bba $\oplus[m_i|1 \leq i \leq N]$ defined by:

$$\forall\phi \in< \Theta >_\Gamma\,,\ \oplus[m_i|1 \leq i \leq N](\phi) = \sum_{\bigwedge_{i=1}^N \psi_i=\phi} f_o(\psi_i|1 \leq i \leq N)\ ,$$

where:

$$\begin{aligned}
&f_o \in \arg\max_f -\sum_\psi f(\psi_i|1 \leq i \leq N) \ln f(\psi_i|1 \leq i \leq N)\\
&\text{under constraints:}\\
&\qquad f \geq 0\ , \quad \forall i, \sum_{\psi_j|j\neq i} f(\psi_j|1 \leq j \leq N) = m_i(\psi_i)\ ,\\
&\qquad \text{and} \quad \forall\psi \in< \Theta >_\Gamma^N\,,\ \bigwedge_{i=1}^N \psi_i = \bot \Longrightarrow f(\psi_i|1 \leq i \leq N) = 0\,.
\end{aligned} \tag{6.9}$$

6.3.4.2 Approximation of the rule

The definition of our rule needs the maximization of the entropy of f under various constraints. An algorithm for solving this maximization is proposed in section 6.4. The problem is solved by means of a variational method. By the way, it may be interesting to have a more direct computation of the rule. In particular, better computations of the rule could be obtained by approximating the optimization problem.

As soon as a solution is feasible, there are many ways to approximate the rules. The main point is to approximate the entropy $H(f) = -\sum_{\psi,\eta} f(\psi,\eta)\ln f(\psi,\eta)$ by a function $\widetilde{H}(f)$ such that $\widetilde{H}(f) \simeq H(f)$. Then, the rule is just rewritten:

$$\forall\phi \in< \Theta >_\Gamma\,,\ m_1 \oplus m_2(\phi) = \sum_{\psi\wedge\eta=\phi} f_o(\psi,\eta)\ ,$$

where:

$$\begin{aligned}
&f_o \in \arg\max_f \widetilde{H}(f)\ ,\\
&\text{under constraints:}\\
&\qquad f \geq 0\ , \quad \sum_\psi f(\psi,\eta) = m_2(\eta)\ , \quad \sum_\eta f(\psi,\eta) = m_1(\psi)\ ,\\
&\qquad \text{and} \quad \forall\psi,\eta \in< \Theta >_\Gamma\,,\ \psi\wedge\eta = \bot \Longrightarrow f(\psi,\eta) = 0\,.
\end{aligned} \tag{6.10}$$

An example of approximation is $\widetilde{H}(f) = -\sum_{\psi,\eta} f^2(\psi,\eta)$, which is obtained by a first order derivation of ln. Approximated rules will not be investigated in the present chapter.

6.4 Implementation and properties

This section is devoted to the development of basic properties of the rule EMR and to practical implementation on examples.

6.4.1 Properties

Commutativity. Let m_1 and m_2 be two bbas over $< \Theta >_\Gamma$. By definition (6.7), the fused bba $m_1 \oplus m_2$ exists if and only if $m_2 \oplus m_1$ exists. Then $m_1 \oplus m_2 = m_2 \oplus m_1$.

Neutral element. Define the bba of total ignorance ν by $\nu(\top) = 1$. Let m be a bba over $< \Theta >_\Gamma$. Then the fused bba $m \oplus \nu$ exists, and $m \oplus \nu = m$.

Proof. Since $\sum_\phi f_o(\phi, \psi) = \nu(\psi)$ and $f_o \geq 0$, it is deduced $f_o(\phi, \psi) = 0$ unless $\psi = \top$.
Now, since $\sum_\psi f_o(\phi, \psi) = m(\phi)$, it is concluded:

$$f_o(\phi, \top) = m(\phi) \quad \text{and} \quad f_o(\phi, \psi) = 0 \text{ for } \psi \neq \top\,.$$

This function satisfies the hypotheses of (6.7), thus implying the existence of $m \oplus \nu$.
Then the result $m \oplus \nu = m$ is immediate.

□□□

Belief enhancement. Let be given two bbas m_1 and m_2, and assume that there exists a fused bba $m_1 \oplus m_2$ computed by (6.7). Denote by $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2$ the belief function related to $m_1 \oplus m_2$. Then:

$$\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) \geq \max\bigl\{\mathrm{Bel}_1(\phi), \mathrm{Bel}_2(\phi)\bigr\} \quad \text{for any } \phi \in < \Theta >_\Gamma\ . \tag{6.11}$$

Proof.
Proof of $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) \geq \mathrm{Bel}_1(\phi)$.
Let f_o be a function satisfying to (6.7).
Then $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) = \sum_{\psi \subset \phi} m_1 \oplus m_2(\psi) = \sum_{\eta \wedge \xi \subset \phi} f_o(\eta, \xi)$.
In particular, $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) \geq \sum_{\psi \subset \phi} \sum_\eta f_o(\psi, \eta)$.
At last, $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) \geq \sum_{\psi \subset \phi} m_1(\psi) = \mathrm{Bel}_1(\phi)$.

Conclusion. It is similarly proved $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\phi) \geq \mathrm{Bel}_2(\phi)$ and then the final result.

□□□

Corollary. Let be given two bbas m_1 and m_2, and assume that there exists a fused bba $m_1 \oplus m_2$ computed by (6.7). Let $\phi_1, \ldots, \phi_n \in < \Theta >_\Gamma$ be such that $\phi_i \wedge \phi_j = \bot$ for any $i \neq j$. Then the property $\sum_{i=1}^n \max\bigl\{\mathrm{Bel}_1(\phi_i), \mathrm{Bel}_2(\phi_i)\bigr\} \leq 1$ *is necessarily true.*

This result is a direct consequence of the belief enhancement. It could be used as a criterion for proving the non existence of the fusion rule.

Associativity. The computed rule $\oplus$ is not associative.

Proof. Consider the bbas m_1, m_2 and m_3 defined on the Boolean algebra $\{\bot, a, \neg a, \top\}$ by:

$$\begin{cases} m_1(a) = m_2(a) = 0.5 \quad \text{and} \quad m_1(\top) = m_2(\top) = 0.5\,, \\ m_3(\neg a) = 0.5 \quad \text{and} \quad m_3(\top) = 0.5\,. \end{cases}$$

We are now comparing the fusions $(m_1 \oplus m_2) \oplus m_3$ and $m_1 \oplus (m_2 \oplus m_3)$.

Computing $(m_1 \oplus m_2) \oplus m_3$.

First it is noticed that there is no possible conflict between m_1 and m_2, so that $m_1 \oplus m_2$ could be obtained by means of the usual conjunctive rule:

$$m_1 \oplus m_2(a) = 0.5 \times 0.5 + 0.5 \times 0.5 + 0.5 \times 0.5 = 0.75 \quad \text{and} \quad m_1 \oplus m_2(\top) = 0.5 \times 0.5 \; .$$

As a consequence:

$$\max\bigl\{\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(a), \mathrm{Bel}_3(a)\bigr\} + \max\bigl\{\mathrm{Bel}_1 \oplus \mathrm{Bel}_2(\neg a), \mathrm{Bel}_3(\neg a)\bigr\} = 0.75 + 0.5 > 1 \; .$$

It is concluded that $\underline{(m_1 \oplus m_2) \oplus m_3 \text{ does not exist.}}$

Computing $m_1 \oplus (m_2 \oplus m_3)$.
It is known that $\mathrm{Bel}_2 \oplus \mathrm{Bel}_3 \geq \max\bigl\{\mathrm{Bel}_2, \mathrm{Bel}_3\bigr\}$ when $m_2 \oplus m_3$ exists.
Since $\max\bigl\{\mathrm{Bel}_2(a), \mathrm{Bel}_3(a)\bigr\} = \max\bigl\{\mathrm{Bel}_2(\neg a), \mathrm{Bel}_3(\neg a)\bigr\} = 0.5$, it is deduced that necessarily $m_2 \oplus m_3(a) = m_2 \oplus m_3(\neg a) = 0.5$
It appears that $m_2 \oplus m_3(a) = m_2 \oplus m_3(\neg a) = 0.5$ is actually a valid solution, since it is derived from f_o such that $f_o(a, \top) = f_o(\top, \neg a) = 0.5$ (zeroed on the other cases).
It is also deduced by a similar reasoning that $m_1 \oplus (m_2 \oplus m_3)$ exists and is necessary defined by $\underline{m_1 \oplus (m_2 \oplus m_3)(a) = m_1 \oplus (m_2 \oplus m_3)(\neg a) = 0.5}$.

The associativity thus fails clearly on this example.

□□□

Compatibility with the probabilistic bound hypothesis. A temptation in evidence theories is to link the notion of probability with the notion of belief by means of the relation:

$$\mathrm{Bel}(\phi) \leq p(\phi) \quad \text{for any } \phi \in <\Theta>_\Gamma \; . \tag{6.12}$$

In general, this relation is not compatible with the fusion rules.

For example, let us test Dempster-Shafer rule on the relation (6.12)
Let be given m_1 and m_2 defined on $\{\bot, a, \neg a, \top\}$ by $m_1(a) = m_1(\neg a) = 0.5$ and $m_2(a) = m_2(\top) = 0.5$.
It is deduced $\mathrm{Bel}_1(a) = \mathrm{Bel}_1(\neg a) = 0.5$, $\mathrm{Bel}_2(a) = 0.5$ and $\mathrm{Bel}_2(\neg a) = 0$.
The choice of m_1 and m_2 is thus compatible with the bound hypothesis (6.12), and it follows $p(a) = p(\neg a) = 0.5$.
Now, Dempster-Shafer rule implies $m_1 \oplus m_2(a) = 2/3$ and $m_1 \oplus m_2(\neg a) = 1/3$.
Confronting $m_1 \oplus m_2$ to (6.12), it comes $p(a) \geq 2/3$.
This is contradictory with the previously obtained relation $p(\neg a) = 0.5$.

This difficulty is avoided by some theorists by saying that the notion of probability is dependent of the considered sensor, or that belief and probability are two separated notions.

In our opinion, probability should be considered as an absolute notion. We will see in section 6.5, that the belief could be considered as a probabilistic modal proposition. Then there are two cases:

- If the modal propositions are not comparable to the propositions without modality, then there is no obvious relation between the belief and the probability. This is particularly the case of the TBM paradigm.

- If the modal propositions are comparable to the propositions without modality (axiom m.iii of section 6.5), then the bound hypothesis (6.12) is recovered. Moreover, the fusion rule EMR is then logically derived.

This anticipatory logical result has the following implication:

The rule EMR is compatible with the bound hypothesis (6.12).

But this result is already foreseeable from the property (6.11). Indeed, property (6.11) makes impossible the construction of a counter-example like the previous one of this paragraph.

Idempotence. The rule is not idempotent: it always increases the precision of a bba, when possible. However it will be idempotent, when the precision of the bba cannot be increased (*e.g.* a probability).

This obvious property is just illustrated on examples subsequently.

6.4.2 Algorithm

The optimization (6.7) is convex and is not difficult. The implemented algorithm is based on Rosen's gradient projection method. Now, the gradient of $H(f) = \sum_{\psi,\eta} -f(\psi,\eta)\ln f(\psi,\eta)$ is characterized by:

$$D_f H(f) = \sum_{\psi,\eta} -(1+\ln f(\psi,\eta))\, df(\psi,\eta)\,.$$

Then, the algorithm follows the synopsis:

1. Find a feasible solution f_0 to the simplex:

$$f_0 \geq 0\,, \quad \sum_{\psi} f_0(\psi,\eta) = m_2(\eta)\,, \quad \sum_{\eta} f_0(\psi,\eta) = m_1(\psi)\,,$$
$$\text{and} \quad \forall \psi,\eta \in <\Theta>_\Gamma\,,\ \psi\wedge\eta = \perp \Longrightarrow f_0(\psi,\eta) = 0\,.$$

   If such a solution does not exist, then stop: *the fusion is not possible.*

   Otherwise, set $t = 0$ and continue on next step.

2. Build Δf_t by solving the linear program:

$$\max_{\Delta f_t} \sum_{\psi,\eta} -(1+\ln f_t(\psi,\eta))\,\Delta f_t(\psi,\eta)\,,$$

   under constraints:

$$f_t + \Delta f_t \geq 0\,, \quad \sum_{\psi} \Delta f_t(\psi,\eta) = \sum_{\eta} \Delta f_t(\psi,\eta) = 0\,,$$
$$\text{and} \quad \forall \psi,\eta \in <\Theta>_\Gamma\,,\ \psi\wedge\eta = \perp \Longrightarrow \Delta f_t(\psi,\eta) = 0\,.$$

3. Repeat $\Delta f_t := \Delta f_t/2$ until $H(f_t+\Delta f_t) > H(f_t)$.

4. Set $f_{t+1} = f_t + \Delta f_t$. Then set $t := t+1$.

5. Reiterate from step 2 until full convergence.

The linear programming library Coin-LP has been used in our implementation.

6.4.3 Examples

In this section is studied the fusion of bbas m_i defined over $\mathcal{P}\Big(\{a,b,c\}\Big)$ by:

$$\begin{cases} m_1(a)=\alpha_1\,,\ m_1(b)=0\,,\ m_1(c)=\gamma_1 \text{ and } m_1(\{a,b,c\})=1-\alpha_1-\gamma_1\,,\\ m_2(a)=0\,,\ m_2(b)=\beta_2\,,\ m_2(c)=\gamma_2 \text{ and } m_2(\{a,b,c\})=1-\beta_2-\gamma_2\,. \end{cases}$$

This is a slight generalization of Zadeh's example. The fusion $m_1\oplus m_2$ is solved by the algorithm, but also mathematically. The solutions were identical by the both methods. The results of fusion are presented for particular choices of the parameters α,β,γ.

Mathematical solution. Assume that f is is a function over $\mathcal{P}\Big(\{a,b,c\}\Big)^2$ verifying the conditions (6.8) of the rule. The marginal constraints say:

$$\begin{cases} \displaystyle\sum_{B\subset\{a,b,c\}} f(a,B)=\alpha_1\,,\ \sum_{B\subset\{a,b,c\}} f(c,B)=\gamma_1\,,\ \sum_{B\subset\{a,b,c\}} f(\{a,b,c\},B)=1-\alpha_1-\gamma_1\,,\\ \displaystyle\sum_{A\subset\{a,b,c\}} f(A,b)=\beta_2\,,\ \sum_{A\subset\{a,b,c\}} f(A,c)=\gamma_2\,,\ \sum_{A\subset\{a,b,c\}} f(A,\{a,b,c\})=1-\beta_2-\gamma_2\,,\\ \displaystyle\sum_{B\subset\{a,b,c\}} f(A,B)=0 \quad\text{and}\quad \sum_{A\subset\{a,b,c\}} f(A,B)=0 \text{ in any other cases.} \end{cases} \tag{6.13}$$

Since $f(A,B)=0$ for any (A,B) such that $A\cap B=\emptyset$ and $f\geq 0$, it is deduced that :

$$f\Big(a,\{a,b,c\}\Big)\,,\ f\Big(\{a,b,c\},b\Big)\,,\ f(c,c)\,,\ f\Big(c,\{a,b,c\}\Big)\,,\ f\Big(\{a,b,c\},c\Big) \text{ and } f\Big(\{a,b,c\},\{a,b,c\}\Big)$$

are the only values of f, which are possibly non zero. Then the system (6.13) reduces to the linear solution:

$$\begin{cases} f\Big(a,\{a,b,c\}\Big)=\alpha_1\,,\ f\Big(\{a,b,c\},b\Big)=\beta_2\,,\ f(c,c)=\theta\,,\ f\Big(c,\{a,b,c\}\Big)=\gamma_1-\theta\,,\\ f\Big(\{a,b,c\},c\Big)=\gamma_2-\theta\,,\ f\Big(\{a,b,c\},\{a,b,c\}\Big)=1-\alpha_1-\beta_2-\gamma_1-\gamma_2+\theta\,,\\ f(A,B)=0 \text{ for any other case.} \end{cases} \tag{6.14}$$

This solution depends on the only unknow parameter θ. The optimal parameter θ_o is obtained by solving:

$$\max_\theta\big(h(\theta)+h(\gamma_1-\theta)+h(\gamma_2-\theta)+h(1-\alpha_1-\beta_2-\gamma_1-\gamma_2+\theta)\big)\quad\text{where } h(\tau)=-\tau\ln\tau\,.$$

The function f_o is then computed by using θ_o in (6.14). And $m_1\oplus m_2$ is of course obtained by $m_1\oplus m_2(\phi)=\sum_{\psi\wedge\eta=\phi} f_o(\psi,\eta)$.

It is sometimes impossible to find a valid parameter θ. The condition of existence is easily derived:

$$\theta \text{ exists}\quad\text{if and only if}\quad \max\{0,\alpha_1+\beta_2+\gamma_1+\gamma_2-1\}\leq\min\{\gamma_1,\gamma_2\}\,. \tag{6.15}$$

When this condition is fulfilled, the optimal parameter is given by[2]:

$$\theta_o = \frac{\gamma_1\gamma_2}{1-\alpha_1-\beta_2} \text{ when } \alpha_1+\beta_2<1\ , \quad \theta_o = 0 \text{ when } \alpha_1+\beta_2=1\ . \tag{6.16}$$

Then, it is not difficult to check that θ_o is bounded accordingly to the existence condition:

$$\max\{0,\alpha_1+\beta_2+\gamma_1+\gamma_2-1\} \leq \theta_o \leq \min\{\gamma_1,\gamma_2\}\ .$$

Experimentation.

Zadeh's example.

Zadeh's example is defined by $\alpha_1=\beta_2=0,99$ and $\gamma_1=\gamma_2=0.01$. This fusion is *rejected by EMR.*

More generally, assume $\gamma_1=1-\alpha_1$ and $\gamma_2=1-\alpha_2$. Then the condition:

$$\max\{0,\alpha_1+\beta_2+\gamma_1+\gamma_2-1\} \leq \min\{\gamma_1,\gamma_2\}$$

fails unless when $\gamma_1=\gamma_2=1$. The case $\gamma_1=\gamma_2=1$ is trivial, since it means a perfect agreement between the two sources. Thus, Zadeh's example is rejected by EMR, even if there is a negligible conflict between the two sources.

By the way, this is not surprising. In Zadeh's example, the bbas are put on the singletons only. Then, the bbas work like probabilities, thus defining an uncertainty but without any imprecision. Since the information provided by the sources are free from any imprecision, there are only two possible cases: either the information are the same, either some information are false.

Now, imagine again that our information come with a negligible conflict:

$$m_1(a)=m_2(b)=\epsilon \quad\text{and}\quad m_1(c)=m_2(c)=1-\epsilon\ .$$

This could indeed happen, when our information have been obtained from slightly distorted sources. Now, it has been seen that EMR rejects this fusion. Thus, we have to be cautious when using EMR and the following recommendation has to be considered:

> If the sources of information are distorted, even slightly, these distortions **have to** be encoded in the bbas by an additional imprecision.

Typically, by weakening the bbas as follows:

$$\begin{cases} m_1(a)=m_2(b)=\rho\epsilon\ ,\ m_1(c)=m_2(c)=\rho(1-\epsilon)\ , \\ m_1(\{a,b,c\})=m_2(\{a,b,c\})=1-\rho\ ,\ \text{with } \rho\leq\frac{1}{1+\epsilon}\ , \end{cases}$$

the fusion is again possible by means of EMR.

Extended example.

The following table summarizes the fusion by EMR for various values of α,β,γ:

α_1	γ_1	β_2	γ_2	$m=m_1\oplus m_2$
0.501	0	0.501	0	Rejection
0.499	0	0.499	0	$m(a)=m(b)=0.499\ ,\ m(\{a,b,c\})=0.002$
0.3	0.1	0.3	0.1	$m(a)=m(b)=0.3\ ,\ m(c)=0.175\ ,\ m(\{a,b,c\})=0.225$
0.3	0.05	0.3	0.05	$m(a)=m(b)=0.3\ ,\ m(c)=0.09375\ ,\ m(\{a,b,c\})=0.30625$
0.3	0.01	0.3	0.01	$m(a)=m(b)=0.3\ ,\ m(c)=0.01975\ ,\ m(\{a,b,c\})=0.38025$

[2]Other cases are not compliant with the existence condition (6.15).

Comparison.
In this last example, we compare the fusion by EMR and by Dempster-Shafer of the bbas m_1 and m_2 defined by:

$$\begin{cases} m_1(a) = m_1(\{a,b\}) = m_1(\{a,c\}) = m_1(\{b,c\}) = m_1(\{a,b,c\}) = 0.2 \ , \\ m_2(a) = m_2(\{a,b\}) = m_2(\{a,c\}) = m_2(\{b,c\}) = m_2(\{a,b,c\}) = 0.2 \ . \end{cases}$$

The following table presents the fusion results for Dempster-Shafer (DST) and for EMR:

A	a	b	c	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$m_1 \oplus m_2(A)$ / DST	0.390	0.087	0.087	0.131	0.131	0.131	0.043
$m_1 \oplus m_2(A)$ / EMR	0.411	0.093	0.093	0.107	0.107	0.153	0.036

We will not argue about the advantage of EMR compared to DST on such simple example. The important point is to notice how the bba concentration subtly changes from DST to EMR. In general, the belief enforcement of the small propositions is stronger in EMR. But the case of proposition $\{b,c\}$ is different, since it is made weaker by DST than by EMR. This is perhaps a consequence of a greater belief attraction of proposition a compared to b and c, during the fusion process.

6.5 Logical ground of the rule

This section justifies logically the definition (6.7) of EMR. This logical development, based on modal logics, is quite similar to what have been previously done in the DSmT book 1, chapter 8 [3]. Actually, the modal operators will be applied to non modal proposition only for the simplicity of the exposition (the definitions of [3] were more general), but there is no significant change in the logic. Now, the reader could also refer to [3] since it introduces the logic on examples. Subsequently, the logic behind EMR will be exposed directly.

In [3] the definition of the logic was axiomatic. Since logic is not usual for many readers, such axiomatic definition is avoided in this presentation. A model viewpoint is considered now. More precisely, the description of our modal logic is made in the framework of a Boolean algebra. Typically, a logical relation $\vdash \phi$, *i.e.* ϕ *is proved*, is replaced by $\phi = \top$ in the Boolean algebra. In the same way, a relation $\vdash \phi \to \psi$ is replaced by $\phi \subset \psi$. Moreover, the modal relations are directly used within the Boolean algebra, *i.e.* $\Box\phi \subset \phi$ is the Boolean counterpart of the logical axiom $\vdash \Box\phi \to \phi$. We will not justify here the soundness of this Boolean framework, in regards to the modal propositions. But such framework is of course tightly related to an implied Kripke's model. By the way, it is also assumed that our model is complete for the logic. But all these technical considerations should be ignored by most readers.

Notation. Subsequently, the proposition $\phi\setminus\psi$ is a shortcut for $\phi\wedge\neg\psi$. Moreover, the notation $\phi \subsetneq \psi$ is used as a shortcut for $\phi \subset \psi$ and $\phi \neq \psi$.

6.5.1 The belief as a probability of a modal proposition

Many evidence theorists are particularly cautious, when comparing belief and probabilities. On the first hand, there is a historical reason. As new theories of uncertainty, the evidence theories had to confront the already existing theory of probability. On the second hand, the sub-additivity property of the belief is often proposed as a counter-argument against the probabilistic hypothesis. In this introductory subsection, it is seen that this property is easily fulfilled by means of modal propositions.

Sub-additivity, modality and probability. Subsequently, the belief $\mathrm{Bel}_i(\phi)$ of a proposition ϕ according to a sensor i is interpreted as the probability $p(\Box_i\phi)$ of the modal proposition $\Box_i\phi$. It is not the purpose of this paragraph to make a full justification of this interpretation, but rather to explain how the modal interpretation is able to overcome the sub-additivity property.

First at all, it is important to understand the meaning of the modal operator $\Box_i$. The modal proposition $\Box_i\phi$ characterizes all the information, which sensor i can associate for sure to the proposition ϕ.

> For example, the equation $\Box_i\phi = \top$ means: *sensor i considers ϕ as true in any configuration of the system.*

Then, it is noticed that the modal propositions fulfill a *logical sub-additivity* property:

$$(\Box_i\phi \vee \Box_i\psi) \subset \Box_i(\phi \vee \psi) \,. \tag{6.17}$$

The converse is false in general. This well-known property will be proved in the subsequent section. It has a concrete interpretation. The proposition $\Box_i\phi$ describes logically the information about ϕ which is granted as sure by the sensor i. But there are information which sensor i can attribute to $\phi \vee \psi$ for sure, but which cannot be attributed without ambiguity to ϕ or to ψ alone. These ambiguous information are exactly described by the non empty proposition $\Box_i(\phi \vee \psi) \setminus (\Box_i\phi \vee \Box_i\psi)$.

The important point now is that the logical sub-additivity directly implies the sub-additivity of the belief. From (6.17), it is derived: $p(\Box_i\phi \vee \Box_i\psi) \leq p\big(\Box_i(\phi \vee \psi)\big)$. Assume now that ϕ and ψ are disjoint, *i.e.* $\phi \wedge \psi = \bot$. It is also hypothesized that $\Box_i$ is coherent, which implies $\Box_i\phi \wedge \Box_i\psi = \bot$ *(coherent sensors will consider disjoint propositions as disjoint)*. Then it comes: $p(\Box_i\phi) + p(\Box_i\psi) \leq p\big(\Box_i(\phi \vee \psi)\big)$, and it is finally deduced: $\mathrm{Bel}_i(\phi) + \mathrm{Bel}_i(\psi) \leq \mathrm{Bel}_i(\phi \vee \psi)$, from the modal definition of Bel_i. At last, we have recovered the sub-additivity of Bel_i from the logical sub-additivity of the modal operator $\Box_i$.

It appears that the sub-additivity is not incompatible with a probabilistic interpretation of the belief. The sub-additivity seems rather related to a modal nature of the belief. At the end of this paragraph, a last word has to be said about the TBM paradigm. Belief as defined in (6.1), that is including the mass assignment of the empty set, is not sub-additive in TBM. Only the truncated belief Bel_S is sub-additive. This is a consequence of the possible non-zero mass assignment of the empty set. By the way, there is also a modal interpretation of this fact. It is seen subsequently that the TBM paradigm is equivalent to remove the coherence hypothesis about the modality $\Box_i$. But the incoherence of $\Box_i$ allows $\Box_i\phi \wedge \Box_i\psi \neq \bot$ even when $\phi \wedge \psi = \bot$. As a consequence, the sub-additivity cannot be recovered from the modal hypothesis either, when considering incoherent sensors.

This introduction has revealed some similar characteristics of the belief functions and the modal operators. The forthcoming sections establish a complete correspondence between the evidence theories and the probabilistic modal logic.

6.5.2 Definition of the logic

Let Θ be a set of atomic propositions. Let Γ be a set of Boolean constraints built from Θ and the Boolean operators. The constraints of Γ are describing the logical priors which are known about the atomic propositions. Then, define $\mathcal{B}_\Gamma(\Theta)$ the Boolean algebra generated by Θ and compliant with the constraints of Γ.[3] The algebra $\mathcal{B}_\Gamma(\Theta)$ will describe the real world.

The real world will be observed by sensors. Let I be the set of the sensors. The sensors of I may be combined by pairs, thus constituting composite sensors. The set of composite sensors, denoted J, is defined recursively as follows:

$$I \subset J \quad \text{and} \quad (i,j) \in J \text{ for any } i,j \in J\,.$$

Among the (composite) sensors of J, it will be assumed that some pairs of sensors are mutually independent.

For $i,j \in J$, the notation $i \times j$ indicates that the sensors i and j are independent. (6.18)

To any sensor $i \in J$ is associated a modal operators $\Box_i$ (for short, the notation $\Box_{i,j}$ will be used instead of $\Box_{(i,j)}$). The modal operators $\Box_i$ will describe logically the information received and interpreted by sensor i. The modal operators will act on any proposition of $\mathcal{B}_\Gamma(\Theta)$. For any $\phi \in \mathcal{B}_\Gamma(\Theta)$, the proposition $\Box_i\phi$ is an interpretation of the real proposition ϕ by the sensor i. It is noticed that $\Box_i\phi$ is not necessarily a proposition of $\mathcal{B}_\Gamma(\Theta)$, so that $\Box_i$ should be considered as an external operator. In order to manipulate these modal propositions, we will consider the Boolean algebra $\mathcal{B}_\Gamma(\Theta,\Box)$ generated by Θ and $\Box_i\phi$ where $\phi \in \mathcal{B}_\Gamma(\Theta)$ and $i \in J$. It is also assumed that the algebra $\mathcal{B}_\Gamma(\Theta,\Box)$ is compliant with the following constraints on the modal propositions:

m.i. $\Box_i\top = \top$, for any $i \in J$,

m.ii. $\Box_i(\neg\phi \vee \psi) \subset (\neg\Box_i\phi \vee \Box_i\psi)$, for any $\phi,\psi \in \mathcal{B}_\Gamma(\Theta)$ and $i \in J$,

m.iii. (optional) $\Box_i\phi \subset \phi$, for any $\phi \in \mathcal{B}_\Gamma(\Theta)$ and $i \in J$,

m.iv. $\Box_i\phi \subset \Box_{i,j}\phi$, for any $\phi \in \mathcal{B}_\Gamma(\Theta)$ and $i,j \in J$,

m.indep. $\Box_{i,j}\phi \subset (\Box_i\phi \vee \Box_j\phi)$, for any $\phi \in \mathcal{B}_\Gamma(\Theta)$ and $i,j \in J$ such that $i \times j$.

Together with the axioms $m.*$, the algebra $\mathcal{B}_\Gamma(\Theta,\Box)$ is a model characterizing our modal logic. It is necessary to explain the signification of these axioms.

Axiom m.i explains that the sensors hold any tautology (trivially true propositions) for true.

Axiom m.ii is the model counterpart of the axiom $\vdash \Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$ of modal logic, which is a *modus ponens* encoded within the modality. In other word, axiom m.ii just says that the sensors make logical deductions. Together, axioms m.i and m.ii express that the sensors are reasoning logically.

Axiom m.iii says that the sensors always say the truth. More precisely, it says that ϕ *is true* whenever *sensor i considers ϕ as true*. This axiom implies the coherence of the sensors. By the way, it is probably stronger than the coherence hypothesis. Axiom m.iii is considered as optional. It will be suppressed in the case of the TBM paradigm, but used otherwise.

[3] $\mathcal{B}_\Gamma(\Theta)$ could be obtained by propagating the constraints of Γ within the free Boolean algebra $\mathcal{B}(\Theta)$.

Axiom m.iv says that the knowledge increases with the number of sensors used. More precisely, m.iv means that the surety of a proposition ϕ is greater according to i, j than according to i only. Although this axiom seems quite natural, it is noticed that it is not necessarily compatible with fusion rules involving a redistribution of the conflict.

Axiom m.indep expresses a logical independence of the sensors. Assuming i, j to be independent sensors (*i.e.* $i \times j$), the axiom m.indep then says that the information obtained from the *joint* sensor (i, j) could be obtained separately from one of the sensors i or j. In other word, there is no possible interaction between the sensors i and j during the observation process.

m.i, m.ii and m.iii are typically the axioms of the system T of modal logic. Before concluding this subsection, some useful logical theorems are now derived.

6.5.2.1 Useful theorems

The following theorems will be deduced without the help of the optional axiom m.iii. The axioms used for each theorem will be indicated at the end of the proof.

The modality is non decreasing. Let $i \in J$ and $\phi, \psi \in \mathcal{B}_\Gamma(\Theta)$. Then:

$$\phi \subset \psi \quad \text{implies} \quad \Box_i\phi \subset \Box_i\psi\,.$$

Proof. $\phi \subset \psi$ implies $\neg\phi \vee \psi = \top$.
By applying axiom m.i, it comes $\Box_i\big(\neg\phi \vee \psi\big) = \top$.
Now, m.ii implies $\Box_i\big(\neg\phi \vee \psi\big) \subset \neg\Box_i\phi \vee \Box_i\psi$.
Consequently $\neg\Box_i\phi \vee \Box_i\psi = \top$.
As a conclusion, $\Box_i\phi \subset \Box_i\psi$.

□□□

The proof requested the axioms m.i and m.ii.

Modality and conjunction. $(\Box_i\phi \wedge \Box_i\psi) = \Box_i(\phi \wedge \psi)$ for any $\phi, \psi \in \mathcal{B}_\Gamma(\Theta)$ and $i \in J$.

Proof.
Proof of $(\Box_i\phi \wedge \Box_i\psi) \subset \Box_i(\phi \wedge \psi)$.
Since $\neg\phi \vee \neg\psi \vee (\phi \wedge \psi) = \top$, axiom m.i implies $\Box_i\big(\neg\phi \vee \neg\psi \vee (\phi \wedge \psi)\big) = \top$.
Now, m.ii implies $\Box_i\big(\neg\phi \vee \neg\psi \vee (\phi \wedge \psi)\big) \subset \neg\Box_i\phi \vee \neg\Box_i\psi \vee \Box_i(\phi \wedge \psi)$.
Consequently $\neg\Box_i\phi \vee \neg\Box_i\psi \vee \Box_i(\phi \wedge \psi) = \top$.
As a conclusion, $(\Box_i\phi \wedge \Box_i\psi) \subset \Box_i(\phi \wedge \psi)$.

Proof of $(\Box_i\phi \wedge \Box_i\psi) \supset \Box_i(\phi \wedge \psi)$.
Since $\Box_i$ is non decreasing, it is proved $\Box_i(\phi \wedge \psi) \subset \Box_i\phi$ and $\Box_i(\phi \wedge \psi) \subset \Box_i\psi$.

□□□

The proof requested the axioms m.i and m.ii.

Modality and disjunction. $(\Box_i\phi \vee \Box_i\psi) \subset \Box_i(\phi \vee \psi)$ for any $\phi, \psi \in \mathcal{B}_\Gamma(\Theta)$ and $i \in J$. In general, $(\Box_i\phi \vee \Box_i\psi) \neq \Box_i(\phi \vee \psi)$.

Proof.

Proof of $(\Box_i\phi \vee \Box_i\psi) \subset \Box_i(\phi \vee \psi)$.
Since $\Box_i$ is non decreasing, it is proved $\Box_i\phi \subset \Box_i(\phi \vee \psi)$ and $\Box_i\psi \subset \Box_i(\phi \vee \psi)$.
Then the result.

A counter-example for $(\Box_i\phi\vee\Box_i\psi) = \Box_i(\phi\vee\psi)$ *needs the construction of a dedicated model of the logic. This construction is outside the scope of the chapter. Readers interested in models constructions for modal logics could refer to [2].*

□□□

The proof requested the axioms m.i and m.ii.

Conjunction of heterogeneous modalities. $(\Box_i\phi\wedge\Box_j\psi) \subset \Box_{i,j}(\phi\wedge\psi)$ for any $\phi, \psi \in \mathcal{B}_\Gamma(\Theta)$ and $i, j \in J$. *In other words, if sensor* i *asserts* ϕ *and sensor* j *asserts* ψ*, then the fused sensor asserts* $\phi \wedge \psi$ *.*

Proof. Axioms m.iv says $\Box_i\phi \subset \Box_{i,j}\phi$ and $\Box_j\psi \subset \Box_{i,j}\psi$.
Since $(\Box_{i,j}\phi \wedge \Box_{i,j}\psi) = \Box_{i,j}(\phi \wedge \psi)$, it is deduced $(\Box_i\phi \wedge \Box_j\psi) \subset \Box_{i,j}(\phi \wedge \psi)$.

□□□

The proof requested the axioms m.i, m.ii and m.iv.

Disjunction of heterogeneous modalities. $(\Box_i\phi \vee \Box_j\phi) \subset \Box_{i,j}\phi$ for any $\phi \in \mathcal{B}_\Gamma(\Theta)$ and $i, j \in J$. *In other words, if sensor* i *or sensor* j *assert* ϕ*, then the fused sensor asserts* ϕ *.*

Proof. Axioms m.iv says $\Box_i\phi \subset \Box_{i,j}\phi$ and $\Box_j\phi \subset \Box_{i,j}\phi$.
Then, the result is immediate.

□□□

The proof requested the axiom m.iv.

The converse of this property is obtained by means of axiom m.indep, when the sensors i, j are independent.

6.5.3 Fusion rule

The purpose of this subsection is to derive logically the fusion rule on $< \Theta >_\Gamma$, the pre-Boolean algebra generated by Θ within the Boolean algebra $\mathcal{B}_\Gamma(\Theta)$.[4] In a first step, the fusion will be derived in a strict logical acceptation, by means of the modal operators. In a second step, the notion of belief is also introduced by means of probabilistic modal propositions. But as a preliminary, we are beginning by introducing the notion of partitions.

6.5.3.1 Preliminary definitions.

Partition. Let $\Pi \subset \mathcal{B}(\Theta)$ be a set of propositions. The set Π is a partition of $\top$ if it satisfies the following properties:

[4]In this case Γ may contain constraints outside $< \Theta >$. But this is the same notion of pre-Boolean algebra discussed earlier.

- The propositions of Π are exclusive (*i.e.* disjoint): $\phi \wedge \psi = \perp$ for any $\phi, \psi \in \Pi$ such that $\phi \neq \psi$,
- The propositions of Π are exhaustive: $\bigvee_{\phi\in\Pi} \phi = \top$.

Notice that Π may contain $\perp$, in this definition.

Partition and complement. Let Π be a partition and let $A \subset \Pi$ and $B \subset \Pi$. Then:

$$\left(\bigvee_{\phi\in A} \phi\right) \setminus \left(\bigvee_{\phi\in B} \phi\right) = \left(\bigvee_{\phi\in A\setminus B} \phi\right) .$$

This property just tells that the Boolean algebra generated by Π is isomorph to the Boolean structure implied by the set $\Pi \setminus \{\perp\}$. The proof of this result is obvious from the definition.

Partitions and probabilities. Partitions are useful since they make possible the definition of a probability by means of elementary density. More precisely, for any partition Π and any subset $A \subset \Pi$, the probability of the proposition $\bigvee_{\phi\in A} \phi$ is given by:

$$p\left(\bigvee_{\phi\in A} \phi\right) = \sum_{\phi\in A} p(\phi) .$$

This property will be particularly useful subsequently for linking the logical fusion to the belief fusion.

Joint partitions. Let Π and Λ be two partitions of $\top$. Let $\Gamma = \{\phi \wedge \psi / \phi \in \Pi \text{ and } \psi \in \Lambda\}$ be the set of joint propositions obtained from Π and Λ. Then Γ is a partition.

Proof. Let $\phi, \phi' \in \Pi$ and $\psi, \psi' \in \Lambda$ be such that $(\phi, \psi) \neq (\phi', \psi')$.

The exclusivity of $(\phi \wedge \psi)$ and $(\phi' \wedge \psi')$ is a direct consequence of:

$$(\phi \wedge \psi) \wedge (\phi' \wedge \psi') = (\phi \wedge \phi') \wedge (\psi \wedge \psi') = \perp .$$

The exhaustivity is derived from:

$$\bigvee_{\phi\in\Pi} \bigvee_{\psi\in\Lambda} (\phi \wedge \psi) = \left(\bigvee_{\phi\in\Pi} \phi\right) \wedge \left(\bigvee_{\psi\in\Lambda} \psi\right) = \top \wedge \top = \top .$$

□□□

Corollary of the proof. $(\phi \wedge \psi) = (\phi' \wedge \psi')$ and $(\phi, \psi) \neq (\phi', \psi')$ imply $(\phi \wedge \psi) = (\phi' \wedge \psi') = \perp$. This corollary will be useful for the computation of $\phi^{(i,j)}$, subsequently.

6.5.3.2 Logical fusion

Definition of the logical fusion. Logically, the information provided by the sensor $i \in J$ is described by the modal propositions $\Box_i \phi$, where $\phi \in <\Theta>_\Gamma$. The propositions of $\mathcal{B}_\Gamma(\Theta) \setminus <\Theta>_\Gamma$ are not considered explicitly, since our *discernment* is restricted to $<\Theta>_\Gamma$.

Let $i, j \in J$ be two sensors which are independent, *i.e.* such that $i \times j$. The fusion of i and j is simply defined as the composite sensor (i, j). Now arises the following issue: *How to characterize the fused information $\Box_{i,j}\phi$ from the primary information $\Box_i \phi$ and $\Box_j \phi$?* In order to solve this question, we introduce first the notion of *basic propositional assignments* which constitute the elementary logical components of the information.

Definition of the basic propositional assignments. Let $i \in J$ be a sensor. The basic propositional assignments (bpa) related to sensor i are the modal propositions $\phi^{(i)}$ defined for any $\phi \in <\Theta>_\Gamma$ by:

$$\phi^{(i)} = \Box_i \phi \setminus \left(\bigvee_{\psi \in <\Theta>_\Gamma : \psi \subsetneq \phi} \Box_i \psi \right) . \tag{6.19}$$

The bpa $\phi^{(i)}$ is the logical information, which sensor i attributes to proposition ϕ intrinsically. The information of $\phi^{(i)}$ cannot be attributed to smaller propositions than ϕ.

Subsequently, the bpas appear as essential tools for characterizing the fusion rule.

Logical properties of the bpa.
Exclusivity. The bpas $\phi^{(i)}$, where $\phi \in <\Theta>_\Gamma$, are exclusive for any given sensor $i \in J$:

$$\forall \phi, \psi \in <\Theta>_\Gamma \,,\; \phi \neq \psi \Rightarrow \phi^{(i)} \wedge \psi^{(i)} = \bot \,. \tag{6.20}$$

Proof. From the definition, it is deduced:

$$\phi^{(i)} \wedge \psi^{(i)} = \Box_i(\phi \wedge \psi) \wedge \left(\bigwedge_{\eta \in <\Theta>_\Gamma : \eta \subsetneq \phi} \neg \Box_i \eta \right) \wedge \left(\bigwedge_{\eta \in <\Theta>_\Gamma : \eta \subsetneq \psi} \neg \Box_i \eta \right) .$$

Since $\phi \wedge \psi \subsetneq \phi$ or $\phi \wedge \psi \subsetneq \psi$ when $\phi \neq \psi$, it comes $\phi^{(i)} \wedge \psi^{(i)} = \bot$.

□□□

Exhaustivity. The bpas $\phi^{(i)}$, where $\phi \in <\Theta>_\Gamma$, are exhaustive for any given sensor $i \in J$:

$$\bigvee_{\psi \in <\Theta>_\Gamma : \psi \subset \phi} \psi^{(i)} = \Box_i \phi \,, \quad \text{and in particular:} \bigvee_{\psi \in <\Theta>_\Gamma} \psi^{(i)} = \top \,. \tag{6.21}$$

Proof. The proof is recursive.
It is first noticed that $\Box_i \bot = \bot^{(i)}$.

Now, let $\phi \in <\Theta>_\Gamma$ and assume $\bigvee_{\eta \subset \psi} \eta^{(i)} = \Box_i \psi$ for any $\psi \subsetneq \phi$.
Then:

$$\bigvee_{\psi \subset \phi} \psi^{(i)} = \phi^{(i)} \vee \left(\bigvee_{\psi \subsetneq \phi} \; \bigvee_{\eta \subset \psi} \eta^{(i)} \right) = \phi^{(i)} \vee \left(\bigvee_{\psi \subsetneq \phi} \Box_i \psi \right) .$$

It follows $\bigvee_{\psi\subset\phi}\psi^{(i)} = \left(\Box_i\phi \setminus \left(\bigvee_{\psi\subsetneq\phi}\Box_i\psi\right)\right) \vee \left(\bigvee_{\psi\subsetneq\phi}\Box_i\psi\right) = \Box_i\phi \vee \left(\bigvee_{\psi\subsetneq\phi}\Box_i\psi\right)$.
Since $\Box_i$ is non decreasing, it is deduced $\bigvee_{\psi\subset\phi}\psi^{(i)} = \Box_i\phi$.

□□□

Partition. Being both disjoint and exhaustive, the bpas $\phi^{(i)}$, where $\phi \in <\Theta>_\Gamma$, constitute a partition of $\top$.

Joint partition. Let $i, j \in J$. The propositions $\phi^{(i)} \wedge \psi^{(j)}$, where $\phi, \psi \in <\Theta>_\Gamma$, constitute a partition of $\top$.

Computing the fusion. Let $i, j \in J$ be such that $i \times j$. Then, the following property holds for any $\phi \in <\Theta>_\Gamma$:

$$\phi^{(i,j)} = \bigvee_{\psi,\eta\in<\Theta>_\Gamma:\psi\wedge\eta=\phi} \left(\psi^{(i)} \wedge \eta^{(j)}\right) . \tag{6.22}$$

Proof.

Lemma.

$$\Box_{i,j}\phi = \bigvee_{\psi\wedge\eta\subset\phi} (\Box_i\psi \wedge \Box_j\eta) = \bigvee_{\psi\wedge\eta\subset\phi} \left(\psi^{(i)} \wedge \eta^{(j)}\right) .$$

Proof of lemma. From the property $\Box_i\psi \wedge \Box_j\eta \subset \Box_{i,j}(\psi\wedge\eta)$ of section 6.5.2.1 and the non decreasing property of $\Box_{i,j}$, it is deduced:

$$\bigvee_{\psi\wedge\eta\subset\phi} (\Box_i\psi \wedge \Box_j\eta) \subset \Box_{i,j}\phi .$$

Now, the axiom m.indep implies $\Box_{i,j}\phi \subset (\Box_i\phi \vee \Box_j\phi)$ and then:

$$\Box_{i,j}\phi \subset \big((\Box_i\phi \wedge \Box_j\top) \vee ((\Box_i\top \wedge \Box_j\phi)\big) .$$

As a consequence, $\bigvee_{\psi\wedge\eta\subset\phi}(\Box_i\psi \wedge \Box_j\eta) = \Box_{i,j}\phi$.

Now, since $\Box_i\psi = \bigvee_{\xi\subset\psi}\xi^{(i)}$ and $\Box_j\eta = \bigvee_{\zeta\subset\eta}\zeta^{(j)}$ (refer to the exhaustivity property), it comes also:

$$\Box_{i,j}\phi = \bigvee_{\psi\wedge\eta\subset\phi}\ \bigvee_{\xi\subset\psi}\ \bigvee_{\zeta\subset\eta} \left(\xi^{(i)} \wedge \zeta^{(j)}\right) = \bigvee_{\xi\wedge\zeta\subset\phi} \left(\xi^{(i)} \wedge \zeta^{(j)}\right) .$$

□□

From the definition of the bpa, it is deduced:

$$\phi^{(i,j)} = \Box_{i,j}\phi \setminus \left(\bigvee_{\psi\subsetneq\phi}\Box_{i,j}\psi\right) = \bigvee_{\eta\wedge\xi\subset\phi}\left(\eta^{(i)} \wedge \xi^{(j)}\right) \setminus \left(\bigvee_{\eta\wedge\xi\subsetneq\phi}\left(\eta^{(i)} \wedge \xi^{(j)}\right)\right) .$$

Now, since the propositions $\eta^{(i)} \wedge \xi^{(j)}$ constitute a partition (and taking into account the *corollary of the proof* in section 6.5.3.1), it comes:

$$\phi^{(i,j)} = \bigvee_{\eta\wedge\xi=\phi}\left(\eta^{(i)} \wedge \xi^{(j)}\right) .$$

□□□

Conclusion. The sensors $i, j \in J$ being independent, the fused sensor (i, j) is computed from i and j accordingly to the following process:

- Build $\phi^{(i)} = \Box_i\phi \setminus \left(\bigvee_{\psi\in<\Theta>_\Gamma:\psi\subsetneq\phi} \Box_i\psi\right) = \Box_i\phi \setminus \left(\bigvee_{\psi\in<\Theta>_\Gamma:\psi\subsetneq\phi} \psi^{(i)}\right)$ and $\phi^{(j)} = \Box_j\phi \setminus \left(\bigvee_{\psi\in<\Theta>_\Gamma:\psi\subsetneq\phi} \Box_j\psi\right) = \Box_j\phi \setminus \left(\bigvee_{\psi\in<\Theta>_\Gamma:\psi\subsetneq\phi} \psi^{(j)}\right)$ for any $\phi \in <\Theta>_\Gamma$,

- Compute $\phi^{(i,j)} = \bigvee_{\eta,\xi\in<\Theta>_\Gamma:\eta\wedge\xi=\phi} \left(\eta^{(i)} \wedge \xi^{(j)}\right)$ for any $\phi \in <\Theta>_\Gamma$,

- Derive $\Box_{i,j}\phi = \bigvee_{\psi\in<\Theta>_\Gamma:\psi\subset\phi} \psi^{(i,j)}$ for any $\phi \in <\Theta>_\Gamma$.

Obviously, this process is almost identical to the computation of the fused belief $\mathrm{Bel}_i \oplus \mathrm{Bel}_j$ in free DSmT or in the TBM paradigm (while including the empty proposition in the definition of the belief function):

- Set $m_i(\phi) = \mathrm{Bel}_i(\phi) - \sum_{\psi\subsetneq\phi} m_i(\psi)$ and $m_j(\phi) = \mathrm{Bel}_j(\phi) - \sum_{\psi\subsetneq\phi} m_j(\psi)$,

- Compute $m_i \oplus m_j(\phi) = \sum_{\eta\wedge\xi=\phi} m_i(\eta)m_j(\xi)$,

- Get back $\mathrm{Bel}_i \oplus \mathrm{Bel}_j(\phi) = \sum_{\psi\subset\phi} m_i \oplus m_j(\psi)$.

It is yet foreseeable that $m_i \oplus m_j(\phi)$ could be interpreted as $p\left(\phi^{(i,j)}\right)$ owing to some additional hypotheses about the *probabilistic independence* of the propositions. This idea will be combined with the entropic maximization method described in section 6.3, resulting in a logically interpreted fusion rule for the evidence theories.

For now, we are discussing about the signification of optional axiom m.iii which has not been used until now.

The consequence of axiom m.iii. Axiom m.iii says $\Box_i\phi \subset \phi$ and in particular implies $\Box_i\bot \subset \bot$ and then $\Box_i\bot = \bot$. Thus, there are two important properties related to m.iii:

- It establishes a comparison of the propositions ϕ and their interpretation $\Box_i\phi$ by means of $\Box_i\phi \subset \phi$,

- It makes the sensors i coherent by implying $\Box_i\bot = \bot$.

By removing m.iii, the incoherence $\Box_i\bot \neq \bot$ is made possible, and this has a fundamental interpretation in term of evidence theories.

- Allowing the incoherence $\Box_i\bot \neq \bot$ is a logical counterpart of the TBM paradigm,

- Hypothesizing the coherence $\Box_i\bot = \bot$ is a logical counterpart of the DSmT or DST paradigm.

Next section establishes the connection between the logical fusion and the belief fusion.

6.5.3.3 From logical to belief fusion

Subsequently, we are assuming that a probability p is defined over the Boolean algebra $\mathcal{B}_\Gamma(\Theta, \Box)$. This probability is known partially by means of the sensors. For any $i \in J$ and any $\phi \in <\Theta>_\Gamma$ are then defined:

- The belief $\mathrm{Bel}_i(\phi) = p(\Box_i\phi)$,
- The basic belief assignment $m_i(\phi) = p\big(\phi^{(i)}\big)$.

For any $i, j \in J$ such that $i \times j$ (independent sensors), the fused bba and belief are defined by:

$$m_i \oplus m_j = m_{i,j} \quad \text{and} \quad \mathrm{Bel}_i \oplus \mathrm{Bel}_j = \mathrm{Bel}_{i,j} \; . \tag{6.23}$$

The propositions $\phi^{(i)}$ constituting a partition of $\top$, the logical property

$$\phi^{(i)} = \Box_i\phi \setminus \left(\bigvee_{\psi \in <\Theta>_\Gamma : \psi \subsetneq \phi} \psi^{(i)} \right) \tag{6.24}$$

implies:

$$m_i(\phi) = \mathrm{Bel}_i(\phi) - \sum_{\psi \subsetneq \phi} m_i(\psi) \; .$$

From the exhaustivity property, *i.e.* $\Box_i\phi = \bigvee_{\psi \subset \phi} \psi^{(i)}$, is derived:

$$\mathrm{Bel}_i(\phi) = \sum_{\psi \subset \phi} m_i(\psi) \; .$$

By the way, two fundamental properties of evidence theories have been recovered from our logical approach. Now, the remaining question is about the fusion rule.

From the definition and the computation of $\phi^{(i,j)}$, it is deduced:

$$m_i \oplus m_j(\phi) = p\big(\phi^{(i,j)}\big) = p\left(\bigvee_{\eta \wedge \xi = \phi} \left(\eta^{(i)} \wedge \xi^{(j)}\right) \right) \; .$$

Since the propositions $\eta^{(i)} \wedge \xi^{(j)}$ are constituting a partition (and owing to the *corollary of the proof* in section 6.5.3.1), it is obtained:

$$m_i \oplus m_j(\phi) = \sum_{\eta \wedge \xi = \phi} p\left(\eta^{(i)} \wedge \xi^{(j)}\right) \; . \tag{6.25}$$

It is not possible to reduce (6.25) anymore, without an additional hypothesis. In order to compute $p\left(\eta^{(i)} \wedge \xi^{(j)}\right)$, the independence of sensors i and j will be again instrumental. But this time, the independence is considered from an entropic viewpoint, and the probabilities $p\left(\eta^{(i)} \wedge \xi^{(j)}\right)$ are computed by maximizing the entropy of p over the propositions $\eta^{(i)} \wedge \xi^{(j)}$. Denoting $\mathcal{P}(\Theta) = \mathcal{P}\big(\mathcal{B}_\Gamma(\Theta, \Box)\big)$ the set of all probabilities over $\mathcal{B}_\Gamma(\Theta, \Box)$, the probabilities $p\left(\eta^{(i)} \wedge \xi^{(j)}\right)$ are obtained by means of the program:

$$\begin{aligned} &p \in \max_{q \in \mathcal{P}(\Theta)} \sum_{\eta, \xi \in <\Theta>_\Gamma} -q\left(\eta^{(i)} \wedge \xi^{(j)}\right) \ln q\left(\eta^{(i)} \wedge \xi^{(j)}\right) \; , \\ &\text{under constraints:} \quad q\left(\phi^{(i)}\right) = m_i(\phi) \quad \text{and} \quad q\left(\phi^{(j)}\right) = m_j(\phi) \quad \text{for any } \phi \in <\Theta>_\Gamma \; . \end{aligned} \tag{6.26}$$

Combining (6.25) and (6.26), it becomes possible to derive $m_i \oplus m_j$ from m_i and m_j. Three different cases arise.

- Axiom m.iii is removed. Then, the fusion rule (6.4) of TBM is recovered,
- Axiom m.iii is used, but $< \Theta >_\Gamma$ verifies the insulation property (6.2). Then, the fusion rule (6.3) of free DSmT is recovered,
- Axiom m.iii is used in the general case. Then, the definition (6.7) of EMR is recovered. Moreover, $\Box_i(\phi) \subset \phi$ implies $\mathrm{Bel}_i(\phi) \leq p(\phi)$, which is exactly the bound hypothesis (6.12). (Notice that the constraints $\mathrm{Bel}_{i,j}(\phi) \leq p(\phi)$ could be discarded from (6.7) because of the belief enhancement property of section 6.4)

The logical justification of rule EMR is now completed.

6.6 Conclusion

In this chapter, a new fusion rule have been defined for evidence theories. This rule is computed in order to maximize the entropy of the joint information. This method provides an adaptive implementation of the independence hypothesis of the sensors. The rule has been tested on typical examples by means of an algorithmic optimization and by means of a direct computation. It has been shown that it does not generate conflicts and is compatible with a probabilistic bound interpretation of the belief function. It is still able to detect truly conflicting sources however, since the optimization may be unfeasible on these cases. At last, a main contribution of this rule is also that it is derived from an interpretation of evidence theories by means of modal logics.

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Chapter 7

DSm models and Non-Archimedean Reasoning

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Abstract: *The Dezert-Smarandache theory of plausible and paradoxical reasoning is based on the premise that some elements θ_i of a frame Θ have a non-empty intersection. These elements are called exhaustive. In number theory, this property is observed only in non-Archimedean number systems, for example, in the ring $\mathbf{Z}_p$ of p-adic integers, in the field ${}^*\mathbf{Q}$ of hyperrational numbers, in the field ${}^*\mathbf{R}$ of hyperreal numbers, etc. In this chapter, I show that non-Archimedean structures are infinite DSm models in that each positive exhaustive element is greater (or less) than each positive exclusive element. Then I consider three principal versions of the non-Archimedean logic: p-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$, hyperrational valued logic $\mathfrak{M}_{^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{^*\mathbf{R}}$, and their applications to plausible reasoning. These logics are constructed for the first time.*

7.1 Introduction

The development of fuzzy logic and fuzziness was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Recall that fuzzy logic was introduced by Lofti Zadeh in 1965 (see [20]) to represent data and information possessing nonstatistical uncertainties. Florentin Smarandache had generalized fuzzy logic and introduced two new concepts (see [16], [18], [17]):

1. neutrosophy as study of neutralities;
2. neutrosophic logic and neutrosophic probability as a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, etc.

Neutrosophy is a new branch of philosophy, which studies the nature of neutralities, as well as their logical applications. This branch represents a version of paradoxism studies. The essence of paradoxism studies is that there is a neutrality for any two extremes. For example, denote by A an idea (or proposition, event, concept), by *Anti-A* the opposite to A. Then there exists a neutrality *Neut-A* and this means that something is neither A nor *Anti-A*. It is readily seen that the paradoxical reasoning can be modeled if some elements θ_i of a frame Θ are not exclusive, but exhaustive, i. e., here θ_i have a non-empty intersection. A mathematical model that has such a property is called the Dezert-Smarandache model (DSm model). A theory of plausible and paradoxical reasoning that studies DSm models is called the Dezert-Smarandache theory (DSmT). It is totally different from those of all existing approaches managing uncertainties and fuzziness. In this chapter, I consider plausible reasoning on the base of particular case of infinite DSm models, namely, on the base of non-Archimedean structures.

Let us remember that Archimedes' axiom is the formula of infinite length that has one of two following notations:

- for any ε that belongs to the interval $[0, 1]$, we have

$$(\varepsilon > 0) \supset [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots], \tag{7.1}$$

- for any positive integer ε, we have

$$[(1 \geq \varepsilon) \vee (1 + 1 \geq \varepsilon) \vee (1 + 1 + 1 \geq \varepsilon) \vee \ldots]. \tag{7.2}$$

Formulas (7.1) and (7.2) are valid in the field $\mathbf{Q}$ of rational numbers and as well as in the field $\mathbf{R}$ of real numbers. In the ring $\mathbf{Z}$ of integers, only formula (7.2) has a nontrivial sense, because $\mathbf{Z}$ doesn't contain numbers of the open interval $(0, 1)$.

Also, Archimedes' axiom affirms the existence of an integer multiple of the smaller of two numbers which exceeds the greater: for any positive real or rational number ε, there exists a positive integer n such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$.

The negation of Archimedes' axiom has one of two following forms:

- there exists ε that belongs to the interval $[0, 1]$ such that

$$(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots], \tag{7.3}$$

- there exists a positive integer ε such that

$$[(1 < \varepsilon) \wedge (1 + 1 < \varepsilon) \wedge (1 + 1 + 1 < \varepsilon) \wedge \ldots]. \tag{7.4}$$

Let us show that (7.3) is the negation of (7.1). Indeed,

$$\begin{gathered}
\neg\forall\varepsilon\,[(\varepsilon > 0) \supset [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] \equiv \\
\exists\varepsilon\,\neg\neg[(\varepsilon > 0) \wedge \neg[(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] \equiv \\
\exists\varepsilon\,(\varepsilon > 0) \wedge [\neg(\varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon + \varepsilon \geq 1) \wedge \ldots]] \equiv \\
\exists\varepsilon\,(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots]]
\end{gathered}$$

It is obvious that formula (7.3) says that there exist *infinitely small numbers* (or *infinitesimals*), i. e., numbers that are smaller than all real or rational numbers of the open interval $(0, 1)$. In other words, ε is said to be an infinitesimal if and only if, for all positive integers n, we have $|\varepsilon| < \frac{1}{n}$. Further, formula (7.4) says that there exist *infinitely large integers* that are greater than all positive integers. Infinitesimals and infinitely large integers are called *nonstandard numbers* or *actual infinities*.

The field that satisfies all properties of $\mathbf{R}$ without Archimedes' axiom is called the field of *hyperreal numbers* and it is denoted by ${}^*\mathbf{R}$. The field that satisfies all properties of $\mathbf{Q}$ without Archimedes' axiom is called the field of hyperrational numbers and it is denoted by ${}^*\mathbf{Q}$. By definition of field, if $\varepsilon \in \mathbf{R}$ (respectively $\varepsilon \in \mathbf{Q}$), then $1/\varepsilon \in \mathbf{R}$ (respectively $1/\varepsilon \in \mathbf{Q}$). Therefore ${}^*\mathbf{R}$ and ${}^*\mathbf{Q}$ contain simultaneously infinitesimals and infinitely large integers: for an infinitesimal ε, we have $N = \frac{1}{\varepsilon}$, where N is an infinitely large integer.

The ring that satisfies all properties of $\mathbf{Z}$ without Archimedes' axiom is called the ring of hyperintegers and it is denoted by ${}^*\mathbf{Z}$. This ring includes infinitely large integers. Notice that there exists a version of ${}^*\mathbf{Z}$ that is called the ring of p-adic integers and is denoted by $\mathbf{Z}_p$.

I shall show in this chapter that nonstandard numbers (actual infinities) are exhaustive elements (see section 7.3). This means that their intersection isn't empty with some other elements. Therefore non-Archimedean structures of the form ${}^*\mathbf{S}$ (where we obtain ${}^*\mathbf{S}$ on the base of the set $\mathbf{S}$ of exclusive elements) are particular case of the DSm model. These structures satisfy the properties:

1. all members of $\mathbf{S}$ are exclusive and $\mathbf{S} \subset {}^*\mathbf{S}$,

2. all members of ${}^*\mathbf{S} \backslash \mathbf{S}$ are exhaustive,

3. if a member a is exhaustive, then there exists a exclusive member b such that $a \cap b \neq \emptyset$,

4. there exist exhaustive members a, b such that $a \cap b \neq \emptyset$,

5. each positive exhaustive member is greater (or less) than each positive exclusive member.

I shall consider three principal versions of the logic on non-Archimedean structures: hyperrational valued logic $\mathfrak{M}_{{}^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{{}^*\mathbf{R}}$, p-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$, and their applications to plausible and fuzzy reasoning.

7.2 Standard many-valued logics

Let us remember that a *first-order logical language* $\mathcal{L}$ consists of the following symbols:

1. Variables:

   (i) Free variables: $a_0, a_1, a_2, \ldots, a_j, \ldots$ $(j \in \omega)$

   (ii) Bound variables: $x_0, x_1, x_2, \ldots, x_j, \ldots$ $(j \in \omega)$

2. Constants:

   (i) Function symbols of arity i ($i \in \omega$): $F_0^i, F_1^i, F_2^i, \ldots, F_j^i, \ldots$ ($j \in \omega$). Nullary function symbols are called constants.

   (ii) Predicate symbols of arity i ($i \in \omega$): $P_0^i, P_1^i, P_2^i, \ldots, P_j^i, \ldots$ ($j \in \omega$).

3. Logical symbols:

   (i) Propositional connectives of arity n_j : $\Box_0^{n_0}, \Box_1^{n_1}, \ldots, \Box_r^{n_r}$, which are built by superposition of negation $\neg$ and implication $\supset$.

   (ii) Quantifiers: $\mathrm{Q}_0, \mathrm{Q}_1, ..., \mathrm{Q}_q$.

4. Auxiliary symbols: (,), and , (comma).

Terms are inductively defined as follows:

1. Every individual constant is a term.

2. Every free variable (and every bound variable) is a term.

3. If F^n is a function symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $F^n(t_1, \ldots, t_n)$ is a term.

Formulas are inductively defined as follows:

1. If P^n is a predicate symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a formula. It is called atomic or an atom. It has no outermost logical symbol.

2. If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are formulas and $\Box^n$ is a propositional connective of arity n, then $\Box^n(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a formula with outermost logical symbol $\Box^n$.

3. If φ is a formula not containing the bound variable x, a is a free variable and Q is a quantifier, then $\mathrm{Q}x\varphi(x)$, where $\varphi(x)$ is obtained from φ by replacing a by x at every occurrence of a in φ, is a formula. Its outermost logical symbol is Q.

A formula is called *open* if it contains free variables, and *closed* otherwise. A formula without quantifiers is called *quantifier-free*. We denote the set of formulas of a language $\mathcal{L}$ by L. We will write $\varphi(x)$ for a formula possibly containing the bound variable x, and $\varphi(a)$ respectively $\varphi(t)$ for the formula obtained from φ by replacing every occurrence of the variable x by the free variable a respectively the term t. Hence, we shall need meta-variables for the symbols of a language $\mathcal{L}$. As a notational convention we use letters $\varphi, \phi, \psi, \ldots$ to denote formulas.

A *matrix*, or *matrix logic*, $\mathfrak{M}$ for a language $\mathcal{L}$ is given by:

1. a non-empty set of truth values V of cardinality $|V| = m$,

2. a subset $D \subseteq V$ of designated truth values,

3. an algebra with domain V of appropriate type: for every n-place connective $\Box$ of $\mathcal{L}$ there is an associated truth function f: $V^n \mapsto V$, and

4. for every quantifier Q, an associated truth function $\widetilde{\mathrm{Q}}$: $\wp(V)\backslash\emptyset \mapsto V$

Notice that a truth function for quantifiers is a mapping from non-empty sets of truth values to truth values: for a non-empty set $M \subseteq V$, a quantified formula $\mathrm{Q}x\varphi(x)$ takes the truth value $\widetilde{\mathrm{Q}}(M)$ if, for every truth value $v \in V$, it holds that $v \in M$ iff there is a domain element d such that the truth value of φ in this point d is v (all relative to some interpretation). The set M is called the distribution of φ. For example, suppose that there are only the universal quantifier $\forall$ and the existential quantifier $\exists$ in $\mathcal{L}$. Further, we have the set of truth values $V = \{\top, \bot\}$, where $\bot$ is false and $\top$ is true, i. e., the set of designated truth values $D = \{\top\}$. Then we define the truth functions for the quantifiers $\forall$ and $\exists$ as follows:

1. $\widetilde{\forall}(\{\top\}) = \top$
2. $\widetilde{\forall}(\{\top, \bot\}) = \widetilde{\forall}(\{\bot\}) = \bot$
3. $\widetilde{\exists}(\{\bot\}) = \bot$
4. $\widetilde{\exists}(\{\top, \bot\}) = \widetilde{\exists}(\{\top\}) = \top$

Also, a matrix logic $\mathfrak{M}$ for a language $\mathcal{L}$ is an algebraic system denoted

$$\mathfrak{M} = <V, f_0, f_1, \ldots, f_r, \widetilde{\mathrm{Q}}_0, \widetilde{\mathrm{Q}}_1, \ldots, \widetilde{\mathrm{Q}}_q, D>$$

where

1. V is a non-empty set of truth values for well-formed formulas of $\mathcal{L}$,
2. $f_0, f_1, \ldots, f_r$ are a set of matrix operations defined on the set V and assigned to corresponding propositional connectives $\square_0^{n_0}, \square_1^{n_1}, \ldots, \square_r^{n_r}$ of $\mathcal{L}$,
3. $\widetilde{\mathrm{Q}}_0, \widetilde{\mathrm{Q}}_1, \ldots, \widetilde{\mathrm{Q}}_q$ are a set of matrix operations defined on the set V and assigned to corresponding quantifiers $\mathrm{Q}_0, \mathrm{Q}_1, ..., \mathrm{Q}_q$ of $\mathcal{L}$,
4. D is a set of designated truth values such that $D \subseteq V$.

Now consider $(n+1)$*-valued Łukasiewicz's matrix logic* $\mathfrak{M}_{n+1}$ defined as the ordered system $<V_{n+1}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{n\}>$ for any $n \geqslant 2$, $n \in \mathbf{N}$, where

1. $V_{n+1} = \{0, 1, \ldots, n\}$,
2. for all $x \in V_{n+1}$, $\neg x = n - x$,
3. for all $x, y \in V_{n+1}$, $x \supset y = \min(n, n - x + y)$,
4. for all $x, y \in V_{n+1}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{n+1}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{n+1}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{n+1}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{n\}$ is the set of designated truth values.

The truth value $0 \in V_{n+1}$ is false, the truth value $n \in V_{n+1}$ is true, and other truth values $x \in V_{n+1}$ are neutral.

The ordered system $<V_{\mathbf{Q}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}>$ is called *rational valued Łukasiewicz's matrix logic* $\mathfrak{M}_{\mathbf{Q}}$, where

1. $V_{\mathbf{Q}} = \{x \colon x \in \mathbf{Q}\} \cap [0, 1]$,
2. for all $x \in V_{\mathbf{Q}}$, $\neg x = 1 - x$,
3. for all $x, y \in V_{\mathbf{Q}}$, $x \supset y = \min(1, 1 - x + y)$,
4. for all $x, y \in V_{\mathbf{Q}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{\mathbf{Q}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{Q}}$ is false, the truth value $1 \in V_{\mathbf{Q}}$ is true, and other truth values $x \in V_{\mathbf{Q}}$ are neutral.

Real valued Łukasiewicz's matrix logic $\mathfrak{M}_{\mathbf{R}}$ is the ordered system $<V_{\mathbf{R}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}>$, where

1. $V_{\mathbf{R}} = \{x \colon x \in \mathbf{R}\} \cap [0, 1]$,
2. for all $x \in V_{\mathbf{R}}$, $\neg x = 1 - x$,
3. for all $x, y \in V_{\mathbf{R}}$, $x \supset y = \min(1, 1 - x + y)$,
4. for all $x, y \in V_{\mathbf{R}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. for all $x, y \in V_{\mathbf{R}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{R}}$ is false, the truth value $1 \in V_{\mathbf{R}}$ is true, and other truth values $x \in V_{\mathbf{R}}$ are neutral.

Notice that the elements of truth value sets V_{n+1}, $V_{\mathbf{Q}}$, and $V_{\mathbf{R}}$ are exclusive: for any members x, y we have $x \cap y = \emptyset$. Therefore Łukasiewicz's logics are based on the premise of existence *Shafer's model*. In other words, these logics are built on the families of exclusive elements (see [15], [14]).

However, for a wide class of fusion problems, “the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated” (see [19]). This means that if some elements θ_i of a frame Θ have non-empty intersection, then sources of evidence don't provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of Θ (see [3], [4]).

7.3 Many-valued logics on DSm models

Definition 1. *A many-valued logic is said to be a many-valued logic on DSm model if some elements of its set V of truth values are not exclusive, but exhaustive.*

Recall that a DSm model (*Dezert-Smarandache model*) is formed as a hyper-power set. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements. The *hyper-power set* D^Θ is defined as the set of all composite propositions built from elements of Θ with $\cap$ and $\cup$ operators such that:

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$;
2. if $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$;
3. no other elements belong to D^Θ, except those obtained by using rules 1 or 2.

The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i. e. $|\Theta| = n$. Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$, we call D^Θ the hyper-power set of Θ. Also, D^Θ constitutes what is called the *DSm model* $\mathcal{M}^f(\Theta)$. However elements θ_i can be truly exclusive. In such case, the hyper-power set D^Θ reduces naturally to the classical power set 2^Θ and this constitutes the most restricted hybrid DSm model, denoted by $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an example, suppose that $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, where θ_1 and θ_2 are truly exclusive (i. e., Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = 2^\Theta$.

Now let us show that every non-Archimedean structure is an infinite DSm model, but no vice versa. The most easy way of setting non-Archimedean structures was proposed by Abraham Robinson in [13]. Consider a set Θ consisting only of exclusive members. Let I be any infinite index set. Then we can construct an indexed family Θ^I, i. e., we can obtain the set of all functions: $f\colon I \mapsto \Theta$ such that $f(\alpha) \in \Theta$ for any $\alpha \in I$.

A *filter* $\mathcal{F}$ on the index set I is a family of sets $\mathcal{F} \subset \wp(I)$ for which:

1. $A \in \mathcal{F}$, $A \subset B \Rightarrow B \in \mathcal{F}$;
2. $A_1, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcap\limits_{k=1}^{n} A_k \in \mathcal{F}$;
3. $\emptyset \notin \mathcal{F}$.

The set of all complements for finite subsets of I is a filter and it is called a *Frechet filter*. A maximal filter (ultrafilter) that contains a Frechet filter is called a *Frechet ultrafilter* and it is denoted by $\mathcal{U}$.

Let $\mathcal{U}$ be a Frechet ultrafilter on I. Define a new relation $\backsim$ on the set Θ^I by

$$f \backsim g \equiv \{\alpha \in I \colon f(\alpha) = g(\alpha)\} \in \mathcal{U}. \tag{7.5}$$

It is easily be proved that the relation $\backsim$ is an equivalence. Notice that formula (7.5) means that f and g are equivalent iff f and g are equal on an infinite index subset. For each $f \in \Theta^I$ let $[f]$ denote the equivalence class of f under $\backsim$. The *ultrapower* $\Theta^I/\mathcal{U}$ is then defined to be the set of all equivalence classes $[f]$ as f ranges over Θ^I:

$$\Theta^I/\mathcal{U} \triangleq \{[f] \colon f \in \Theta^I\}.$$

Also, Robinson has proved that each non-empty set Θ has an ultrapower with respect to a Frechet ultrafilter $\mathcal{U}$. This ultrapower $\Theta^I/\mathcal{U}$ is said to be a *proper nonstandard extension* of Θ and it is denoted by $^*\Theta$.

Proposition 1. *Let X be a non-empty set. A nonstandard extension of X consists of a mapping that assigns a set $^*A$ to each $A \subseteq X^m$ for all $m \geq 0$, such that $^*X$ is non-empty and the following conditions are satisfied for all $m, n \geq 0$:*

1. *The mapping preserves Boolean operations on subsets of X^m: if $A \subseteq X^m$, then $^*A \subseteq (^*X)^m$; if $A, B \subseteq X^m$, then $^*(A \cap B) = (^*A \cap {}^*B), {}^*(A \cup B) = (^*A \cup {}^*B)$, and $^*(A \backslash B) = (^*A) \backslash (^*B)$.*

2. *The mapping preserves Cartesian products: if $A \subseteq X^m$ and $B \subseteq X^n$, then $^*(A \times B) = {}^*A \times {}^*B$, where $A \times B \subseteq X^{m+n}$.* □

This proposition is proved in [5].

Recall that each element of $^*\Theta$ is an equivalence class $[f]$ as $f \colon I \mapsto \Theta$. There exist two groups of members of $^*\Theta$ (see Fig. 7.1):

1. functions that are constant, e. g., $f(\alpha) = m \in \Theta$ for infinite index subset $\{\alpha \in I\}$. A constant function $[f = m]$ is denoted by $^*m$,

2. functions that aren't constant.

The set of all constant functions of $^*\Theta$ is called *standard set* and it is denoted by $^\sigma\Theta$. The members of $^\sigma\Theta$ are called *standard*. It is readily seen that $^\sigma\Theta$ and Θ are isomorphic: $^\sigma\Theta \simeq \Theta$.

The following proposition can be easily proved:

Proposition 2. *For any set Θ such that $|\Theta| \geq 2$, there exists a proper nonstandard extension $^*\Theta$ for which $^*\Theta \backslash {}^\sigma\Theta \neq \emptyset$.*

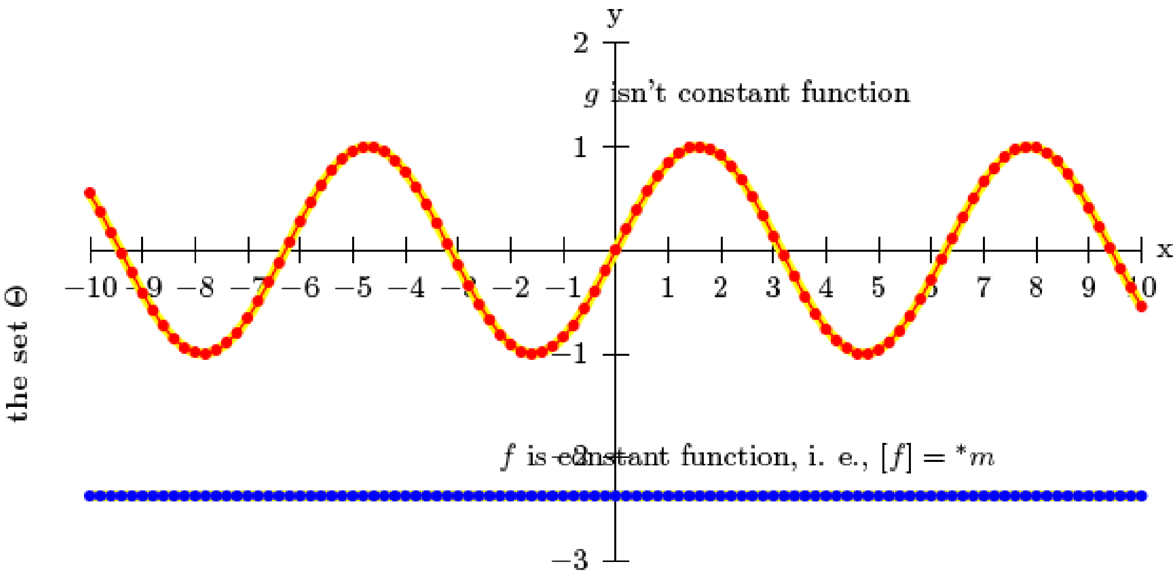

Figure 7.1: The members of $^*\Theta$: constant and non-constant functions.

Proof. Let $I_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \subset I$ be an infinite set and let $\mathcal{U}$ be a Frechet ultrafilter. Suppose that $\Theta_1 = \{m_1, \ldots, m_n\}$ such that $|\Theta_1| \geq 1$ is the subset of Θ and there is a mapping:

$$f(\alpha) = \begin{cases} m_k & \text{if } \alpha = \alpha_k; \\ m_0 \in \Theta & \text{if } \alpha \in I \backslash I_1 \end{cases}$$

and $f(\alpha) \neq m_k$ if $\alpha = \alpha_k \mod (n+1)$, $k = 1, \ldots, n$ and $\alpha \neq \alpha_k$.

Show that $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. The proof is by reductio ad absurdum. Suppose there is $m \in \Theta$ such that $m \in [f(\alpha)]$. Consider the set:

$$I_2 = \{\alpha \in I \colon f(\alpha) = m\} = \begin{cases} \{\alpha_k\} & \text{if } m = m_k,\ k = 1, \ldots, n; \\ I \backslash I_1 & \text{if } m = m_0. \\ \emptyset & \text{if } m \notin \{m_0, m_1, \ldots, m_n\}. \end{cases}$$

In any case $I_2 \notin \mathcal{U}$, because $\{\alpha_k\} \notin \mathcal{U}$, $\emptyset \notin \mathcal{U}$, $I \backslash I_1 \notin \mathcal{U}$. Thus, $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. □

The standard members of $^*\Theta$ are exclusive, because their intersections are empty. Indeed, the members of Θ were exclusive, therefore the members of $^\sigma\Theta$ are exclusive too. However the members of $^*\Theta \backslash {}^\sigma\Theta$ are exhaustive. By definition, if a member $a \in {}^*\Theta$ is nonstandard, then there exists a standard member $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$ (for example, see the proof of proposition 2). We can also prove that there exist exhaustive members $a \in {}^*\Theta$, $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$.

Proposition 3. *There exist two inconstant functions f_1, f_2 such that the intersection of f_1, f_2 isn't empty.*

Proof. Let $f_1 \colon I \mapsto \Theta$ and $f_2 \colon I \mapsto \Theta$. Suppose that $[f_i \neq n]$, $\forall n \in \Theta$, $i = 1, 2$, i. e., f_1, f_2 aren't constant. By proposition 2, these functions are nonstandard members of $^*\Theta$. Further consider an indexed family $F(\alpha)$ for all $\alpha \in I$ such that $\{\alpha \in I \colon f_i(\alpha) \in F(\alpha)\} \in \mathcal{U} \equiv [f_i] \in B$ as $i = 1, 2$.

Consequently it is possible that, for some $\alpha_j \in I$, $f_1(\alpha_j) \cap f_2(\alpha_j) = n_j$ and $n_j \in F(\alpha_j)$. $\square$

Thus, **non-Archimedean structures are infinite DSm-models, because these contain exhaustive members**. In next sections, we shall consider the following non-Archimedean structures:

1. the nonstandard extension ${}^*\mathbf{Q}$ (called the field of hyperrational numbers),
2. the nonstandard extension ${}^*\mathbf{R}$ (called the field of hyperreal numbers),
3. the nonstandard extension $\mathbf{Z}_p$ (called the ring of p-adic integers) that we obtain as follows. Let the set $\mathbf{N}$ of natural numbers be the index set and let $\Theta = \{0, \dots, p-1\}$. Then the nonstandard extension $\Theta^{\mathbf{N}} \backslash \mathcal{U} = \mathbf{Z}_p$.

Further, we shall set the following logics on non-Archimedean structures: hyperrational valued logic $\mathfrak{M}_{{}^*\mathbf{Q}}$, hyperreal valued logic $\mathfrak{M}_{{}^*\mathbf{R}}$, p-adic valued logic $\mathfrak{M}_{\mathbf{Z}_p}$. Note that these many-valued logics are the particular cases of logics on DSm models.

7.4 Hyper-valued Reasoning

7.4.1 Hyper-valued matrix logics

Assume that ${}^*\mathbf{Q}_{[0,1]} = \mathbf{Q}_{[0,1]}^{\mathbf{N}}/\mathcal{U}$ is a nonstandard extension of the subset $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$ of rational numbers and ${}^\sigma\mathbf{Q}_{[0,1]} \subset {}^*\mathbf{Q}_{[0,1]}$ is the subset of standard members. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{Q}_{[0,1]}$:

1. for rational numbers x, $y \in \mathbf{Q}_{[0,1]}$ we have $x \leq y$ in $\mathbf{Q}_{[0,1]}$ iff $[f] \leq [g]$ in ${}^*\mathbf{Q}_{[0,1]}$, where $\{\alpha \in \mathbf{N} \colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N} \colon g(\alpha) = y\} \in \mathcal{U}$,

   i. e., f and g are constant functions such that $[f] = {}^*x$ and $[g] = {}^*y$,
2. each positive rational number ${}^*x \in {}^\sigma\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$,

   i. e., ${}^*x > [f]$ for any positive $x \in \mathbf{Q}_{[0,1]}$ and $[f] \in {}^*\mathbf{Q}_{[0,1]}$, where $[f]$ isn't constant function.

These conditions have the following informal sense:

1. The sets ${}^\sigma\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have isomorphic order structure.
2. The set ${}^*\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^\sigma\mathbf{Q}_{[0,1]}$.

Define this partial order structure on ${}^*\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{{}^*\mathbf{Q}}$ 1. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

and $[f] \neq [g]$, i. e., $\{\alpha \in \mathbf{N} \colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.

3. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

This ordering relation is not linear, but partial, because there exist elements $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, which are incompatible.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{Q}}$:

1. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
2. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\max([f],[g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
3. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = \max([f],[g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
4. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\min([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N} \colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$
5. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\max([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N} \colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

Note there exist the maximal number ${}^*1 \in {}^*\mathbf{Q}_{[0,1]}$ and the minimal number ${}^*0 \in {}^*\mathbf{Q}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{Q}}$. Therefore, for any $[f] \in {}^*\mathbf{Q}_{[0,1]}$, we have: $\max({}^*1, [f]) = {}^*1$, $\max({}^*0, [f]) = [f]$, $\min({}^*1, [f]) = [f]$ and $\min({}^*0, [f]) = {}^*0$.

Now define *hyperrational-valued matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$:

Definition 2. *The ordered system* $<V_{^*\mathbf{Q}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{{}^*1\}>$ *is called hyperrational valued matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$*, where*

1. $V_{^*\mathbf{Q}} = {}^*\mathbf{Q}_{[0,1]}$ *is the subset of hyperrational numbers,*
2. *for all* $x \in V_{^*\mathbf{Q}}$, $\neg x = {}^*1 - x$,
3. *for all* $x, y \in V_{^*\mathbf{Q}}$, $x \supset y = \min({}^*1, {}^*1 - x + y)$,
4. *for all* $x, y \in V_{^*\mathbf{Q}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. *for all* $x, y \in V_{^*\mathbf{Q}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,
7. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,
8. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{Q}}$ is false, the truth value $^*1 \in V_{^*\mathbf{Q}}$ is true, and other truth values $x \in V_{^*\mathbf{Q}}$ are neutral.

Let us consider a nonstandard extension $^*\mathbf{R}_{[0,1]} = \mathbf{R}_{[0,1]}^{\mathbf{N}}/\mathcal{U}$ for the subset $\mathbf{R}_{[0,1]} = \mathbf{R} \cap [0,1]$ of real numbers. Let $^\sigma\mathbf{R}_{[0,1]} \subset {}^*\mathbf{R}_{[0,1]}$ be the subset of standard members. We can extend the usual order structure on $\mathbf{R}_{[0,1]}$ to a partial order structure on $^*\mathbf{R}_{[0,1]}$:

1. for real numbers x, $y \in \mathbf{R}_{[0,1]}$ we have $x \leq y$ in $\mathbf{R}_{[0,1]}$ iff $[f] \leq [g]$ in $^*\mathbf{R}_{[0,1]}$, where $\{\alpha \in \mathbf{N}\colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N}\colon g(\alpha) = y\} \in \mathcal{U}$,
2. each positive real number $^*x \in {}^\sigma\mathbf{R}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{R}_{[0,1]} \backslash {}^\sigma\mathbf{R}_{[0,1]}$,

As before, these conditions have the following informal sense:

1. The sets $^\sigma\mathbf{R}_{[0,1]}$ and $\mathbf{R}_{[0,1]}$ have isomorphic order structure.
2. The set $^*\mathbf{R}_{[0,1]}$ contains actual infinities that are less than any positive real number of $^\sigma\mathbf{R}_{[0,1]}$.

Define this partial order structure on $^*\mathbf{R}_{[0,1]}$ as follows:

$\mathcal{O}_{^*\mathbf{R}}$ 1. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

and $[f] \neq [g], i.e., \{\alpha \in \mathbf{N}\colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.

3. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{R}}$:

1. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f], [g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
2. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\max([f], [g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
3. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f], [g]) = \max([f], [g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,
4. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\min([f], [g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that

$$\{\alpha \in \mathbf{N}\colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

5. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\max([f], [g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that

$$\{\alpha \in \mathbf{N}\colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

Note there exist the maximal number $^*1 \in {}^*\mathbf{R}_{[0,1]}$ and the minimal number $^*0 \in {}^*\mathbf{R}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{R}}$.

As before, define *hyperreal valued matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$:

Definition 3. *The ordered system* $<V_{^*\mathbf{R}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{^*1\}>$ *is called hyperreal valued matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$*, where*

1. $V_{^*\mathbf{R}} = {}^*\mathbf{R}_{[0,1]}$ *is the subset of hyperreal numbers,*
2. *for all* $x \in V_{^*\mathbf{R}}$, $\neg x = {}^*1 - x$,
3. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \supset y = \min({}^*1, {}^*1 - x + y)$,
4. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,
5. *for all* $x, y \in V_{^*\mathbf{R}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,
6. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,
7. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,
8. $\{^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{R}}$ is false, the truth value $^*1 \in V_{^*\mathbf{R}}$ is true, and other truth values $x \in V_{^*\mathbf{R}}$ are neutral.

7.4.2 Hyper-valued probability theory and hyper-valued fuzzy logic

Let X be an arbitrary set and let $\mathcal{A}$ be an *algebra of subsets* $A \subset X$, i. e.

1. union, intersection, and difference of two subsets of X also belong to $\mathcal{A}$;
2. $\emptyset, X$ belong to $\mathcal{A}$.

Recall that a *finitely additive probability measure* is a nonnegative set function $\mathbf{P}(\cdot)$ defined for sets $A \in \mathcal{A}$ that satisfies the following properties:

1. $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{A}$,
2. $\mathbf{P}(X) = 1$ and $\mathbf{P}(\emptyset) = 0$,
3. if $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$. In particular $\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$ for all $A \in \mathcal{A}$.

The algebra $\mathcal{A}$ is called a *σ-algebra* if it is assumed to be closed under countable union (or equivalently, countable intersection), i. e. if for every n, $A_n \in \mathcal{A}$ causes $A = \bigcup_n A_n \in \mathcal{A}$.

A set function $\mathbf{P}(\cdot)$ defined on a σ-algebra is called a *countable additive probability measure* (or a *σ-additive probability measure*) if in addition to satisfying equations of the definition of finitely additive probability measure, it satisfies the following countable additivity property: for any sequence of pairwise disjoint sets A_n, $\mathbf{P}(A) = \sum_n \mathbf{P}(A_n)$. The ordered system $(X, \mathcal{A}, \mathbf{P})$ is called a *probability space*.

Now consider hyper-valued probabilities. Let I be an arbitrary set, let $\mathcal{A}$ be an algebra of subsets $A \subset I$, and let $\mathcal{U}$ be a Frechet ultrafilter on I. Set for $A \in \mathcal{A}$:

$$\mu_{\mathcal{U}}(A) = \begin{cases} 1, & A \in \mathcal{U}; \\ 0, & A \notin \mathcal{U}. \end{cases}$$

Hence, there is a mapping $\mu_{\mathcal{U}} \colon \mathcal{A} \mapsto \{0, 1\}$ satisfying the following properties:

1. $\mu_{\mathcal{U}}(\emptyset) = 0$, $\mu_{\mathcal{U}}(I) = 1$;

2. if $\mu_{\mathcal{U}}(A_1) = \mu_{\mathcal{U}}(A_2) = 0$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = 0$;

3. if $A_1 \cap A_2 = \emptyset$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = \mu_{\mathcal{U}}(A_1) + \mu_{\mathcal{U}}(A_2)$.

This implies that $\mu_{\mathcal{U}}$ is a probability measure. Notice that $\mu_{\mathcal{U}}$ isn't σ-additive. As an example, if A is the set of even numbers and B is the set of odd numbers, then $A \in \mathcal{U}$ implies $B \notin \mathcal{U}$, because the filter $\mathcal{U}$ is maximal. Thus, $\mu_{\mathcal{U}}(A) = 1$ and $\mu_{\mathcal{U}}(B) = 0$, although the cardinalities of A and B are equal.

Definition 4. *The ordered system $(I, \mathcal{A}, \mu_{\mathcal{U}})$ is called a probability space.*

Let's consider a mapping: $f \colon I \ni \alpha \mapsto f(\alpha) \in M$. Two mappings f, g are equivalent: $f \backsim g$ if $\mu_{\mathcal{U}}(\{\alpha \in I \colon f(\alpha) = g(\alpha)\}) = 1$. An equivalence class of f is called a probabilistic events and is denoted by $[f]$. The set ${}^*M$ is the *set of all probabilistic events* of M. This ${}^*M$ is a proper nonstandard extension defined above.

Under condition 1 of proposition 1, we can obtain a nonstandard extension of an algebra $\mathcal{A}$ denoted by ${}^*\mathcal{A}$. Let ${}^*X$ be an arbitrary nonstandard extension. Then the nonstandard algebra ${}^*\mathcal{A}$ is an algebra of subsets $A \subset {}^*X$ if the following conditions hold:

1. union, intersection, and difference of two subsets of ${}^*X$ also belong to ${}^*\mathcal{A}$;

2. $\emptyset, {}^*X$ belong to ${}^*\mathcal{A}$.

Definition 5. *A hyperrational (respectively hyperreal) valued finitely additive probability measure is a nonnegative set function ${}^*\mathbf{P} \colon {}^*\mathcal{A} \mapsto V_{{}^*\mathbf{Q}}$ (respectively ${}^*\mathbf{P} \colon {}^*\mathcal{A} \mapsto V_{{}^*\mathbf{R}}$) that satisfies the following properties:*

1. ${}^*\mathbf{P}(A) \geq {}^*0$ *for all* $A \in {}^*\mathcal{A}$,

2. ${}^*\mathbf{P}({}^*X) = {}^*1$ *and* ${}^*\mathbf{P}(\emptyset) = {}^*0$,

3. *if* $A \in {}^*\mathcal{A}$ *and* $B \in {}^*\mathcal{A}$ *are disjoint, then* ${}^*\mathbf{P}(A \cup B) = {}^*\mathbf{P}(A) + {}^*\mathbf{P}(B)$. *In particular* ${}^*\mathbf{P}(\neg A) = {}^*1 - {}^*\mathbf{P}(A)$ *for all* $A \in {}^*\mathcal{A}$.

Now consider hyper-valued fuzzy logic.

Definition 6. *Suppose ${}^*X$ is a nonstandard extension. Then a hyperrational (respectively hyperreal) valued fuzzy set A in ${}^*X$ is a set defined by means of the membership function ${}^*\mu_A$: ${}^*X \mapsto V_{{}^*\mathbf{Q}}$ (respectively by means of the membership function ${}^*\mu_A$: ${}^*X \mapsto V_{{}^*\mathbf{R}}$).*

A set $A \subset {}^*X$ is called *crisp* if ${}^*\mu_A(u) = {}^*1$ or ${}^*\mu_A(u) = {}^*0$ for any $u \in {}^*X$.

The logical operations on hyper-valued fuzzy sets are defined as follows:

1. ${}^*\mu_{A\cap B}(x) = \min({}^*\mu_A(x), {}^*\mu_B(x))$;
2. ${}^*\mu_{A\cup B}(x) = \max({}^*\mu_A(x), {}^*\mu_B(x))$;
3. ${}^*\mu_{A+B}(x) = {}^*\mu_A(x) + {}^*\mu_B(x) - {}^*\mu_A(x) \cdot {}^*\mu_B(x)$;
4. ${}^*\mu_{\neg A}(x) = \neg {}^*\mu_A(x) = {}^*1 - {}^*\mu_A(x)$.

7.5 p-Adic Valued Reasoning

Let us remember that the expansion

$$n = \alpha_{-N} \cdot p^{-N} + \alpha_{-N+1} \cdot p^{-N+1} + \ldots + \alpha_{-1} \cdot p^{-1} + \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k,$$

where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$, is called the *canonical expansion of p-adic number n* (or p-adic expansion for n). The number n is called p-adic. This number can be identified with sequences of digits: $n = \ldots \alpha_2\alpha_1\alpha_0, \alpha_{-1}\alpha_{-2} \ldots \alpha_{-N}$. We denote the set of such numbers by $\mathbf{Q}_p$.

The expansion $n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum\limits_{k=0}^{\infty} \alpha_k \cdot p^k$, where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{N} \cup \{0\}$, is called the *expansion of p-adic integer n*. The integer n is called p-adic. This number sometimes has the following notation: $n = \ldots \alpha_3\alpha_2\alpha_1\alpha_0$. We denote the set of such numbers by $\mathbf{Z}_p$.

If $n \in \mathbf{Z}_p$, $n \neq 0$, and its canonical expansion contains only a finite number of nonzero digits α_j, then n is natural number (and vice versa). But if $n \in \mathbf{Z}_p$ and its expansion contains an infinite number of nonzero digits α_j, then n is an infinitely large natural number. Thus the set of p-adic integers contains actual infinities $n \in \mathbf{Z}_p \backslash \mathbf{N}$, $n \neq 0$. This is one of the most important features of non-Archimedean number systems, therefore it is natural to compare $\mathbf{Z}_p$ with the set of nonstandard numbers ${}^*\mathbf{Z}$. Also, the set $\mathbf{Z}_p$ contains *exhaustive elements.*

7.5.1 p-Adic valued matrix logic

Extend the standard order structure on $\{0, \ldots, p-1\}$ to a partial order structure on $\mathbf{Z}_p$. Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{\mathbf{Z}_p}$ Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$.

1. We set $x \leq y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$
2. We set $x < y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and there exists n_0 such that $x_{n_0} < y_{n_0}$.
3. We set $x = y$ if $x_n = y_n$ for each $n = 0, 1, \ldots$

Now introduce two operations max, min in the partial order structure on $\mathbf{Z}_p$:

1 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\min(x, y) = x$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

2 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = y$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

3 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = \min(x, y) = x = y$ if and only if $x = y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$.

The ordering relation $\mathcal{O}_{\mathbf{Z}_p}$ is not linear, but partial, because there exist elements $x, z \in \mathbf{Z}_p$, which are incompatible. As an example, let $p = 2$ and let $x = -\frac{1}{3} = \ldots 10101 \ldots 101$, $z = -\frac{2}{3} = \ldots 01010 \ldots 010$. Then the numbers x and z are incompatible.

Thus,

4 Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$ and x, y are incompatible under condition $\mathcal{O}_{\mathbf{Z}_p}$. We get $\min(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \geq y_n$,
2. $z_n = x_n$ if $x_n \leq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

We get $\max(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \leq y_n$,
2. $z_n = x_n$ if $x_n \geq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

It is important to remark that there exists the maximal number $N_{max} \in \mathbf{Z}_p$ under condition $\mathcal{O}_{\mathbf{Z}_p}$. It is easy to see:

$$N_{max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots = \sum_{k=0}^{\infty} (p-1) \cdot p^k$$

Therefore

5 $\min(x, N_{max}) = x$ and $\max(x, N_{max}) = N_{max}$ for any $x \in \mathbf{Z}_p$.

Now consider *p-adic valued matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$.

Definition 7. *The ordered system* $<V_{\mathbf{Z}_p}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}>$ *is called p-adic valued matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$, *where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg x = N_{max} - x$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \supset y = (N_{max} - \max(x, y) + y)$,
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$,

5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$,

6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{N_{max}\}$ *is the set of designated truth values.*

The truth value $0 \in \mathbf{Z}_p$ is false, the truth value $N_{max} \in \mathbf{Z}_p$ is true, and other truth values $x \in \mathbf{Z}_p$ are neutral.

Proposition 4. *The logic* $\mathfrak{M}_{\mathbf{Z}_2} = <V_{\mathbf{Z}_2}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}>$ *is a Boolean algebra.*

Proof. Indeed, the operation $\neg$ in $\mathfrak{M}_{\mathbf{Z}_2}$ is the Boolean complement:

1. $\max(x, \neg x) = N_{max}$,

2. $\min(x, \neg x) = 0$. □

7.5.2 p-Adic probability theory

7.5.2.1 Frequency theory of p-adic probability

Let us remember that the frequency theory of probability was created by Richard von Mises in [10]. This theory is based on the notion of a collective: "We will say that a collective is a mass phenomenon or a repetitive event, or simply a long sequence of observations for which there are sufficient reasons to believe that the relative frequency of the observed attribute would tend to a fixed limit if the observations were infinitely continued. This limit will be called the probability of the attribute considered within the given collective" [10].

As an example, consider a random experiment $\mathcal{S}$ and by $L = \{s_1, \ldots, s_m\}$ denote the set of all possible results of this experiment. The set $\mathcal{S}$ is called the label set, or the set of attributes. Suppose there are N realizations of $\mathcal{S}$ and write a result x_j after each realization. Then we obtain the finite sample: $x = (x_1, \ldots, x_N), x_j \in L$. A collective is an infinite idealization of this finite sample: $x = (x_1, \ldots, x_N, \ldots), x_j \in L$. Let us compute frequencies $\nu_N(\alpha; x) = n_N(\alpha; x)/N$, where $n_N(\alpha; x)$ is the number of realizations of the attribute α in the first N tests. There exists the statistical stabilization of relative frequencies: the frequency $\nu_N(\alpha; x)$ approaches a limit as N approaches infinity for every label $\alpha \in L$. This limit $\mathbf{P}(\alpha) = \lim \nu_N(\alpha; x)$ is said to be the probability of the label α in the frequency theory of probability. Sometimes this probability is denoted by $\mathbf{P}_x(\alpha)$ to show a dependence on the collective x. Notice that the limits of relative frequencies have to be stable with respect to a place selection (a choice of a subsequence) in the collective. A. Yu. Khrennikov developed von Mises' idea and proposed the frequency theory of p-adic probability in [6, 7]. We consider here Khrennikov's theory.

We shall study some ensembles $S = S_N$, which have a p-dic volume N, where N is the p-adic integer. If N is finite, then S is the ordinary finite ensemble. If N is infinite, then S has essentially p-adic structure. Consider a sequence of ensembles M_j having volumes $l_j \cdot p^j$, $j = 0, 1, \ldots$ Get $S = \cup_{j=0}^{\infty} M_j$. Then the cardinality $|S| = N$. We may imagine an ensemble S as being the population of a tower $T = T_S$, which has an infinite number of floors with the following distribution of population through floors: population of j-th floor is M_j. Set $T_k = \cup_{j=0}^{k} M_j$.

This is population of the first $k+1$ floors. Let $A \subset S$ and let there exists: $n(A) = \lim_{k\to\infty} n_k(A)$, where $n_k(A) = |A \cap T_k|$. The quantity $n(A)$ is said to be a p-adic volume of the set A.

We define the probability of A by the standard proportional relation:

$$\mathbf{P}(A) \triangleq \mathbf{P}_S(A) = \frac{n(A)}{N}, \tag{7.6}$$

where $|S| = N$, $n(A) = |A \cap S|$.

We denote the family of all $A \subset S$, for which $\mathbf{P}(A)$ exists, by $\mathcal{G}_S$. The sets $A \in \mathcal{G}_S$ are said to be events. The ordered system $(S, \mathcal{G}_S, \mathbf{P}_S)$ is called a *p-adic ensemble probability space for the ensemble S.*

Proposition 5. *Let F be the set algebra which consists of all finite subsets and their complements. Then $F \subset \mathcal{G}_S$.*

Proof. Let A be a finite set. Then $n(A) = |A|$ and the probability of A has the form:

$$\mathbf{P}(A) = \frac{|A|}{|S|}$$

Now let $B = \neg A$. Then $|B \cap T_k| = |T_k| - |A \cap T_k|$. Hence there exists $\lim_{k\to\infty} |B \cap T_k| = N - |A|$. This equality implies the standard formula:

$$\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$$

In particular, we have: $\mathbf{P}(S) = 1$. □

The next propositions are proved in [6]:

Proposition 6. *Let $A_1, A_2 \in \mathcal{G}_S$ and $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2 \in \mathcal{G}_S$ and*

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2).$$

□

Proposition 7. *Let $A_1, A_2 \in \mathcal{G}_S$. The following conditions are equivalent:*

1. $A_1 \cup A_2 \in \mathcal{G}_S$,
2. $A_1 \cap A_2 \in \mathcal{G}_S$,
3. $A_1 \backslash A_2 \in \mathcal{G}_S$,
4. $A_2 \backslash A_1 \in \mathcal{G}_S$. □

But it is possible to find sets $A_1, A_2 \in \mathcal{G}_S$ such that, for example, $A_1 \cup A_2 \notin \mathcal{G}_S$. Thus, the family $\mathcal{G}_S$ is not an algebra, but a semi-algebra (it is closed only with respect to a finite unions of sets, which have empty intersections). $\mathcal{G}_S$ is not closed with respect to countable unions of such sets.

Proposition 8. *Let $A \in \mathcal{G}_S$, $\mathbf{P}(A) \neq 0$ and $B \in \mathcal{G}_A$. Then $B \in \mathcal{G}_S$ and the following Bayes formula holds:*

$$\mathbf{P}_A(B) = \frac{\mathbf{P}_S(B)}{\mathbf{P}_S(A)} \tag{7.7}$$

Proof. The tower T_A of the A has the following population structure: there are M_{A_j} elements on the j-th floor. In particular, $T_{A_k} = T_k \cap A$. Thus

$$n_{A_k}(B) = |B \cap T_{A_k}| = |B \cap T_k| = n_k(B)$$

for each $B \subset A$. Hence the existence of $n_A(B) = \lim_{k\to\infty} n_{A_k}(B)$ implies the existence of $n_S(B)$ with $n_S(B) = \lim_{k\to\infty} n_k(B)$. Moreover, $n_S(B) = n_A(B)$. Therefore,

$$\mathbf{P}_A(B) = \frac{n_A(B)}{n_S(A)} = \frac{n_A(B)/|S|}{n_S(A)/|S|}.$$

□

Proposition 9. *Let $N \in \mathbf{Z}_p$, $N \neq 0$ and let the ensemble S_{-1} have the p-adic volume $-1 = N_{max}$ (it is the largest ensemble).*

1. *Then $S_N \in \mathcal{G}_{S_{-1}}$ and*

$$\mathbf{P}_{S_{-1}}(S_N) = \frac{|S_N|}{|S_{-1}|} = -N$$

2. *Then $\mathcal{G}_{S_N} \subset \mathcal{G}_{S_{-1}}$ and probabilities $\mathbf{P}_{S_N}(A)$ are calculated as conditional probabilities with respect to the subensemble S_N of ensemble S_{-1}:*

$$\mathbf{P}_{S_N}(A) = \mathbf{P}_{S_{-1}}(\frac{A}{S_N}) = \frac{\mathbf{P}_{S_{-1}}(A)}{\mathbf{P}_{S_{-1}}(S_N)}, A \in \mathcal{G}_{S_N}$$

□

7.5.2.2 Logical theory of p-adic probability

Transform the matrix logic $\mathfrak{M}_{\mathbf{Z}_p}$ into a p-adic probability theory. Let us remember that a formula φ has the truth value $0 \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is false, a formula φ has the truth value $N_{max} \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is true, and a formula φ has other truth values $\alpha \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is neutral.

Definition 8. *A function $\mathbf{P}(\varphi)$ is said to be a probability measure of a formula φ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\mathbf{P}(\varphi)$ ranges over numbers of $\mathbf{Q}_p$ and satisfies the following axioms:*

1. $\mathbf{P}(\varphi) = \frac{\alpha}{N_{max}}$, *where α is a truth value of φ;*
2. *if a conjunction $\varphi \wedge \psi$ has the truth value 0, then $\mathbf{P}(\varphi \vee \psi) = \mathbf{P}(\varphi) + \mathbf{P}(\psi)$,*
3. $\mathbf{P}(\varphi \wedge \psi) = \min(\mathbf{P}(\varphi), \mathbf{P}(\psi))$.

Notice that:

1. taking into account condition 1 of our definition, if φ has the truth value N_{max} for any its interpretations, i. e., φ is a tautology, then $\mathbf{P}(\varphi) = 1$ in all possible worlds, and if φ has the truth value 0 for any its interpretations, i. e., φ is a contradiction, then $\mathbf{P}(\varphi) = 0$ in all possible worlds;
2. under condition 2, we obtain $\mathbf{P}(\neg\varphi) = 1 - \mathbf{P}(\varphi)$.

Since $\mathbf{P}(N_{max}) = 1$, we have

$$\mathbf{P}(\max\{x \in V_{\mathbf{Z}_p}\}) = \sum_{x \in V_{\mathbf{Z}_p}} \mathbf{P}(x) = 1$$

All events have a conditional plausibility in the logical theory of p-adic probability:

$$\mathbf{P}(\varphi) \equiv \mathbf{P}(\varphi / N_{max}), \tag{7.8}$$

i. e., for any φ, we consider the conditional plausibility that there is an event of φ, given an event N_{max},

$$\mathbf{P}(\varphi/\psi) = \frac{\mathbf{P}(\varphi \wedge \psi)}{\mathbf{P}(\psi)}. \tag{7.9}$$

7.5.3 p-Adic fuzzy logic

The probability interpretation of the logic $\mathfrak{M}_{\mathbf{Z}_p}$ shows that this logic is a special system of fuzzy logic. Indeed, we can consider the membership function μ_A as a p-adic valued predicate.

Definition 9. *Suppose X is a non-empty set. Then a p-adic-valued fuzzy set A in X is a set defined by means of the membership function μ_A: $X \mapsto \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the set of all p-adic integers.*

It is obvious that the set A is completely determined by the set of tuples $\{<u, \mu_A(u)>: u \in X\}$. We define a norm $|\cdot|_p \colon \mathbf{Q}_p \mapsto \mathbf{R}$ on $\mathbf{Q}_p$ as follows:

$$|n = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k|_p \triangleq p^{-L},$$

where $L = \max\{k \colon n \equiv 0 \mod p^k\} \geq 0$, i. e. L is an index of the first number distinct from zero in p-adic expansion of n. Note that $|0|_p \triangleq 0$. The function $|\cdot|_p$ has values 0 and $\{p^\gamma\}_{\gamma \in \mathbf{Z}}$ on $\mathbf{Q}_p$. Finally, $|x|_p \geq 0$ and $|x|_p = 0 \equiv x = 0$. A set $A \subset X$ is called *crisp* if $|\mu_A(u)|_p = 1$ or $|\mu_A(u)|_p = 0$ for any $u \in X$. Notice that $|\mu_A(u) = 1|_p = 1$ and $|\mu_A(u) = 0|_p = 0$. Therefore our membership function is an extension of the classical characteristic function. Thus, $A = B$ causes $\mu_A(u) = \mu_B(u)$ for all $u \in X$ and $A \subseteq B$ causes $|\mu_A(u)|_p \leqslant |\mu_B(u)|_p$ for all $u \in X$.

In p-adic fuzzy logic, there always exists a non-empty intersection of two crisp sets. In fact, suppose the sets A, B have empty intersection and A, B are crisp. Consider two cases under condition $\mu_A(u) \neq \mu_B(u)$ for any u. First, $|\mu_A(u)|_p = 0$ or $|\mu_A(u)|_p = 1$ for all u and secondly $|\mu_B(u)|_p = 0$ or $|\mu_B(u)|_p = 1$ for all u. Assume we have $\mu_A(u_0) = N_{max}$ for some u_0, i. e., $|\mu_A(u_0)|_p = 1$. Then $\mu_B(u_0) \neq N_{max}$, but this doesn't mean that $\mu_B(u_0) = 0$. It is possible that $|\mu_A(u_0)|_p = 1$ and $|\mu_B(u_0)|_p = 1$ for u_0.

Now we set logical operations on p-adic fuzzy sets:

1. $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$;

2. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$;

3. $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \min(\mu_A(x), \mu_B(x))$;

4. $\mu_{\neg A}(x) = \neg\mu_A(x) = N_{max} - \mu_A(x) = -1 - \mu_A(x)$.

7.6 Conclusion

In this chapter, one has constructed on the basis of infinite DSm models three logical many-valued systems: $\mathfrak{M}_{\mathbf{Z}_p}$, $\mathfrak{M}_{^*\mathbf{Q}}$, and $\mathfrak{M}_{^*\mathbf{R}}$. These systems are principal versions of the non-Archimedean logic and they can be used in probabilistic and fuzzy reasoning. Thus, the DSm models assumes many theoretical and practical applications.

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Chapter 8

An In-Depth Look at Quantitative Information Fusion Rules

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Abstract: *This chapter may look like a glossary of the fusion rules and we also introduce new ones presenting their formulas and examples: Conjunctive, Disjunctive, Exclusive Disjunctive, Mixed Conjunctive-Disjunctive rules, Conditional rule, Dempster's, Yager's, Smets' TBM rule, Dubois-Prade's, Dezert-Smarandache classical and hybrid rules, Murphy's average rule, Inagaki-Lefevre-Colot-Vannoorenberghe Unified Combination rules [and, as particular cases: Iganaki's parameterized rule, Weighted Average Operator, minC (M. Daniel), and newly Proportional Conflict Redistribution rules (Smarandache-Dezert) among which PCR5 is the most exact way of redistribution of the conflicting mass to non-empty sets following the path of the conjunctive rule], Zhang's Center Combination rule, Convolutive x-Averaging, Consensus Operator (Jøsang), Cautious Rule (Smets), α-junctions rules (Smets), etc. and three new T-norm & T-conorm rules adjusted from fuzzy and neutrosophic sets to information fusion (Tchamova-Smarandache). Introducing the degree of union and degree of inclusion with respect to the cardinal of sets not with the fuzzy set point of view, besides that of intersection, many fusion rules can be improved. There are corner cases where each rule might have difficulties working or may not get an expected result. As a conclusion, since no theory neither rule fully satisfy all needed applications, the author proposes a Unification of Fusion Theories extending the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closures of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers a fuzzy or neutrosophic complement). And, at each application, one selects the most appropriate model, rule, and algorithm of implementation.*

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8.1 Introduction

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n \geq 2$, and two sources of information:

$$m_1(\cdot), m_2(\cdot) : S^\Theta \to [0, 1].$$

For the simplest frame $\Theta = \{\theta_1, \theta_2\}$ one can define a mass matrix as follows:

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\theta_1 \cap \theta_2$	$\mathcal{C}\theta_1$	$\mathcal{C}\theta_2$	$\mathcal{C}(\theta_1 \cap \theta_2)$	$\emptyset$
$m_1(\cdot)$	m_{11}	m_{12}	m_{13}	m_{14}	m_{15}	m_{16}	m_{17}	m_{18}
$m_2(\cdot)$	m_{21}	m_{22}	m_{23}	m_{24}	m_{25}	m_{26}	m_{27}	m_{28}

In calculations we take into account only the focal elements, i.e. those for which $m_1(\cdot)$ or $m_2(\cdot) > 0$. In the Shafer's model one only has the first three columns of the mass matrix, corresponding to θ_1, θ_2, $\theta_1 \cup \theta_2$, while in the Dezert-Smarandache free model only the first four columns corresponding to θ_1, θ_2, $\theta_1 \cup \theta_2$, $\theta_1 \cap \theta_2$. But here we took the general case in order to include the possible complements (negations) as well.

We note the combination of these bba's, using any of the below rule "r", by

$$m_r = m_1 \otimes_r m_2.$$

All the rules below are extended from their power set $2^\Theta = (\Theta, \cup) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, which is a set closed under union, or hyper-power set $D^\Theta = (\Theta, \cup, \cap) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2, \theta_1 \cap \theta_2\}$ which is a distributive lattice called hyper-power set, to the super-power set $S^\Theta = (\Theta, \cup, \cap, \mathcal{C}) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2, \theta_1 \cap \theta_2, C\theta_1, \mathcal{C}\theta_2, \mathcal{C}(\theta_1 \cap \theta_2)\}$, which is a Boolean algebra with respect to the union, intersection, and complement ($\mathcal{C}$ is the complement).

Of course, all of these can be generalized for Θ of dimension $n \geq 2$ and for any number of sources $s \geq 2$.

Similarly one defines the mass matrix, power-set, hyper-power set, and super-power set for the general frame of discernment.

A list of the main rules we have collected from various sources available in the open literature is given in the next sections.

8.2 Conjunctive Rule

If both sources of information are telling the truth, then we apply the conjunctive rule, which means consensus between them (or their common part):

$$\forall A \in S^\Theta, \text{ one has } m_1(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2),$$

where the *Total Conflicting Mass* is:

$$k_{12} = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

8.3 Disjunctive Rule

If at least one source of information is telling the truth, we use the optimistic disjunctive rule proposed by Dubois and Prade in [7]:

$$m_{\cup}(\emptyset) = 0, \text{ and } \forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cup}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A}} m_1(X_1)m_2(X_2).$$

8.4 Exclusive Disjunctive Rule

If only one source of information is telling the truth, but we don't know which one, then one uses the exclusive disjunctive rule [7] based on the fact that $X_1 \veebar X_2$ means either X_1 is true, or X_2 is true, but not both in the same time (in set theory let's use $X_1 \veebar X_2$ for exclusive disjunctive):

$$m_{\veebar}(\emptyset) = 0, \text{ and } \forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\veebar}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \veebar X_2 = A}} m_1(X_1)m_2(X_2).$$

8.5 Mixed Conjunctive-Disjunctive Rule

This is a mixture of the previous three rules in any possible way [7]. As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth. The mixed formula becomes:

$$m_{\cap\cup}(\emptyset) = 0,$$

and

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cap\cup}(A) = \sum_{\substack{X_1,X_2,X_3,X_4 \in S^{\Theta} \\ ((X_1 \cap X_2) \cup X_3) \veebar X_4 = A}} m_1(X_1)m_2(X_2)m_3(X_3)m_4(X_4).$$

8.6 Conditioning Rule

This classical *conditioning* rule proposed by Glenn Shafer in Dempster-Shafer Theory [26] looks like the conditional probability (when dealing with Bayesian belief functions) but it is different. Shafer's conditioning rule is commonly used when there exists a bba, say $m_S(\cdot)$, such that for an hypothesis, say A, one has $m_S(A) = 1$ (i.e. when the subjective certainty of an hypothesis to occur is given by an expert). Shafer's conditioning rule consists in combining $m_S(\cdot)$ directly with another given bba for belief revision using Dempster's rule of combination. We point out that this conditioning rule could be used also whatever rule of combination is chosen in any other fusion theory dealing with belief functions. After fusioning $m_1(.)$ with $m_S(A) = 1$, the conflicting mass is transferred to non-empty sets using Dempster's rule in DST, or DSmH or PCR5 in DSmT, etc. Another family of belief conditioning rules (BCR) is proposed as a new alternative in chapter 9 of this book.

8.7 Dempster's Rule

This is the most used fusion rule in applications and this rule influenced the development of other rules. Shafer has developed the Dempster-Shafer Theory of Evidence [26] based on the model that all hypotheses in the frame of discernment are exclusive and the frame is exhaustive. Dempster's rule for two *independent* sources is given by [26]

$$m_D(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_D(A) = \frac{1}{1-k_{12}} \cdot \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

8.8 Modified Dempster-Shafer rule (MDS)

MDS rule was introduced by Dale Fixsen and Ronald P. S. Mahler in 1997 [11] for identifying objects in a finite universe $\mathcal{U}$ containing N elements, and it merges Bayesian and Dempster-Shafer theories.
Let B and C be two bodies of evidence:$B = \{(S_1, m_1), (S_2, m_2), \ldots, (S_b, m_b)\}$ and $C = \{(T_1, n_1), (T_2, n_2), \ldots (T_c, n_c)\}$ where S_i, $1 \leq i \leq b$, T_j , $1 \leq j \leq c$, are subsets of the universe $\mathcal{U}$,and (S_i, m_i) represents for the source B the hypothesis *object is in* S_i with a belief (mass assignment) of m_i with of course $\sum_i m_i = 1$. Similarly for (T_j, n_j) for each j.

Then B and C can be fused just following Dempster's rule and one gets a new body of evidence $B \star C$. The elements of $B \star C$ are intersections of $S_i \cap T_j$ for all $i = 1, \ldots, b$ and $j = 1, \ldots, c$, giving the following masses:

$$r_{ij} = m_i n_j \frac{\alpha_{DS}(S_i, T_j)}{\alpha_{DS}(B, C)}$$

if $\alpha_{DS}(B, C) \neq 0$ (it is zero only in the total degenerate case).
The *Dempster-Shafer agreement* $\alpha_{DS}(.,.)$ is defined by [11]:

$$\alpha_{DS}(B, C) = \sum_{i=1}^{b} \sum_{j=1}^{c} m_i n_j \alpha_{DS}(S_i, T_j) \qquad \text{with} \qquad \alpha_{DS}(S, T) = \frac{\rho(S \cap T)}{\rho(S)\rho(T)}$$

where the set function $\rho(S) = 1$ if $S \neq \emptyset$ and $\rho(\emptyset) = 0$; $\alpha_{DS}(S, T) = 1$ if $S \cap T \neq \emptyset$ and zero otherwise.

The agreement between bodies of evidence is just $1-k$, where k is the conflict from Dempster-Shafer Theory. In 1986, J. Yen had proposed a similar rule, but his probability model was different from Fixsen-Mahler MDS's (see [50] for details).

8.9 Murphy's Statistical Average Rule

If we consider that the bba's are important from a statistical point of view, then one averages them as proposed by Murphy in [24]:

$$\forall A \in S^\Theta, \text{ one has } m_M(A) = \frac{1}{2}[m_1(A) + m_2(A)].$$

Or, more general, $m_{\text{mixing}}(A) = \frac{1}{2}[w_1 m_1(A) + w_2 m_2(A)]$, where w_1, w_2 are weights reflecting the reliability of sources.

8.10 Dezert-Smarandache Classic Rule (DSmC)

DSmC rule [31] is a generalization of the conjunctive rule from the power set to the hyper-power set.

$$\forall A \in S^{\Theta},\ \text{one has } m_{\text{DSmC}}(A) = \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

It can also be extended on the Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$ in order to include the complements (or negations) of elements.

8.11 Dezert-Smarandache Hybrid Rule (DSmH)

DSmH rule [31] is an extension of the Dubois-Prade rule for the dynamic fusion. The middle sum in the below formula does not occur in Dubois-Prade's rule, and it helps in the transfer of the masses of empty sets — whose disjunctive forms are also empty — to the total ignorance.

$$m_{\text{DSmH}}(\emptyset) = 0,$$

and

$$\begin{aligned}
&\forall A \in S^{\Theta} \setminus \emptyset \text{ one has} \\
m_{\text{DSmH}}(A) = &\sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in \emptyset \\ (A=U) \vee \{U \in \emptyset \wedge A = I\}}} m_1(X_1) m_2(X_2) \\
&+ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 \in \emptyset}} m_1(X_1) m_2(X_2)
\end{aligned}$$

where all sets are in canonical form (i.e. for example the set $(A \cap B) \cap (A \cup B \cup C))$ will be replaced by its canonical form $A \cap B$), and U is the disjunctive form of $X_1 \cap X_2$ and is defined as follows:

$$\begin{aligned}
U(X) &= X \text{ if } X \text{ is a singleton,} \\
U(X_1 \cap X_2) &= U(X_1) \cup U(X_2), \text{ and} \\
U(X_1 \cup X_2) &= U(X_1) \cup U(X_2);
\end{aligned}$$

while $I = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_n$ is the total ignorance.

Formally the canonical form has the properties:

i) $c(\emptyset) = \emptyset$;

ii) if A is a singleton, then $c(A) = A$;

iii) if $A \subseteq B$, then $c(A \cap B) = A$ and $c(A \cup B) = B$;

iv) the second and third properties apply for any number of sets.

8.12 Smets' TBM Rule

In the TBM (Transferable Belief model) approach, Philippe Smets [36] does not transfer the conflicting mass, but keeps it on the empty set, meaning that $m(\emptyset) > 0$ signifies that there might exist other hypotheses we don't know of in the frame of discernment (this is called an *open world*).

$$m_S(\emptyset) = k_{12} = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_S(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

8.13 Yager's Rule

R. Yager transfers the total conflicting mass to the total ignorance [44], i.e.

$$m_Y(\emptyset) = 0, \qquad m_Y(I) = m_1(I) m_2(I) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

where $I =$ total ignorance, and

$$\forall A \in S^\Theta \setminus \{\emptyset, I\}, \text{ one has } m_Y(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

8.14 Dubois-Prade's Rule

This rule [8] is based on the principle that if two sources are in conflict, then at least one is true, and thus transfers the conflicting mass $m(A \cap B) > 0$ to $A \cup B$.

$$m_{DP}(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset \text{ one has}$$

$$m_{DP}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

8.15 Weighted Operator (Unification of the Rules)

The Weighted Operator (WO) proposed by T. Inagaki in [15] and later by Lefevre-Colot-Vannoorenberghe in [19] is defined as follows:

$$m_{WO}(\emptyset) = w_m(\emptyset) \cdot k_{12},$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_{WO}(A) = \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2) + w_m(A) \cdot k_{12}$$

where $w_m(A) \in [0,1]$ for any $A \in S^\Theta$ and $\sum_{X \in S^\Theta} w_m(X) = 1$ and $w_m(A)$ are called weighting factors.

8.16 Inagaki's Unified Parameterized Combination Rule

Inagaki's Unified Parameterized Combination Rule [15] is defined by

$$\forall A \in S^\Theta \setminus \{\emptyset, I\}, \text{ one has } m_p^U(A) = [1 + p \cdot k_{12}] \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2),$$

and

$$m_p^U(\emptyset) = 0, m_p^U(I) = [1 + p \cdot k_{12}] \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = I}} m_1(X_1)m_2(X_2) + [1 + p \cdot k_{12} - p]k_{12}$$

where the parameter $0 \leq p \leq 1/[1 - k_{12} - m_\cap(I)]$, and k_{12} is the conflict.

The determination of parameter p, used for normalization, is not well justified in the literature, but may be found through experimental data, simulations, expectations [39]. The greater is the parameter p, the greater is the change to the evidence.

8.17 The Adaptive Combination Rule (ACR)

Mihai Cristian Florea, Anne-Laure Jousselme, Dominic Grenier and Eloi Bossé propose a new class of combination rules for the evidence theory as a mixing between the disjunctive (p) and conjunctive (q) rules [12, 13]. The adaptive combination rule (ACR) between m_1 and m_2 is defined by $(m_1 \diamond m_2)(\emptyset) = 0$ and :

$$(m_1 \diamond m_2)(A) = \alpha(k)p(A) + \beta(k)q(A) \ , \qquad \forall A \subseteq \Theta, A \neq \emptyset \tag{8.1}$$

Here, α and β are functions of the conflict $k = q(\varnothing)$ from $[0,1]$ to $[0,+\infty[$. The ACR may be expressed according only to the function β (because of the condition $\sum_{A \subseteq \Theta}(m_1 \diamond m_2)(A) = 1$) as follows:

$$(m_1 \diamond m_2)(A) = [1 - (1-k)\beta(k)]p(A) + \beta(k)q(A) \ , \qquad \forall A \subseteq \Theta, A \neq \emptyset \tag{8.2}$$

and $(m_1 \diamond m_2)(\varnothing) = 0$ where β is any function such that $\beta : [0,1] \to [0,+\infty[$.

In the general case α and β are functions of k with no particular constraint. However, a desirable behaviour of the ACR is that it should act more like the disjunctive rule p whenever k is close to 1 (i.e. at least one source is unreliable), while it should act more like the conjunctive rule q, if k is close to 0 (i.e. both sources are reliable). This amounts to add three conditions on the general formulation:

(C1) α is an increasing function with $\alpha(0) = 0$ and $\alpha(1) = 1$;

(C2) β is a decreasing function with $\beta(0) = 1$ and $\beta(1) = 0$.

(C3) $\alpha(k) = 1 - (1-k)\beta(k)$

In particular, when $k = 0$ the sources are in total agreement and $(m_1 \diamond m_2)(A) = p(A), \forall A \subseteq \Theta$, the conjunction represents the combined evidence, while when $k = 1$ the sources are in total conflict and $(m_1 \diamond m_2)(A) = q(A), \forall A \subseteq \Theta$, the disjunction is the best choice considering that one of them is wrong.

Note that the three conditions above are dependent and (C1) can be removed, since it is a consequence of the (C2) and (C3). The particular case of the adaptive combination rule can be stated as Equation (8.2), $\forall A \subseteq \Theta, A \neq \varnothing$ and $m(\varnothing) = 0$, where β is any decreasing function such that $\beta : [0,1] \to [0,1]$ and $\beta(0) = 1$ and $\beta(1) = 0$.

8.18 The Weighted Average Operator (WAO)

The Weighted Average Operator (WAO) for two sources proposed in [17] consists in first, applying the conjunctive rule to the bba's $m_1(\cdot)$ and $m_2(\cdot)$ and second, redistribute the total conflicting mass k_{12} to all nonempty sets in S^Θ proportionally with their mass averages, i.e. for the set, say A, proportionally with the weighting factor:

$$w_{JDV}(A, m_1, m_2) = \frac{1}{2}(m_1(A) + m_2(A)).$$

The authors do not give an analytical formula for it. WAO does not work in degenerate cases as shown in chapter 1.

8.19 The Ordered Weighted Average operator (OWA)

It was introduced by Ronald R. Yager [46, 48]. The OWA of dimension n is defined as
$F : R^n \mapsto R$ such that

$$F(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j$$

where $b_1 \geq b_2 \dots \geq b_n$ and the weights $w_j \in [0,1]$ with $\sum_{j=1}^{n} w_j = 1$.

OWA satisfies the following properties:
a) Symmetry: For any permutation Π, one has $F(a_{\Pi(1)}, a_{\Pi(2),\dots,a_{\Pi(n)}}) = F(a_1, a_2, \dots, a_n)$.
b) Monotonicity: If $\forall j$, $a_j \geq d_j$ then $F(a_1, a_2, \dots, a_n) \geq F(d_1, d_2, \dots, d_n)$.
c) Boundedness: $\min_j(a_j) \leq F(a_1, a_2, \dots, a_n) \leq \max_j(a_j)$.
d) Idempotency: If $\forall j$, $a_j = a$, then $F(a_1, a_2, \dots, a_n) = F(a, a, \dots, a) = a$.

A measure associated with this operator with weighting vector $W = (w_1, w_2, \dots, w_n)$ is the Attitude-Character (AC) defined as: $AC(W) = \sum_{j=1}^{n} w_j \frac{n-j}{n-1}$.

8.20 The Power Average Operator (PAO)

It was introduced by Ronald Yager in [47] in order to allow values being aggregated to support and reinforce each other. This operator is defined by:

$$PAO(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^{n}(1 + T(a_i)a_i)}{\sum_{i=1}^{n}(1 + T(a - i))}$$

where $T(a_i) = \sum_{j=1 j\neq i}^{n} \sup(a_i, a_j)$ and $\sup(a, b)$ denotes the support for a from b which satisfies the properties:
a) $\sup(a, b) \in [0, 1]$;
b) $\sup(a, b) = \sup(b, a)$;
c) $\sup(a, b) \geq \sup(x, y)$ if $|a - b| < |x - y|$.

8.21 Proportional Conflict Redistribution Rules (PCR)

8.21.1 PCR1 Fusion rule

In 2004, F. Smarandache and J. Dezert independently developed a *Proportional Conflict Redistribution Rule (PCR1)*, which similarly consists in first, applying the conjunctive rule to the bba's $m_1(\cdot)$ and $m_2(\cdot)$ and second, redistribute the total conflicting mass k_{12} to all nonempty sets in S^Θ proportionally with their nonzero mass sum, i.e. for the set, say A, proportionally with the weighting factor:

$$w_{SD}(A, m_1, m_2) = m_1(A) + m_2(A) \neq 0.$$

The analytical formula for PCR1, non-degenerate and degenerate cases, is:

$$m_{PCR1}(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_{PCR1}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \frac{c_{12}(A)}{d_{12}} \cdot k_{12},$$

where $c_{12}(A)$ is the sum of masses corresponding to the set A, i.e. $c_{12}(A) = m_1(A) + m_2(A) \neq 0$, d_{12} is the sum of nonzero masses of all nonempty sets in S^Θ assigned by the sources $m_1(\cdot)$ and $m_2(\cdot)$ [in many cases $d_{12} = 2$, but in degenerate cases it can be less], and k_{12} is the total conflicting mass.

Philippe Smets pointed out that PCR1 gives the same result as the WAO for non-degenerate cases, but PCR1 extends actually WAO, since PCR1 works also for the degenerate cases when all column sums of all non-empty sets are zero because in such cases, the conflicting mass is transferred to the non-empty disjunctive form of all non-empty sets together; when this disjunctive form happens to be empty, then one can consider an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

For the cases of the combination of only one non-vacuous belief assignment $m_1(\cdot)$ with the vacuous belief assignment[1] $m_v(\cdot)$ where $m_1(\cdot)$ has mass assigned to an empty element, say $m_1(\cdot) > 0$ as in Smets' TBM, or as in DSmT dynamic fusion where one finds out that a previous non-empty element A, whose mass $m_1(A) > 0$, becomes empty after a certain time, then this mass of an empty set has to be transferred to other elements using PCR1, but for such case $[m_1 \otimes m_v](\cdot)$ is different from $m_1(\cdot)$. This severe draw-back of WAO and PCR1 forces us to develop more sophisticated PCR rules satisfying the neutrality property of VBA with better redistributions of the conflicting information.

[1]The VBA (vacuous belief assignment) is the bba $m_v(\text{total ignorance}) = 1$.

8.21.2 PCR2-PCR4 Fusion rules

F. Smarandache and J. Dezert then developed more improved versions of *Proportional Conflict Redistribution Rule (PCR2-4)*. A detailed presentation of these rules can be found in Chapter 1 of this book.

In the *PCR2* fusion rule, the total conflicting mass k_{12} is redistributed only to the non-empty sets involved in the conflict (not to all non-empty sets as in WAO and PCR1) proportionally with respect to their corresponding non-empty column sum in the mass matrix. The redistribution is then more exact (accurate) than in PCR1 and WAO. A nice feature of PCR2 is the preservation of the neutral impact of the VBA and of course its ability to deal with all cases/models.

$$m_{PCR2}(\emptyset) = 0,$$

and $\forall A \in S^{\Theta} \setminus \emptyset$ and A involved in the conflict, one has

$$m_{PCR2}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2) + \frac{c_{12}(A)}{e_{12}} \cdot k_{12},$$

while for a set $B \in S^{\Theta} \setminus \emptyset$ not involved in the conflict one has:

$$m_{PCR2}(B) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cap X_2 = B}} m_1(X_1)m_2(X_2),$$

where $c_{12}(A)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12}(A) = m_1(A) + m_2(A) \neq 0$, k_{12} is the total conflicting mass, and e_{12} is the sum of all non-zero column sums of all non-empty sets only involved in the conflict (in many cases $e_{12} = 2$, but in some degenerate cases it can be less). In the degenerate case when all column sums of all non-empty sets involved in the conflict are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets together which were involved in the conflict. But if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

A non-empty set $X \in S^{\Theta}$ is considered involved in the conflict if there exists another set $Y \in S^{\Theta}$ such that $X \cap Y = 0$ and $m_{12}(X \cap Y) > 0$. This definition can be generalized for $s \geq 2$ sources.

PCR3 transfers partial conflicting masses, instead of the total conflicting mass. If an intersection is empty, say $A \cap B = \emptyset$, then the mass $m(A \cap B) > 0$ of the partial conflict is transferred to the non-empty sets A and B proportionally with respect to the non-zero sum of masses assigned to A and respectively to B by the bba's $m_1(\cdot)$ and $m_2(\cdot)$. The PCR3 rule works if at least one set between A and B is non-empty and its column sum is non-zero. When both sets A and B are empty, or both corresponding column sums of the mass matrix are zero, or only one set is non-empty and its column sum is zero, then the mass $m(A \cap B)$ is transferred to the non-empty disjunctive form $u(A) \cup u(B)$ [which is defined as follows: $u(A) = A$ if A is a singleton, $u(A \cap B) = u(A \cup B) = u(A) \cup u(B)$]; if this disjunctive form is empty then $m(A \cap B)$ is transferred to the non-empty total ignorance in a closed world approach or to the empty set if one prefers to adopt the Smets' open world approach; but if even the total ignorance is empty (a completely degenerate case) then one considers an open world (i.e. new hypotheses might be in the frame of discernment) and the conflicting mass is transferred to the empty set, which means that the original problem has no solution in the close world initially chosen for the problem.

$$m_{PCR3}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$, one has

$$\begin{aligned} m_{PCR3}(A) = & \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2) + c_{12}(A) \\ & \cdot \sum_{\substack{X \in S^\Theta \setminus A \\ X \cap A = \emptyset}} \frac{m_1(A)m_2(X) + m_2(A)m_1(X)}{c_{12}(A) + c_{12}(X)} \\ & + \sum_{\substack{X_1,X_2 \in S^\Theta \setminus A \\ X_1 \cap X_2 = \emptyset \\ u(X_1) \cup u(X_2) = A}} [m_1(X_1)m_2(X_2) + m_1(X_2)m_2(X_1)] \\ & + \Psi_\Theta(A) \cdot \sum_{\substack{X_1,X_2 \in S^\Theta \setminus A \\ X_1 \cap X_2 = \emptyset \\ u(X_1) = u(X_2) = A}} [m_1(X_1)m_2(X_2) + m_2(X_1)m_1(X_2)] \end{aligned}$$

where $c_{12}(A)$ is the non-zero sum of the mass matrix column corresponding to the set A, and the total ignorance characteristic function $\Psi_\Theta(A) = 1$ if A is the total ignorance, and 0 otherwise.

The PCR4 fusion rule improves Milan Daniel's minC rule [3–5]. After applying the conjunctive rule, Daniel uses the proportionalization with respect to the results of the conjunctive rule, and not with respect to the masses assigned to each nonempty set by the sources of information as done in PCR1-3 or the next PCR5. PCR4 also uses the proportionalization with respect to the results of the conjunctive rule, but with PCR4 the conflicting mass $m_{12}(A \cap B) > 0$ when $A \cap B = \emptyset$ is distributed to A and B only because only A and B were involved in the conflict $\{A \cup B$ was not involved in the conflict since $m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B)\}$, while minC [both its versions a) and b)] redistributes the conflicting mass $m_{12}(A \cap B)$ to A, B, and $A \cup B$. Also, for the mixed sets such as $C \cap (A \cup B) = \emptyset$ the conflicting mass $m_{12}(C \cap (A \cup B)) > 0$ is distributed to C and $A \cup B$ because only them were involved in the conflict by PCR4, while minC version a) redistributes $m_{12}(C \cap (A \cup B))$ to C, $A \cup B$, $C \cup A \cup B$ and minC version b) redistributes $m_{12}(C \cap (A \cup B))$ even worse to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$. The PCR5 formula for the fusion of two sources is given by

$$m_{PCR4}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$, one has

$$m_{PCR4}(A) = m_{12}(A) + \sum_{\substack{X \in S^\Theta \setminus A \\ X \cap A = \emptyset}} m_{12}(X) \frac{m_{12}(A \cap X)}{m_{12}(A) + m_{12}(X)},$$

where $m_{12}(\cdot)$ is the conjunctive rule, and all denominators $m_{12}(A) + m_{12}(X) \neq 0$; (if a denominator corresponding to some X is zero, the fraction it belongs to is discarded and the mass $m_{12}(A \cap X)$ is transferred to A and X using PCR3.

8.21.3 PCR5 Fusion Rule

PCR5 fusion rule is the most mathematically exact form of redistribution of the conflicting mass to non-empty sets which follows backwards the tracks of the conjunctive rule formula. But it is the most difficult to implement. In order to better understand it, let's start with some examples:

- **Example 1**:

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0	0.3	0.7

The conjunctive rule yields:

m_{12}	0.42	0.12	0.28

and the conflicting mass $k_{12} = 0.18$.

Only A and B were involved in the conflict,

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B) = m_1(A)m_2(B) = 0.6 \cdot 0.3 = 0.18.$$

Therefore, 0.18 should be distributed to A and B proportionally with respect to 0.6 and 0.3 {i.e. the masses assigned to A and B by the sources $m_1(\cdot)$ and $m_2(\cdot)$} respectively. Let x be the conflicting mass to be redistributed to A and y the conflicting mass to be redistributed to B (out of 0.18), then:

$$\frac{x}{0.6} = \frac{y}{0.3} = \frac{x+y}{0.6+0.3} = \frac{0.18}{0.9} = 0.2,$$

whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$, which is normal since 0.6 is twice bigger than 0.3. Thus:

$$\begin{aligned} m_{PCR4}(A) &= 0.42 + 0.12 = 0.54, \\ m_{PCR4}(B) &= 0.12 + 0.06 = 0.18, \\ m_{PCR4}(A \cup B) &= 0.28 + 0 = 0.28. \end{aligned}$$

This result is the same as PCR2-3.

- **Example 2**:

Let's modify a little the previous example and have the mass matrix

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0.2	0.3	0.5

The conjunctive rule yields:

$$m_{12} \quad 0.50 \quad 0.12 \quad 0.20$$

and the conflicting mass $k_{12} = 0.18$.

The conflict k_{12} is the same as in previous example, which means that $m_2(A) = 0.2$ did not have any impact on the conflict; why?, because $m_1(B) = 0$.

A and B were involved in the conflict, $A \cup B$ is not, hence only A and B deserve a part of the conflict, $A \cup B$ does not deserve.

With PCR5 one redistributes the conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ respectively, i.e. identically as above. The mass $m_2(A) = 0.2$ is not considered to the weighting factors of redistribution since it did not increase or decrease the conflicting mass. One obtains $x = 0.12$ and $y = 0.06$, which added to the previous masses yields:

$$\begin{aligned}
m_{PCR4}(A) &= 0.50 + 0.12 = 0.62,\\
m_{PCR4}(B) &= 0.12 + 0.06 = 0.18,\\
m_{PCR4}(A \cup B) &= 0.20.
\end{aligned}$$

This result is different from all PCR1-4.

- **Example 3**:

  Let's modify a little the previous example and have the mass matrix

	A	B	$A \cup B$
m_1	0.6	0.3	0.1
m_2	0.2	0.3	0.5

  The conjunctive rule yields:

$$m_{12} \quad 0.44 \quad 0.27 \quad 0.05$$

  and the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B) = 0.6 \cdot 0.3 + 0.2 \cdot 0.3 = 0.18 + 0.06 = 0.24.$$

  Now the conflict is different from the previous two examples, because $m_2(A)$ and $m_1(B)$ are both non-null. Then the partial conflict 0.18 should be redistributed to A and B proportionally to 0.6 and 0.3 respectively (as done in previous examples, and we got $x1 = 0.12$ and $y1 = 0.06$), while 0.06 should be redistributed to A and B proportionally to 0.2 and 0.3 respectively.

  For the second redistribution one similarly calculate the proportions:

$$\frac{x2}{0.2} = \frac{y2}{0.3} = \frac{x2 + y2}{0.2 + 0.3} = \frac{0.06}{0.5} = 0.12,$$

  whence $x = 0.2 \cdot 0.12 = 0.024$, $y = 0.3 \cdot 0.12 = 0.036$. Thus:

$$\begin{aligned}
m_{PCR4}(A) &= 0.44 + 0.12 + 0.024 = 0.584,\\
m_{PCR4}(B) &= 0.27 + 0.06 + 0.036 = 0.366,\\
m_{PCR4}(A \cup B) &= 0.05 + 0 = 0.050.
\end{aligned}$$

  This result is different from PCR1-4.

The formula of the PCR5 fusion rule for two sources is given by [35]:

$$m_{PCR5}(\emptyset) = 0,$$

and $\forall A \in S^{\Theta} \setminus \emptyset$, one has

$$m_{PCR5}(A) = m_{12}(A) + \sum_{\substack{X \in S^{\Theta} \setminus \{A\} \\ X \cap A = \emptyset}} \left[\frac{m_1(A)^2 \cdot m_2(X)}{m_1(A) + m_2(X)} + \frac{m_2(A)^2 \cdot m_1(X)}{m_2(A) + m_1(X)} \right],$$

where $m_{12}(\cdot)$ is the conjunctive rule, and all denominators are different from zero; if a denominator is zero, the fraction it belongs to is discarded.

The general PCR5 formula for $s \geq 2$ sources is given by (see Chapter 1)

$$m_{PCR5}(\emptyset) = 0,$$

and $\forall A \in S^{\Theta} \setminus \emptyset$ by

$$m_{PCR5}(A) = m_{12\ldots s}(A) + \sum_{\substack{2 \leq t \leq s \\ 1 \leq r_1, \ldots, r_t \leq s \\ 1 \leq r_1 < r_2 < \ldots < r_{t-1} < (r_t = s)}} \sum_{\substack{X_{j_2}, \ldots, X_{j_t} \in S^{\Theta} \setminus \{A\} \\ \{j_2, \ldots, j_t\} \in \mathcal{P}^{t-1}(\{1, \ldots, n\}) \\ A \cap X_{j_2} \cap \ldots \cap X_{j_s} = \emptyset \\ \{i_1, \ldots, i_s\} \in \mathcal{P}^s(\{1, \ldots, s\})}} \frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(A)^2) \cdot [\prod_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(A)) + [\sum_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]},$$

where i, j, k, r, s and t are integers. $m_{12\ldots s}(A)$ corresponds to the conjunctive consensus on A between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; $\mathcal{P}^k(\{1, 2, \ldots, n\})$ is the set of all subsets of k elements from $\{1, 2, \ldots, n\}$ (permutations of n elements taken by k), the order of elements doesn't count.

8.21.4 PCR6 Fusion Rule

PCR6 was developed by A. Martin and C. Osswald in 2006 (see Chapters [22] and [23] for more details and applications of this new rule) and it is an alternative of PCR5 for the general case when the number of sources to combine become greater than two (i.e. $s \geq 3$). PCR6 does not follow back on the track of conjunctive rule as PCR5 general formula does, but it gets better intuitive results. For $s = 2$ PCR5 and PCR6 coincide. The general formula for PCR6[2] when extended to super-power set S^{Θ} is:

$$m_{PCR6}(\emptyset) = 0,$$

and $\forall A \in S^{\Theta} \setminus \emptyset$

[2] Two extensions of PCR6 (i.e. PCR6f and PCR6g) are also proposed by A. Martin and C. Osswald in [22].

$$m_{\text{PCR6}}(A) = m_{12\ldots s}(A) + \sum_{i=1}^{s} m_i(A)^2 \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)} \cap A \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(s-1)}) \in (S^\Theta)^{s-1}}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right),$$

with $m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$ and where $m_{12\ldots s}(.)$ is the conjunctive consensus rule and σ_i counts from 1 to s avoiding i, i.e.:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j + 1 & \text{if } j \geq i, \end{cases}$$

8.22 The minC Rule

The minC rule (minimum conflict rule) proposed by M. Daniel in [3–5] improves Dempster's rule since the distribution of the conflicting mass is done from each partial conflicting mass to the subsets of the sets involved in partial conflict proportionally with respect to the results of the conjunctive rule results for each such subset. It goes by types of conflicts. The author did not provide an analytical formula for this rule in his previous publications but only in Chapter 4 of this volume. minC rule is commutative, associative, and non-idempotent.

Let $m_{12}(X \cap Y) > 0$ be a conflicting mass, where $X \cap Y = \emptyset$, and X, Y may be singletons or mixed sets (i.e. unions or intersections of singletons).

minC has two versions, minC a) and minC b), which differs from the way the redistribution is done: either to the subsets X, Y, and $X \cup Y$ in version a), or to all subsets of $P(u(X) \cup u(Y))$ in version b).

One applies the conjunctive rule, and then the partial conflict, say $m_{12}(A \cap B)$, when $A \cap B = \emptyset$, is redistributed to A, B, $A \cup B$ proportionally to the masses $m_{12}(A)$, $m_{12}(B)$, and $m_{12}(A \cup B)$ respectively in both versions a) and b). PCR4 redistributes the conflicting mass to A and B since only them were involved in the conflict.

But for a mixed set, as shown above, say $C \cap (A \cup B) = \emptyset$, the conflicting mass $m_{12}(C \cap (A \cup B)) > 0$ is distributed by PCR4 to C and $A \cup B$ because only them were involved in the conflict, while the minC version a) redistributes $m_{12}(C \cap (A \cup B))$ to C, $A \cup B$, $C \cup A \cup B$, and minC version b) redistributes $m_{12}(C \cap (A \cup B))$ even worse to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$.

Another example is that the mass $m_{12}(A \cap B \cap C)) > 0$, when $A \cap B \cap C = \emptyset$, is redistributed in both versions minC a) and minC b) to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$.

When the conjunctive rule results are zero for all the nonempty sets that are redistributed conflicting masses, the conflicting mass is averaged to each such set.

8.23 The Consensus Operator

Consensus Operator (CO) proposed by A. Jøsang in [16] is defined only on binary frames of discernment. CO doesn't work on non-exclusive elements (i.e. on models with nonempty intersections of sets).

On the frame $\Theta = \{\theta_1, \theta_2\}$ of exclusive elements, θ_2 is considered the complement/negation of θ_1.

If the frame of discernment has more than two elements, then by a simple or normal coarsening it is possible to derive a binary frame containing any element A and its complement $\mathcal{C}(A)$. Let $m(\cdot)$ be a bba on a (coarsened) frame $\Theta = \{A, \mathcal{C}(A)\}$, then one defines an opinion resulted from this bba is:

$$w_A = (b_A, d_A, u_A, \alpha_A),$$

where $b_A = m(A)$ is the belief of A, $d_A = m(\mathcal{C}(A))$ is the disbelief of A, $u_A = m(A \cup \mathcal{C}(A))$ is the uncertainty of A, and α_A represents the atomicity of A. Of course $b_A + d_A + u_A = 1$, for $A \neq \emptyset$.

The relative atomicity expresses information about the relative size of the state space (i.e. the frame of discernment). For every operator, the relative atomicity of the output belief is computed as a function of the input belief operands. The relative atomicity of the input operands is determined by the state space circumstances, or by a previous operation in case that operation's output is used as input operand. The relative atomicity itself can also be uncertain, and that's what's called state space uncertainty. Possibly the state space uncertainty is a neglected problem in belief theory. It relates to Smets' open world, and to DSm paradoxical world. In fact, the open world breaks with the "exhaustive" assumption, and the paradoxical world breaks with the "exclusive" assumption of classic belief theory.

CO is commutative, associative, and non-idempotent.

Having two experts with opinions on the same element A,

$$\begin{aligned} w_{1A} &= (b_{1A}, d_{1A}, u_{1A}, \alpha_{1A}) \text{ and} \\ w_{2A} &= (b_{2A}, d_{2A}, u_{2A}, \alpha_{2A}), \text{ one first computes} \\ k &= u_{1A} + u_{2A} - u_{1A} \cdot u_{2A}. \end{aligned}$$

Let's note by $b_{12A} = (b_{12A}, d_{12A}, u_{12A}, \alpha_{12A})$ the consensus opinion between w_{1A} and w_{2A}. Then:

a) for $k \neq 0$ one has:

$$\begin{aligned} b_{12A} &= (b_{1A} \cdot u_{2A} + b_{2A} \cdot u_{1A})/k \\ d_{12A} &= (d_{1A} \cdot u_{2A} + d_{2A} \cdot u_{1A})/k \\ u_{12A} &= (u_{1A} \cdot u_{2A})/k \\ \alpha_{12A} &= \frac{\alpha_{1A}u_{2A} + \alpha_{2A}u_{1A} - (\alpha_{1A} + \alpha_{2A})u_{1A}u_{2A}}{u_{1A} + u_{2A} - 2u_{1A}u_{2A}} \end{aligned}$$

b) for $k = 0$ one has:

$$\begin{aligned} b_{12A} &= (\gamma_{12A} \cdot b_{1A} + b_{2A})/(\gamma_{12A} + 1) \\ d_{12A} &= (\gamma_{12A} \cdot d_{1A} + d_{2A})/(\gamma_{12A} + 1) \\ u_{12A} &= 0 \\ \alpha_{12A} &= (\gamma_{12A} \cdot \alpha_{1A} + \alpha_{2A})/(\gamma_{12A} + 1) \end{aligned}$$

where $\gamma_{12A} = u_{2A}/u_{1A}$ represents the relative dogmatism between opinions b_{1A} and b_{2A}.

The formulas are not justified, and there is not a well-defined method for computing the relative atomicity of an element when a bba is known.

For frames of discernment of size greater than n, or with many sources, or in the open world it is hard to implement CO.

A bba $m(\cdot)$ is called *Bayesian* on the frame $\Theta = \{\theta_1, \theta_2\}$ of exclusive elements if $m(\theta_1 \cup \theta_2) = 0$, otherwise it is called *non Bayesian*.

If one bba is Bayesian, say $m_1(\cdot)$, and another is not, say $m_2(\cdot)$, then the non Bayesian bba is ignored! See below $m_{CO}(\cdot) = m_1(\cdot)$:
Example

	A	B	$A \cup B$
m_1	0.3	0.7	0.0
m_2	0.8	0.1	0.1
m_{CO}	0.3	0.7	0.0

Because

	b_A	d_A	u_A	α_A
m_{1A}	0.3	0.7	0.0	0.5
m_{2A}	0.8	0.1	0.1	0.5

$\alpha_{1A} = \alpha_{2A} = \frac{|A \cap \Theta|}{|\Theta|} = 0.5$, where $|X|$ means the cardinal of X, whence $\alpha_{12A} = 0.5$.
Similarly one computes the opinion on B, because:

	b_B	d_B	u_B	α_B
m_{1B}	0.7	0.3	0.0	0.5
m_{2B}	0.1	0.8	0.1	0.5

If both bba's are Bayseian, then one uses their arithmetic mean.

8.24 Zhang's Center Combination Rule

The Center Combination Rule proposed by L. Zhang in [54] is given by

$$\forall A \in S^\Theta, \text{ one has } m_Z(A) = k \cdot \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1| \cdot |X_2|} m_1(X_1) m_2(X_2).$$

where k is a renormalization factor, $|X|$ is the cardinal of the set X, and

$$r(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1| \cdot |X_2|}$$

represents the degree (measure) of intersection of the sets X_1 and X_2.

In Dempster's approach the degree of intersection was assumed to be 1.

The degree of intersection could be defined in many ways, for example

$$r(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|}$$

could be better defined this way since if the intersection is empty the degree of intersection is zero, while for the maximum intersection, i.e. when $X_1 = X_2$, the degree of intersection is 1.

One can attach the $r(X_1, X_2)$ to many fusion rules.

8.25 The Convolutive x-Averaging

The Convolutive x-Averaging proposed by Ferson-Kreinovich in [10] is defined as

$$\forall A \in S^{\Theta}, \text{ one has } m_X(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ (X_1+X_2)/2=A}} m_1(X_1)m_2(X_2)$$

This rule works for hypotheses defined as subsets of the set of real numbers.

8.26 The α-junctions Rules

The α-junctions rules [37] are generalizations of the above Conjunctive and Disjunctive Rules, and they are parameterized with respect to $\alpha \in [0,1]$. Philippe Smets finds the rules for the elementary frame of discernment Θ with two hypotheses, using a matrix operator $\mathbf{K}_X$, for each $X \in \{\emptyset, A, B, A \cup B\}$ and shows that it is possible to extend them by iteration to larger frames of discernment. These rules are more theoretical and hard to apply.

8.27 The Cautious Rule

The Cautious Rule[3] has been proposed by Philippe Smets in 2000 and is just theoretical. Also, Smets does not provide a formula or a method for calculating this rule. He states [38] this *Theorem*:

Let m_1, m_2 be two bba's, and q_1, q_2 their corresponding commonality functions, and $SP(m_1)$, $SP(m_2)$ the set of specializations of m_1 and m_2 respectively. Then the hyper-cautious combination rule

$$m_{1\circledast 2} = \min\{m \mid m \in SP(m_1) \cap SP(m_2)\},$$

and the commonality of $m_{1\circledast 2}$ is q_{12} where $q_{12}(A) = \min\{q_1(A), q_2(A)\}$.

We recall that the *commonality function* of a bba $m(\cdot)$ is $q : S^{\Theta} \to [0,1]$ such that:

$$q(A) = \sum_{\substack{X \in S^{\Theta} \\ X \supseteq A}} m(X) \text{ for all } A \in S^{\Theta}.$$

Now a few words about the least commitment and specialization.

a) *Least Commitment, or Minimum Principle*, means to assign a missing mass of a bba or to transfer a conflicting mass to the least specific element in the frame of discernment (in most of the cases to the partial ignorances or to the total ignorance). "The Principle of Minimal Commitment consists in selecting the least committed belief function in a set of equally justified belief functions. This selection procedure does not always lead to a unique solution in which case extra requirements are added. The principle formalizes the idea that one should never give more support than justified to any subset of Ω. It satisfies a form of skepticism, of a commitment, of conservatism in the allocation of our belief. In its spirit, it is not far from what the probabilists try to achieve with the maximum entropy principle." [Philippe Smets]

[3] More details about this rule can be found in [6].

b) About *specialization* [18]:

Suppose at time t_o one has the evidence $m_0(\cdot)$ which gives us the value of an hypothesis A as $m_0(A)$. When a new evidence $m_1(\cdot)$ comes in at time $t_1 > t_0$, then $m_0(A)$ might flow down to the subsets of A therefore towards a more specific information. The impact of a new bba might result in a redistribution of the initial mass of A, $m_0(A)$, towards its more specific subsets. Thus $m_1(\cdot)$ is called a specialization of $m_0(\cdot)$.

8.28 Other fusion rules

Yen's rule is related to fuzzy set, while the p-boxes method to upper and lower probabilities (neutrosophic probability is a generalization of upper and lower probability) - see Sandia Tech. Rep.

8.29 Fusion rules based on T-norm and T-conorm

These rules proposed by Tchamova, Dezert and Smarandache in [40] started from the T-norm and T-conorm respectively in fuzzy and neutrosophic logics, where the "and" logic operator $\wedge$ corresponds in fusion to the conjunctive rule, while the "or" logic operator $\vee$ corresponds to the disjunctive rule. While the logic operators deal with degrees of truth and degrees of falsehood, the fusion rules deal with degrees of belief and degrees of disbelief of hypotheses.

A **T-norm** is a function $T_n : [0,1]^2 \to [0,1]$, defined in fuzzy/neutrosophic set theory and fuzzy/neutrosophic logic to represent the "intersection" of two fuzzy/neutrosophic sets and the fuzzy/neutrosophic logical operator "and" respectively. Extended to the fusion theory the T-norm will be a substitute for the conjunctive rule.

The T-norm satisfies the conditions:

a) Boundary Conditions: $T_n(0,0) = 0$, $T_n(x,1) = x$.

b) Commutativity: $T_n(x,y) = T_n(y,x)$.

c) Monotonicity: If $x \leq u$ and $y \leq v$, then $T_n(x,y) \leq T_n(u,v)$.

d) Associativity: $T_n(T_n(x,y),z) = T_n(x,T_n(y,z))$.

There are many functions which satisfy the T-norm conditions. We present below the most known ones:

- The Algebraic Product T-norm: $T_{n-\text{algebraic}}(x,y) = x \cdot y$
- The Bounded T-norm: $T_{n-\text{bounded}}(x,y) = \max\{0, x+y-1\}$
- The Default (min) T-norm [21, 51]: $T_{n-\min}(x,y) = \min\{x,y\}$.

A **T-conorm** is a function $T_c : [0,1]^2 \to [0,1]$, defined in fuzzy/neutrosophic set theory and fuzzy/neutrosophic logic to represent the "union" of two fuzzy/neutrosophic sets and the fuzzy/neutrosophic logical operator "or" respectively. Extended to the fusion theory the T-conorm will be a substitute for the disjunctive rule.

The T-conorm satisfies the conditions:

a) Boundary Conditions: $T_c(1,1) = 1$, $T_c(x,0) = x$.

b) Commutativity: $T_c(x,y) = T_c(y,x)$.

c) Monotonicity: if $x \leq u$ and $y \leq v$, then $T_c(x,y) \leq T_c(u,v)$.

d) Associativity: $T_c(T_c(x,y),z) = T_c(x, T_c(y,z))$.

There are many functions which satisfy the T-conorm conditions. We present below the most known ones:

- The Algebraic Product T-conorm: $T_{c-\text{algebraic}}(x,y) = x + y - x \cdot y$
- The Bounded T-conorm: $T_{c-\text{bounded}}(x,y) = \min\{1, x+y\}$
- The Default (max) T-conorm [21, 51]: $T_{c-\max}(x,y) = \max\{x,y\}$.

Then, the T-norm Fusion rules are defined as follows:

$$m_{\cap 12}(A) = \sum_{\substack{X,Y \in S^{\Theta} \\ X \cap Y = A}} T_n(m_1(X), m_2(Y))$$

and the T-conorm Fusion rules are defined as follows:

$$m_{\cup 12}(A) = \sum_{\substack{X,Y \in S^{\Theta} \\ X \cup Y = A}} T_c(m_1(X), m_2(Y))$$

The min T-norm rule yields results, very closed to Conjunctive Rule. It satisfies the principle of neutrality of the vacuous bba, reflects the majority opinion, converges towards idempotence. It is simpler to apply, but needs normalization.

"What is missed it is a strong justification of the way of presenting the fusion process. But we think, the consideration between two sources of information as a vague relation, characterized with the particular way of association between focal elements, and corresponding degree of association (interaction) between them is reasonable." (Albena Tchamova)

"Min rule can be interpreted as an optimistic lower bound for combination of bba and the below Max rule as a prudent/pessimistic upper bound." (Jean Dezert)

The T-norm and T-conorm are commutative, associative, isotone, and have a neutral element.

8.30 Improvements of fusion rules

Degree of Intersection

The degree of intersection measures the percentage of overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view [27]:

$$d(X_1 \cap X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|},$$

where $|X|$ means cardinal of the set X.

This definition of the degree of intersection is different from Zhang's previous one. For the minimum intersection/overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of intersection is 0, while for the maximum intersection/overlapping, i.e. when $X_1 = X_2$, the degree of intersection is 1.

Degree of Union

The degree of intersection measures the percentage of non-overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view [27]:

$$d(X_1 \cup X_2) = \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|}.$$

For the maximum non-overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of union is 1, while for the minimum non-overlapping, i.e. when $X_1 = X_2$, the degree of union is 0.

The sum of degrees of intersection and union is 1 since they complement each other.

Degree of Inclusion

The degree of intersection measures the percentage of the included region X_1 with respect to the includant region X_2 [27]:

Let $X_1 \subseteq X_2$, then

$$d(X_1 \subseteq X_2) = \frac{|X_1|}{|X_2|}.$$

$d(\emptyset \subseteq X_2) = 0$ because nothing is included in X_2, while $d(X_2 \subseteq X_2) = 1$ because X_2 is fulfilled by inclusion. By definition $d(\emptyset \subseteq \emptyset) = 1$.

And we can generalize the above degree for $n \geq 2$ sets.

Improvements of Credibility, Plausibility and Communality Functions

Thus the Bel($\cdot$), Pl($\cdot$) and Com($\cdot$) functions can incorporate in their formulas the above degrees of inclusion and intersection respectively:

- Credibility function improved:

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } \mathrm{Bel}_d(A) = \sum_{\substack{X \in S^\Theta \\ X \subseteq A}} \frac{|X|}{|A|} m(X)$$

- Plausibility function improved:

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } \mathrm{Pl}_d(A) = \sum_{\substack{X \in S^\Theta \\ X \cap A \neq \emptyset}} \frac{|X \cap A|}{|X \cup A|} m(X)$$

- Communality function improved:

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } \mathrm{Com}_d(A) = \sum_{\substack{X \in S^\Theta \\ A \subseteq X}} \frac{|A|}{|X|} \cdot m(X)$$

Improvements of quantitative fusion rules

- Disjunctive rule improved:

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cup d}(A) = k_{\cup d} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where $k_{\cup d}$ is a constant of renormalization.

- Dezert-Smarandache classical rule improved:

$$\forall A \in S^{\Theta}, \text{ one has } m_{\text{DSmCd}}(A) = k_{\text{DSmCd}} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where k_{DSmCd} is a constant of renormalization. It is similar with the Zhang's Center Combination rule extended on the Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$ and using another definition for the degree of intersection.

- Dezert-Smarandache hybrid rule improved:

$$\begin{aligned} &\forall A \in S^{\Theta} \setminus \emptyset \text{ one has} \\ m_{\text{DSmHd}}(A) &= k_{\text{DSmHd}} \\ &\cdot \Bigg\{ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2) \\ &+ \sum_{\substack{X_1, X_2 \in \emptyset \\ (A=U) \vee \{U \in \emptyset \wedge A = I\}}} m_1(X_1) m_2(X_2) \\ &+ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2) \Bigg\} \end{aligned}$$

  where k_{DSmHd} is a constant of renormalization.

- Smets' rule improved:

$$m_S(\emptyset) = k_{12} = k_{Sd} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  and

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{one has } m_S(A) = k_{Sd} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where k_{Sd} is a constant of renormalization.

- Yager's rule improved:

$$m_Y(\emptyset) = 0, m_Y(I) = k_{Yd} \cdot \{m_1(I)m_2(I) + \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2)\}$$

  and

$$\forall A \in S^\Theta \setminus \{\emptyset, I\}, \text{ one has } m_{Yd}(A) = k_{Yd} \cdot \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2).$$

  where $I =$ total ignorance and k_{Yd} is a constant of renormalization.

- Dubois-Prade's rule improved:

$$m_{DP}(\emptyset) = 0,$$

  and

$$\begin{aligned} &\forall A \in S^\Theta \setminus \emptyset \text{ one has} \\ m_{DP}(A) &= k_{DPd} \\ &\cdot \left\{ \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2) \right. \\ &\left. + \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2) \right\}, \end{aligned}$$

  where k_{DPd} is a constant of renormalization.

8.31 Extension of bba on neutrosophic sets

- Let T, I, $F \subseteq [0,1]$. An element $x(T, I, F)$ belongs to a *neutrosophic set* M as follows: its membership is T, its nonmembership is F, and its indeterminacy is I.

- Define a **neutrosophic basic belief assignment** (nbba):

$$n(\cdot) : S^\Theta \to [0,1]^3, \quad n(A) = (T_A, I_A, F_A),$$

  where $T_A =$ belief in A, $F_A =$ disbelief in A, $I_A =$ indeterminacy on A.

- *Admissibility condition*: For each $A \in S^\Theta$, there exist scalars $t_A \in T_A$, $i_A \in I_A$, $f_A \in F_A$ such that:

$$\sum_{A \in S^\Theta} (t_A + i_A + f_A) = 1.$$

- When $F_A = I_A = \phi$ nbba coincides with bba.

- Intuitionistic Fuzzy Set can not be applied since the sum of components for a single set < 1.
- N-norms/conorms [42, 43] use nbba's for information fusion.
- *All fusion rules and functions can be extended on nbba's.*

N-norms

$$N_n : \big([0,1] \times [0,1] \times [0,1]\big)^2 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$N_n\Big(\big(x_1, x_2, x_3\big), \big(y_1, y_2, y_3\big)\Big) = \Big(N_nT\big(x_1, y_1\big), N_nI\big(x_2, y_2\big), N_nF\big(x_3, y_3\big)\Big)$$

- For each component, $J \in \{T, I, F\}$, N_n satisfies the conditions:

  a) Boundary Conditions: $N_nJ(0,0) = 0$, $N_nJ(x,1) = x$.
  b) Commutativity: $N_nJ(x,y) = N_nJ(y,x)$.
  c) Monotonicity: If $x \leq u$ and $y \leq v$, then $N_nJ(x,y) \leq N_nJ(u,v)$.
  d) Associativity: $N_nJ(N_nJ(x,y),z) = N_nJ(x, N_nJ(y,z))$.

  N_n represents *intersection* in neutrosophic set theory, respectively the *and* operator in neutrosophic logic.

- Most known ones:
  - The Algebraic Product N-norm: $N_{n-\text{algebraic}}J(x,y) = x \cdot y$
  - The Bounded N-Norm: $N_{n-\text{bounded}}J(x,y) = \max\{0, x+y-1\}$
  - The Default (min) N-norm: $N_{n-\min}J(x,y) = \min\{x,y\}$.

N-conorms

$$N_c : \big([0,1] \times [0,1] \times [0,1]\big)^2 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$N_c\Big(\big(x_1, x_2, x_3\big), \big(y_1, y_2, y_3\big)\Big) = \Big(N_cT\big(x_1, y_1\big), N_cI\big(x_2, y_2\big), N_cF\big(x_3, y_3\big)\Big)$$

- For each component, $J \in \{T, I, F\}$, N_c satisfies the conditions:

  a) Boundary Conditions: $N_cJ(1,1) = 1$, $N_cJ(x,0) = x$.
  b) Commutativity: $N_cJ(x,y) = N_cJ(y,x)$.
  c) Monotonicity: if $x \leq u$ and $y \leq v$, then $N_cJ(x,y) \leq N_cJ(u,v)$.
  d) Associativity: $N_cJ(N_cJ(x,y),z) = N_cJ(x, N_cJ(y,z))$.

  N_c represents *union* in neutrosophic set theory, respectively the *or* operator in neutrosophic logic.

- Most known ones:
  - The Algebraic Product N-conorm: $N_{c-\text{algebraic}}J(x,y) = x + y - x \cdot y$
  - The Bounded N-conorm: $N_{c-\text{bounded}}J(x,y) = \min\{1, x+y\}$
  - The Default (max) N-conorm: $N_{c-\max}J(x,y) = \max\{x,y\}$.

N-norm and N-conorm based fusion rules

Let $n_1(\cdot)$ and $n_2(\cdot)$ be nbba's.

N-norm fusion rulesare defined as follows:

$$n_{\cap 12}(A) = \sum_{\substack{X,Y \in S\wedge\Theta \\ X\cap Y=A}} N_n(n_1(X), n_2(Y))$$

N-conorm fusion rules are defined as follows:

$$n_{\cup 12}(A) = \sum_{\substack{X,Y \in S\wedge\Theta \\ X\cup Y=A}} N_c(n_1(X), n_2(Y))$$

- they can replace the conjunctive respectively disjunctive rules (Smarandache 2004)
- need normalizations

Example of N-norm fusion rule

A first doctor's belief about a patient's disease A is 0.4, disbelief 0.2, while about the second disease B his belief is 0.1 with disbelief 0.2 and not sure 0.1. Second doctor is more confident in disease B with 0.6, his disbelief on A is 0.1 and 0.3 not sure on A.

Hence $n_1(A) = (0.4, 0, 0.2)$, $n_1(B) = (0.1, 0.1, 0.2)$, and $n_2(A) = (0, 0.3, 0.1)$, $n_2(B) = (0.6, 0, 0)$ are nbba's and frame of discernment $\{A, B\}$. Using the Algebraic Product N-norm fusion rule we get:

$$n_{12}(A) = (0, 0, 0.02), n_{12}(B) = (0.06, 0, 0),$$
$$n_{12}(A \cap B) = (0.24, 0, 0) + (0, 0.03, 0.02) = (0.24, 0.03, 0.02).$$

Transfer the conflicting mass $n_{12}(A\cap B)$ to A and B proportionally to their mass sums (following PCR3):

$$x_1/(0.4 + 0) = y_1/(0.1 + 0.6) = 0.24/1.1,$$

hence

$$x_1 = 0.4(0.24/1.1) = 0.087273, \quad y_1 = 0.7(0.24/1.1) = 0.152727;$$

$$x_2/0.3 = y_2/0.1 = 0.03/0.4,$$

hence

$$x_2 = 0.3(0.03/0.4) = 0.0225, \quad y_2 = 0.1(0.03/0.4) = 0.0075;$$

$$x_3/0.3 = y_3/0.2 = 0.02/0.5,$$

hence

$$x_3 = 0.3(0.02/0.5) = 0.012, y_3 = 0.2(0.02/0.5) = 0.008.$$

Summing them with the previous one gets: $n_{12tr}(A) = (0.087273, 0.0225,\ 0.032)$, $n_{12tr}(B) = (0.212727, 0.0075, 0.008)$ and renormalize (divide by their sum$= 0.37$):

$$n_{12\text{tr-norm}}(A) = (0.235873, 0.060811, 0.086486),$$
$$n_{12\text{tr-norm}}(B) = (0.574938, 0.02027, 0.021622).$$

Remark: If first done the normalization and second the transfer the result will be the same [20].

8.32 Unification of Fusion Rules (UFR)

If variable y is *directly proportional* with variable p, then $y = k \cdot p$, where k is a constant. If variable y is *inversely proportional* with variable q, then $y = k \cdot \frac{1}{q}$; we can also say that y is directly proportional with variable $\frac{1}{q}$. In a general way, we say that y is directly proportional with variables $p_1, p_2, \ldots, p_m$ and inversely proportionally with variables $q_1, q_2, \ldots, q_n$, where $m, n \geq 1$, then:

$$y = k \cdot \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{q_1 \cdot q_2 \cdot \ldots \cdot q_n} = k \cdot \frac{P}{Q},$$

where $P = \prod_{i=1}^{m} p_i$ and $Q = \prod_{j=1}^{n} q_j$.

Then a Unification of Fusion Rules (UFR) is given by : $m_{UFR}(\emptyset) = 0$ and $\forall A \in S^\Theta \setminus \emptyset$ one has

$$m_{UFR}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \star X_2 = A}} d(X_1 \star X_2) R(X_1, X_2) + \frac{P(A)}{Q(A)} \cdot \sum_{\substack{X \in S^\Theta \setminus A \\ X \star A \in E}} d(X \star A) \cdot \frac{R(A, X)}{P(A)/Q(A) + P(X)/Q(X)}$$

where $\star$ means intersection or union of sets (depending on the application or problem to be solved);
$d(X \star Y)$ is the degree of intersection or union respectively;
$R(X, Y)$ is a T-norm/conorm (or N-norm/conorm in a more general case) fusion combination rule respectively (extension of conjunctive or disjunctive rules respectively to fuzzy or neutrosophic operators) or any other fusion rule; the T-norm and N-norm correspond to the intersection of sets, while the T-conorm and N-conorm to the disjunction of sets;
E is the ensemble of sets (in majority cases they are empty sets) whose masses must be transferred (in majority cases to non-empty sets, but there are exceptions for the open world);
$P(A)$ is the product of all parameters directly proportional with A;
while $Q(A)$ is the product of all parameters inversely proportional with A [in most of the cases $P(A)$ and $Q(A)$ are derived from the masses assigned to the set A by the sources].

8.33 Unification of Fusion Theories (UFT)

As a conclusion, since no theory neither rule fully satisfy all needed applications, the author proposes [27–30] a Unification of (Quantitative) Fusion Theories extending the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closures of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers a fuzzy or neutrosophic complement).

And, at each application, one selects the most appropriate model, rule, and algorithm of implementation.

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or even like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that $X_1 \veebar X_2 \veebar \cdots \veebar X_n$ means either X_1 is reliable, or X_2, or and so on, or X_n, but not two or more in the same time.

4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes the vacuum bba (i.e. $m(\Theta) = 1$, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

6) If all sources are reliable, or the unreliable sources have been discounted (in the *default case*), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$, no matter what contradictions (or model) the problem has. I emphasize that the super-power set S^Θ generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

8) In the case the model is not known (the *default case*), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on S^Θ, and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then:

   9.1) If an intersection $A \cap B$ is not empty, we keep the mass $m(A \cap B)$ on $A \cap B$, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

   9.2) If the intersection $A \cap B = \emptyset$ is empty, meaning contradiction, we do the following :

   9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass $m(A \cap B)$ to $m(A \cup B)$, since $A \cup B$ means at least one is right [here one gets *Yager's* if $n = 2$, or *Dubois-Prade,* or *DSmT*];

   9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass $m(A \cap B)$ to hypothesis A (nothing is transferred to B);

   9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass $m(A \cap B)$ to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2–5*]

   9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass $m(A \cap B)$ to $A \cup B$ (the more pessimistic the further one gets in the Specificity Chains: $(A \cap B) \subset A \subset (A \cup B) \subset I$); this is also the *default case* [using *DP's*, *DSm hybrid rule*, *Yager's*]; if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass $m(A \cap B)$ to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

   9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass $m(A \cap B)$ to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) — different from A, B, $A \cup B$ — for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a *closed world* [DSm hybrid rule]; but, if it is an *open world*, we can transfer the mass $m(A \cap B)$ to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

   9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. $n = 2$ is the size of the frame of discernment), then one considers the *open world* and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: $A \cap B \cap C$, etc. and even for mixed sets: $A \cup (B \cup C)$, etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of *decaying the old sources* because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule.

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Chapter 9

Belief Conditioning Rules

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Abstract: *In this chapter we propose a new family of Belief Conditioning Rules (BCR) for belief revision. These rules are not directly related with the fusion of several sources of evidence but with the revision of a belief assignment available at a given time according to the new truth (i.e. conditioning constraint) one has about the space of solutions of the problem.*

9.1 Introduction

In this chapter we define several Belief Conditioning Rules (BCR) for use in information fusion and for belief revision. Suppose we have a basic belief assignment (bba) $m_1(.)$ defined on hyper-power set D^Θ, and we find out that the truth is in a given element $A \in D^\Theta$. So far in literature devoted to belief functions and the mathematical theory of evidence, there has been used Shafer's Conditioning Rule (SCR) [2], which simply combines the mass $m_1(.)$ with a specific bba focused on A, i.e. $m_S(A) = 1$, and then uses Dempster's rule to transfer the conflicting mass to non-empty sets. But in our opinion this conditioning approach based on the combination of two bba's is *subjective* since in such procedure both sources are subjective. While conditioning a mass $m_1(.)$, *knowing* (or assuming) that the truth is in A, means that we have an absolute (not subjective) information, i.e. the truth is in A has occurred (or is assumed to have occurred), thus A was realized (or is assumed to be realized), hence it is an absolute truth. "Truth in A" must therefore be considered as an absolute truth when conditioning, while $m_S(A) = 1$ used in SCR does not refer to an absolute truth actually, but only to a subjective certainty in the possible occurrence of A given by a second source of evidence. This is the main and fundamental distinction between our approaches (BCRs) and Shafer's (SCR). In our opinion, SCR does not do a conditioning, but only a fusion of $m_1(.)$ with a particular bba $m_S(A) = 1$. The main advantage of SCR is that it is simple and thus very appealing, and in some cases it gives the same results with some BCRs, and it remains coherent with conditional probability when $m_1(.)$ is a Bayesian belief assignment. In the sequel, we will present many (actually thirty one BCR rules, denoted BCR1-BCR31) new alternative issues for belief conditioning. The sequel does not count: a) if we first know the source $m_1(.)$ and then that the truth is in A (or is supposed to be in A), or b) if we first know (or assume) the truth is in A, and then we find the source $m_1()$.The results of conditioning are the same. In addition, we work on a hyper-power set, that is a generalization of the power set. The best among these BCR1-31, that we recommend researchers to use, are: BCR17 for a pessimistic/prudent view on conditioning problem and a more refined redistribution of conflicting masses, or BCR12 for a very pessimistic/prudent view and less refined redistribution. After a short presentation of SCR rule, we present in the following sections all new BCR rules we propose, many examples, and a very important and open challenging question about belief fusion and conditioning.

9.2 Shafer's conditioning rule (SCR)

Before going further in the development of new belief conditioning rules, it is important to recall the conditioning of beliefs proposed by Glenn Shafer in [2] (p.66–67) and reported below.

So, let's suppose that the effect of a new evidence (say source 2) on the frame of discernment Θ is to establish a particular subset $B \subset \Theta$ with certainty. Then Bel_2 will give a degree of belief one to the proposition corresponding to B and to every proposition implied by it:

$$\mathrm{Bel}_2(A) = \begin{cases} 1, & \text{if } B \subset A; \\ 0, & \text{otherwise.} \end{cases}$$

Since the subset B is the only focal element of Bel_2, its basic belief assignment is one, i.e. $m_2(B) = 1$. Such a function Bel_2 is then combinable with the (prior) Bel_1 as long as $\mathrm{Bel}_1(\bar{B}) < 1$, and the Dempster's rule of combination (denoted $\oplus$) provides the conditional belief $\mathrm{Bel}_1(.|B) = \mathrm{Bel}_1 \oplus \mathrm{Bel}_2$ (according to Theorem 3.6 in [2]). More specifically, one gets for all $A \subset \Theta$,

$$\mathrm{Bel}_1(A|B) = \frac{\mathrm{Bel}_1(A \cup \bar{B}) - \mathrm{Bel}_1(\bar{B})}{1 - \mathrm{Bel}_1(\bar{B})}$$

$$\mathrm{Pl}_1(A|B) = \frac{\mathrm{Pl}_1(A \cap B)}{\mathrm{Pl}_1(B)}$$

where Pl(.) denotes the plausibility function.

9.3 Belief Conditioning Rules (BCR)

Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, $n \geq 2$, and the hyper-power set[1] D^Θ. Let's consider a basic belief assignment (bba) $m(.) : D^\Theta \mapsto [0,1]$ such that $\sum_{X \in D^\Theta} m(X) = 1$.

Suppose one finds out that the truth is in the set $A \in D^\Theta \setminus \{\emptyset\}$. Let $\mathcal{P}_\mathcal{D}(A) = 2^A \cap D^\Theta \setminus \{\emptyset\}$, i.e. all non-empty parts (subsets) of A which are included in D^Θ. Let's consider the normal cases when $A \neq \emptyset$ and $\sum_{Y \in \mathcal{P}_\mathcal{D}(A)} m(Y) > 0$. For the degenerate case when the truth

[1]The below formulas can also be defined on the power set 2^Θ and respectively super-power set S^Θ in exactly the same way.

is in $A = \emptyset$, we consider Smets' open-world, which means that there are other hypotheses $\Theta' = \{\theta_{n+1}, \theta_{n+2}, \ldots \theta_{n+m}\}$, $m \geq 1$, and the truth is in $A \in D^{\Theta'} \setminus \{\emptyset\}$. If $A = \emptyset$ and we consider a close-world, then it means that the problem is impossible. For another degenerate case, when $\sum_{Y \in \mathcal{P}_{\mathcal{D}}(A)} m(Y) = 0$, i.e. when the source gave us a totally (100%) wrong information $m(.)$, then, we define: $m(A|A) \triangleq 1$ and, as a consequence, $m(X|A) = 0$ for any $X \neq A$.

Let $s(A) = \{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_p}\}$, $1 \leq p \leq n$, be the singletons/atoms that compose A (For example, if $A = \theta_1 \cup (\theta_3 \cap \theta_4)$ then $s(A) = \{\theta_1, \theta_3, \theta_4\}$.). We consider three subsets of $D^\Theta \setminus \emptyset$, generated by A:

- $D_1 = \mathcal{P}_{\mathcal{D}}(A)$, the parts of A which are included in the hyper-power set, except the empty set;
- $D_2 = \{(\Theta \setminus s(A)), \cup, \cap\} \setminus \{\emptyset\}$, i.e. the sub-hyper-power set generated by $\Theta \setminus s(A)$ under $\cup$ and $\cap$, without the empty set.
- $D_3 = (D^\Theta \setminus \{\emptyset\}) \setminus (D_1 \cup D_2)$; each set from D_3 has in its formula singletons from both $s(A)$ and $\Theta \setminus s(A)$ in the case when $\Theta \setminus s(A)$ is different from empty set.

D_1, D_2 and D_3 have no element in common two by two and their union is $D^\Theta \setminus \{\emptyset\}$.

Examples of decomposition of $D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$: Let's consider $\Theta = \{A, B, C\}$ and the free DSm model.

- If one supposes the truth is in A, then $D_1 = \{A, A \cap B, A \cap C, A \cap B \cap C\} \equiv \mathcal{P}(A) \cap (D^\Theta \setminus \emptyset)$, i.e. D_1 contains all the parts of A which are included in D^Θ, except the empty set. D_2 contains all elements which do not contain the letter A, i.e. $D_2 = (\{B, C\}, \cup, \cap) = D^{\{B,C\}} = \{B, C, B \cup C, B \cap C\}$. $D_3 = \{A \cup B, A \cup C, A \cup B \cup C, A \cup (B \cap C)\}$, i.e. In this case, sets whose formulas contain both the letters A and at least a letter from $\{B, C\}$ and are not included in D_1.
- If one supposes the truth is in $A \cap B$, then one has $D1 = \{A \cap B, A \cap B \cap C\}$, $D_2 = \{C\}$; i.e. D_2 elements do not contain the letters A, B; and $D_3 = \{A, B, A \cup B, A \cap C, B \cap C, \ldots\}$, i.e. what's left from $D^\Theta \setminus \{\emptyset\}$ after removing D_1 and D_2.
- If one supposes the truth is in $A \cup B$, then one has $D_1 = \{A, B, A \cap B, A \cup B\}$, and all other sets included in these four ones, i.e. $A \cap C$, $B \cap C$, $A \cap B \cap C$, $A \cup (B \cap C)$, $B \cup (A \cap C)$, $(A \cap C) \cup (B \cap C)$, etc; $D_2 = \{C\}$, i.e. D_2 elements do not contain the letters A, B and $D_3 = \{A \cup C, B \cup C, A \cup B \cup C, C \cup (A \cap B)\}$.
- If one supposes the truth is in $A \cup B \cup C$, then one has $D_1 = D^\Theta \setminus \{\emptyset\}$. D_2 does not exist since $s(A \cup B \cup C) = \{A, B, C\}$ and $\Theta \setminus \{A, B, C\} = \emptyset$; i.e. D_2 elements do not contain the letters A, B, C. D_3 does not exist since $(D^\Theta \setminus \{\emptyset\}) \setminus D_1 = \emptyset$.
- If one supposes the truth is in $A \cap B \cap C$, then one has $D_1 = \{A \cap B \cap C\}$; D_2 does not exist; i.e. D_2 elements do not contain the letters A, B, C and D_3 equals everything else, i.e. $(D^\Theta \setminus \{\emptyset\}) \setminus D_1 = \{A, B, C, A \cap B, A \cap C, B \cap C, A \cup B, A \cup C, B \cup C, A \cup B \cup C, A \cup (B \cap C), \ldots\}$; D_3 has $19 - 1 - 1 = 17$ elements.

We propose several Belief Conditioning Rules (BCR) for deriving a (posterior) conditioning belief assignment $m(.|A)$ from a (prior) bba $m(.)$ and a conditioning set $A \in D^\Theta \setminus \{\emptyset\}$. For all forthcoming BCR formulas, of course we have:

$$m(X|A) = 0, \qquad \text{if} \quad X \notin D_1 \tag{9.1}$$

For all forthcoming BCR formulas, we suppose all denominators are non-zero.

9.3.1 Belief Conditioning Rule no. 1 (BCR1)

The Belief Conditioning Rule no. 1, denoted BCR1 for short, is defined for $X \in D_1$ by the formula

$$m_{BCR1}(X|A) = m(X) + \frac{m(X) \cdot \sum_{Z \in D_2 \cup D_3} m(Z)}{\sum_{Y \in D_1} m(Y)} = \frac{m(X)}{\sum_{Y \in D_1} m(Y)} \tag{9.2}$$

This is the easiest transfer of masses of the elements from D_2 and D_3 to the non-empty elements from D_1. This transfer is done indiscriminately in a similar way to Dempster's rule transfer, but this transfer is less exact. Therefore the sum of masses of non-empty elements from D_2 and D_3 is transferred to the masses of non-empty elements from D_1 proportionally with respect to their corresponding non-null masses.

In a set of sets, such as D_1, D_2, D_3, D^Θ, we consider the *inclusion* of sets, $\subseteq$, which is a partial ordering relationship. The model of D^Θ generates submodels for D_1, D_2 and D_3 respectively.

Let $W \in D_3$. We say $X \in D_1$ is the *k-largest*, $k \geq 1$, element from D_1 that is included in W, if: $\nexists Y \in D_1 \setminus \{X\}$ with $X \subset Y$, and $Y \subset W$. Depending on the model, there are $k \geq 1$ such elements. Similarly, we say that $X \in D_1$ is the *k-smallest*, $k \geq 1$, element from D_1 that is included in W, if: $\nexists Y \in D_1 \setminus \{X\}$ with $Y \subset X$, and $Y \subset W$. Since in many cases there are $k \geq 1$ such elements, we call each of them a *k-smallest element*.

We recall that the *DSm Cardinal*, i.e. $Card_{DSm}(X)$ for $X \in D^\Theta$, is the number of distinct parts that compose X in the Venn Diagram. It depends on the model and on the cardinal of Θ, see [3] for details.

We partially increasingly order the elements in D_1 using the inclusion relationship and their DSm Cardinals. Since there are elements $X, Y \in D_1$ that are in no relationship with each other (i.e. $X \nsubseteq Y$, $Y \nsubseteq X$), but having the *same* DSm Cardinal, we list them together in a same class. We, similarly as in statistics, say that X is a *k-median*, $k \geq 1$, element if X is in the middle of D_1 in the case when cardinal of D_1, $Card(D_1)$, is odd, or if $Card(D_1)$ is even we take the left and right classes from the middle of D_1 list. We also compute a k-average, $k \geq 1$, element of D_1 by first computing $\sum_{X \in D_1} Card_{DSm}(X)/Card(D_1)$. Then *k-average elements* are those whose DSm Cardinal is close to the atomic average of D_1. For each computation of k-largest, k-smallest, k-median, or k-average we take the whole class of a such element. In a class as stated above, all elements have the same DSm Cardinal and none is included in another one.

Let's see a few examples:

a) Let $\Theta = \{A, B, C\}$, Shafer's model, and the truth is in $B \cup C$.

$$D_1 = \{B, C, B \cup C\} \qquad Card_{DSm}(B) = Card_{DSm}(C) = 1 \qquad Card_{DSm}(B \cup C) = 2$$

In D_1, we have: the 1-largest element is $B \cup C$; the k-smallest (herein 2-smallest) are B, C; the k-median (herein 2-median) is the class of C, which is formed by the elements B,

C; the k-average of D_1 is $(Card_{DSm}(B)+Card_{DSm}(C)+Card_{DSm}(B\cup C))/Card(D_1) = (1+1+2)/3 = 1.333333 \approx 1$ and the k-averages are B, C.

b) Let $\Theta = \{A, B, C\}$, free DSm model, and the truth is in $B \cup C$. Then:

$$D_1 = \{\underbrace{B\cap C\cap A}_{Card_{DSm}=1}, \underbrace{B\cap C, B\cap A, C\cap A}_{Card_{DSm}=2},$$
$$\underbrace{(B\cap C)\cup(B\cap A), (B\cap C)\cup(C\cap A), (B\cap A)\cup(C\cap A)}_{Card_{DSm}=3},$$
$$\underbrace{(B\cap C)\cup(B\cap A)\cup(C\cap A), B, C}_{Card_{DSm}=4}, \underbrace{B\cup(C\cap A), C\cup(B\cap A)}_{Card_{DSm}=5}, \underbrace{B\cup C}_{Card_{DSm}=6}\}$$

Therefore $Card(D_1) = 13$.

$$D_2 = \{\underbrace{A}_{Card_{DSm}=4}\} \quad \text{and} \quad Card(D_2) = 1.$$

$$D_3 = \{\underbrace{A\cup(B\cap C)}_{Card_{DSm}=5}, \underbrace{A\cup B, A\cup C}_{Card_{DSm}=6}, \underbrace{A\cup B\cup C}_{Card_{DSm}=7}\} \quad \text{and} \quad Card(D_3) = 4.$$

One verifies easily that:

$$Card(D^\Theta) = 19 = Card(D_1) + Card(D_2) + Card(D_3) + 1 \text{ element (the empty set)}$$

c) Let $\Theta = \{A, B, C\}$, free DSm model, and the truth is in B.

$$D_1 = \{\underbrace{B\cap C\cap A}_{\text{class 1}}, \underbrace{B\cap A, B\cap C}_{\text{class 2}}, \underbrace{B}_{\text{class 3}}\}$$

$$Card_{DSm}(\text{class 1}) = 1 \qquad Card_{DSm}(\text{class 2}) = 2 \qquad Card_{DSm}(\text{class 3}) = 3$$

The D_1 list is increasingly by class DSm Cardinals. The 1-largest element is B; the 1-smallest is $B\cap C\cap A$; the 2-median elements are $B\cap A$, $B\cap C$; the average of DSm Cardinals is $[1\cdot(1) + 2\cdot(2) + 1\cdot(3)]/4 = 2$. The 2-average elements are $B\cap A$, $B\cap C$.

For the following BCR formulas, the k-largest, k-smallest, k-median, and k-average elements respectively are calculated *only* for those elements from D_1 that are included in a given W (where $W \in D_3$), not for the whole D_1.

9.3.2 Belief Conditioning Rule no. 2 (BCR2)

In Belief Conditioning Rule no. 2, i.e. BCR2 for short, a better transfer is proposed. While the sum of masses of elements from D_2 is redistributed in a similar way to the non-empty elements from D_1 proportionally with respect to their corresponding non-null masses, the masses of elements from D_3 are redistributed differently, i.e. if $W \in D_3$ then its whole mass, $m(W)$, is transferred to the k-largest (with respect to inclusion from D_1) set $X \subset W$; this is considered a *pessimistic/prudent way.* The formula of BCR2 for $X \in D_1$ is defined by:

$$m_{BCR2}(X|A) = m(X) + \frac{m(X) \cdot \sum_{Z \in D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-largest}}} m(W)/k \tag{9.3}$$

or equivalently

$$m_{BCR2}(X|A) = \frac{m(X) \cdot \sum_{Z \in D_1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-largest}}} m(W)/k \tag{9.4}$$

where X is the k-largest (with respect to inclusion) set included in W. The previous formula is actually equivalent in the Shafer's model to the following formula:

$$m_{BCR2}(X|A) = \frac{m(X) \cdot \sum_{Z \in D_1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ W = X \text{ when } \Theta \setminus s(A) \equiv \emptyset}} m(W)/k \tag{9.5}$$

9.3.3 Belief Conditioning Rule no. 3 (BCR3)

The Belief Conditioning Rule no. 3, i.e. BCR3 is a dual of BCR2, but the transfer of $m(W)$ is done to the k-smallest, $k \geq 1$, (with respect to inclusion) set $X \subset W$, i.e. in an *optimistic way*. The formula of BCR3 for $X \in D_1$ is defined by:

$$m_{BCR3}(X|A) = m(X) + \frac{m(X) \cdot \sum_{Z \in D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-smallest}}} m(W)/k \tag{9.6}$$

or equivalently

$$m_{BCR3}(X|A) = \frac{m(X) \cdot \sum_{Z \in D_1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-smallest}}} m(W)/k \tag{9.7}$$

where X is the k-smallest, $k \geq 1$, (with respect to inclusion) set included in W.

There are cases where BCR2 and BCR3 coincide, i.e. when there is only one, or none, $X \subset W$ for each $W \in D_3$.

9.3.4 Belief Conditioning Rule no. 4 (BCR4)

In an average between pessimistic and optimistic ways, we can consider "X k-median" in the previous formulas in order to get the Belief Conditioning Rule no. 4 (BCR4), i.e. for any $X \in D_1$,

$$m_{BCR4}(X|A) = m(X) + \frac{m(X) \cdot \sum_{Z \in D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-median}}} m(W)/k \tag{9.8}$$

or equivalently

$$m_{BCR4}(X|A) = \frac{m(X) \cdot \sum_{Z \in D1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-median}}} m(W)/k \tag{9.9}$$

where X is a k-median element of all elements from D_1 which are included in W. Here we do a medium (neither pessimistic nor optimistic) transfer.

9.3.5 Belief Conditioning Rule no. 5 (BCR5)

We replace "X is k-median" by "X is k-average" in BCR4 formula in order to obtain the BCR5, i.e. for any $X \in D_1$,

$$m_{BCR5}(X|A) = \frac{m(X) \cdot \sum_{Z \in D1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-average}}} m(W)/k \tag{9.10}$$

where X is a k-average element of all elements from D_1 which are included in W. This transfer from D_3 is also medium and the result close to BCR4's.

9.3.6 Belief Conditioning Rule no. 6 (BCR6)

BCR6 does a uniform redistribution of masses of each element $W \in D_3$ to all elements from D_1 which are included in W, i.e. for any $X \in D_1$,

$$m_{BCR6}(X|A) = \frac{m(X) \cdot \sum_{Z \in D1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W}} \frac{m(W)}{Card\{V \in D_1 | V \subset W\}} \tag{9.11}$$

where $Card\{V \in D_1 | V \subset W\}$ is the cardinal (number) of D_1 sets included in W.

9.3.7 Belief Conditioning Rule no. 7 (BCR7)

In our opinion, a better (prudent) transfer is done in the following Belief Conditioning Rule no. 7 (BCR7) defined for any $X \in D_1$ by:

$$m_{BCR7}(X|A) = m(X) + \frac{m(X)\cdot\sum_{Z\in D_2} m(Z)}{\sum_{Y\in D_1} m(Y)} + m(X)\cdot \sum_{\substack{W\in D_3\\ X\subset W, S(W)\neq 0}} \frac{m(W)}{S(W)} + \sum_{\substack{W\in D_3\\ X\subset W, X \text{ is } k\text{-largest}\\ S(W)=0}} m(W)/k$$

where $S(W) \triangleq \sum_{Y\in D_1, Y\subset W} m(Y)$.

Or, simplified we get:

$$m_{BCR7}(X|A) = m(X)\cdot\left[\frac{\sum_{Z\in D_1\cup D_2} m(Z)}{\sum_{Y\in D_1} m(Y)} + \sum_{\substack{W\in D_3\\ X\subset W,\, S(W)\neq 0}} \frac{m(W)}{S(W)}\right] + \sum_{\substack{W\in D_3\\ X\subset W, X \text{ is } k\text{-largest}\\ S(W)=0}} m(W)/k \quad (9.12)$$

The transfer is done in the following way:

- the sum of masses of elements in D_2 are redistributed to the non-empty elements from D_1 proportionally with respect to their corresponding non-null masses (similarly as in BCR1-BCR6 and BCR8-BCR11 defined in the sequel);
- for each element $W \in D_3$, its mass $m(W)$ is distributed to all elements from D_1 which are included in W and whose masses are non-null proportionally with their corresponding masses (according to the second term of the formula (9.12));
- but, if all elements from D_1 which are included in W have null masses, then $m(W)$ is transferred to the k-largest X from D_1, which is included in W (according to the last term of the formula (9.12)); this is the *pessimistic/prudent way.*

9.3.8 Belief Conditioning Rule no. 8 (BCR8)

A dual of BCR7 is the Belief Conditioning Rule no. 8 (BCR8), where we consider the *optimistic/more specialized way*, i.e. "X is k-largest" is replaced by "X is k-smallest", $k \geq 1$ in (9.12). Therefore, BCR8 formula for any $X \in D_1$ is given by :

$$m_{BCR8}(X|A) = m(X) \cdot \left[\frac{\sum_{Z \in D_1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W,\, S(W) \neq 0}} \frac{m(W)}{S(W)} \right] + \sum_{\substack{W \in D_3 \\ X \subset W, X \text{ is } k\text{-smallest} \\ S(W)=0}} m(W)/k \quad (9.13)$$

where $S(W) \triangleq \sum_{Y \in D_1, Y \subset W} m(Y)$.

9.3.9 Belief Conditioning Rule no. 9 (BCR9)

In an average between pessimistic and optimistic ways, we can consider "X k-median" in the previous formulas (9.12) and (9.13) instead of "k-largest" or "k-smallest" in order to get the Belief Conditioning Rule no. 9 (BCR9).

9.3.10 Belief Conditioning Rule no. 10 (BCR10)

BCR10 is similar to BCR9 using an average transfer (neither pessimistic nor optimistic) from D_3 to D_1. We only replace "X k-median" by "X k-average" in BCR9 formula.

9.3.11 Belief Conditioning Rule no. 11 (BCR11)

BCR11 does a uniform redistribution of masses of D_3 to D_1, as BCR6, but when $S(W) = 0$ for $W \in D_3$. BCR11 formula for any $X \in D_1$ is given by:

$$m_{BCR11}(X|A) = m(X) \cdot \left[\frac{\sum_{Z \in D_1 \cup D_2} m(Z)}{\sum_{Y \in D_1} m(Y)} + \sum_{\substack{W \in D_3 \\ X \subset W,\, S(W) \neq 0}} \frac{m(W)}{S(W)} \right] + \sum_{\substack{W \in D_3 \\ X \subset W, \\ S(W)=0}} \frac{m(W)}{Card\{V \in D_1 | V \subset W\}} \quad (9.14)$$

where $Card\{V \in D_1 | V \subset W\}$ is the cardinal (number) of D_1 sets included in W.

9.3.12 More Belief Conditioning Rules (BCR12-BCR21)

More versions of BCRs can be constructed that are distinguished through the way the masses of elements from $D_2 \cup D_3$ are redistributed to those in D_1. So far, in BCR1-11, we have redistributed the masses of D_2 indiscriminately to D_1, but for the free and some hybrid DSm models of D^Θ we can do a more exact redistribution.

There are elements in D_2 that don't include any element from D_1; the mass of these elements will be redistributed identically as in BCR1-. But other elements from D_2 that include at least one element from D_1 will be redistributed as we did before with D_3. So we can improve the last ten BCRs for any $X \in D_1$ as follows:

$$m_{BCR12}(X|A) = m(X) + \big[m(X) \cdot \sum_{\substack{Z\in D_2 \\ \nexists Y\in D_1 \text{ with } Y\subset Z}} m(Z)\big] / \sum_{Y\in D_1} m(Y) + \sum_{\substack{Z\in D_2 \\ X\subset Z, \text{X is } k\text{-largest}}} m(Z)/k + \sum_{\substack{W\in D_3 \\ X\subset W, \text{X is } k\text{-largest}}} m(W)/k \tag{9.15}$$

or equivalently

$$m_{BCR12}(X|A) = \big[m(X) \cdot \sum_{\substack{Z\in D_1, \\ \text{or } Z\in D_2 \,|\, \nexists Y\in D_1 \text{ with } Y\subset Z}} m(Z)\big] / \sum_{Y\in D_1} m(Y) + \sum_{\substack{W\in D_2\cup D_3 \\ X\subset W, \text{X is } k\text{-largest}}} m(W)/k \tag{9.16}$$

$$m_{BCR13}(X|A) = \big[m(X) \cdot \sum_{\substack{Z\in D_1, \\ \text{or } Z\in D_2 \,|\, \nexists Y\in D_1 \text{ with } Y\subset Z}} m(Z)\big] / \sum_{Y\in D_1} m(Y) + \sum_{\substack{W\in D_2\cup D_3 \\ X\subset W, \text{X is } k\text{-smallest}}} m(W)/k \tag{9.17}$$

$$m_{BCR14}(X|A) = \big[m(X) \cdot \sum_{\substack{Z\in D_1, \\ \text{or } Z\in D_2 \,|\, \nexists Y\in D_1 \text{ with } Y\subset Z}} m(Z)\big] / \sum_{Y\in D_1} m(Y) + \sum_{\substack{W\in D_2\cup D_3 \\ X\subset W, \text{X is } k\text{-median}}} m(W)/k \tag{9.18}$$

$$m_{BCR15}(X|A) = \big[m(X) \cdot \sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,|\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z)\big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{X is } k\text{-average}}} m(W)/k \quad (9.19)$$

$$m_{BCR16}(X|A) = \big[m(X) \cdot \sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,|\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z)\big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W}} \frac{m(W)}{Card\{V \in D_1 | V \subset W\}} \quad (9.20)$$

$$m_{BCR17}(X|A) = m(X) \cdot \Bigg[\big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,|\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z)\big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)}\Bigg] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{X is } k\text{-largest} \\ S(W) = 0}} m(W)/k \quad (9.21)$$

$$m_{BCR18}(X|A) = m(X) \cdot \Bigg[\big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \,|\, \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z)\big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)}\Bigg] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{X is } k\text{-smallest} \\ S(W) = 0}} m(W)/k \quad (9.22)$$

$$m_{BCR19}(X|A) = m(X) \cdot \Big[\big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \mid \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z) \big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)} \Big] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{X is } k\text{-median} \\ S(W)=0}} m(W)/k \quad (9.23)$$

$$m_{BCR20}(X|A) = m(X) \cdot \Big[\big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \mid \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z) \big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)} \Big] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W, \text{X is } k\text{-average} \\ S(W)=0}} m(W)/k \quad (9.24)$$

$$m_{BCR21}(X|A) = m(X) \cdot \Big[\big[\sum_{\substack{Z \in D_1, \\ \text{or } Z \in D_2 \mid \nexists Y \in D_1 \text{ with } Y \subset Z}} m(Z) \big] / \sum_{Y \in D_1} m(Y) + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W) \neq 0}} \frac{m(W)}{S(W)} \Big] + \sum_{\substack{W \in D_2 \cup D_3 \\ X \subset W \\ S(W)=0}} \frac{m(W)}{Card\{V \in D_1 | V \subset W\}} \quad (9.25)$$

Surely, other combinations of the ways of redistributions of masses from D_2 and D_3 to D_1 can be done, obtaining new BCR rules.

9.4 Examples

9.4.1 Example no. 1 (free DSm model with non-Bayesian bba)

Let's consider $\Theta = \{A, B, C\}$, the free DSm model (no intersection is empty) and the following prior bba

$$m_1(A) = 0.2 \qquad m_1(B) = 0.1 \qquad m_1(C) = 0.2 \qquad m_1(A \cup B) = 0.1 \qquad m_1(B \cup C) = 0.1$$

$$m_1(A \cup (B \cap C)) = 0.1 \qquad m_1(A \cap B) = 0.1 \qquad m_1(A \cup B \cup C) = 0.1$$

and let's assume that the truth is in $B \cup C$, i.e. the conditioning term is $B \cup C$. Then:

$$\begin{aligned} D_1 = \{&\underbrace{B \cap C \cap A}_{Card_{DSm}=1}, \underbrace{B \cap C, B \cap A, C \cap A}_{Card_{DSm}=2}, \\ &\underbrace{(B \cap C) \cup (B \cap A), (B \cap C) \cup (C \cap A), (B \cap A) \cup (C \cap A)}_{Card_{DSm}=3}, \\ &\underbrace{(B \cap C) \cup (B \cap A) \cup (C \cap A), B, C}_{Card_{DSm}=4}, \underbrace{B \cup (C \cap A), C \cup (B \cap A)}_{Card_{DSm}=5}, \underbrace{B \cup C}_{Card_{DSm}=6} \} \end{aligned}$$

Therefore $Card(D_1) = 13$.

We recall that $\forall X \in D^\Theta$, the *DSm Cardinal* of X, $Card_{DSm}(X)$, is equal to the number of distinct parts that compose X in the Venn Diagram (see below) according to the given model on D^Θ. By definition, $Card_{DSm}(\emptyset) = 0$ (see [3] for examples and details).

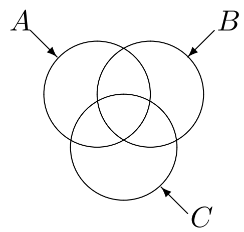

Figure 9.1: Venn Diagram for the 3D free DSm model

$$D_2 = \{ \underbrace{A}_{Card_{DSm}=4} \} \qquad \text{and} \qquad Card(D_2) = 1.$$

$$D_3 = \{\underbrace{A \cup (B \cap C)}_{Card_{DSm}=5}, \underbrace{A \cup B, A \cup C}_{Card_{DSm}=6}, \underbrace{A \cup B \cup C}_{Card_{DSm}=7}\} \qquad \text{and} \qquad Card(D_3) = 4.$$

One verifies easily that:

$$Card(D^\Theta) = 19 = Card(D_1) + Card(D_2) + Card(D_3) + 1 \text{ element (the empty set)}$$

The masses of elements from $D_2 \cup D_3$ are transferred to the elements of D_1. The ways these transfers are done make the distinction between the BCRs.

a) In BCR1, the sum of masses of D_2 and D_3 are indiscriminately distributed to B, C, $B \cup C$, $A \cap B$, proportionally to their corresponding masses 0.1, 0.2, 0.1, and respectively 0.1, i.e.

$$m(D_2 \cup D_3) = m_1(A) + m_1(A \cup B) + m_1(A \cup (B \cap C)) + m_1(A \cup B \cup C) = 0.5$$
$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{w_{B \cap A}}{0.1} = \frac{0.5}{0.5} = 1$$

whence $x_B = 0.1$, $y_C = 0.2$, $z_{B \cup C} = 0.1$ and $w_{B \cap A} = 0.1$ are added to the original masses of B, C, $B \cup C$ and $B \cap A$ respectively.

Finally, one gets with BCR1-based conditioning:

$$\begin{aligned}
m_{BCR1}(B|B \cup C) &= 0.2 \\
m_{BCR1}(C|B \cup C) &= 0.4 \\
m_{BCR1}(B \cup C|B \cup C) &= 0.2 \\
m_{BCR1}(B \cap A|B \cup C) &= 0.2
\end{aligned}$$

b) In BCR2, $m(D_2) = m_1(A) = 0.2$ and is indiscriminately distributed to B, C, $B \cup C$, $A \cap B$, proportionally to their corresponding masses, i.e.

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{w_{B \cap A}}{0.1} = \frac{0.2}{0.5} = 0.4$$

whence $x_B = 0.04$, $y_C = 0.08$, $z_{B \cup C} = 0.04$ and $w_{B \cap A} = 0.04$.

$m(D_3)$ is redistributed, element by element, to the k-largest D_1 element in each case: $m_1(A \cup B) = 0.1$ to $B \cup (C \cap A)$, since $B \cup (C \cap A) \in D_1$ and it is the 1-largest one from D_1 included in $A \cup B$; $m_1(A \cup (B \cap C)) = 0.1$ to $(B \cap A) \cup (C \cap A) \cup (B \cap C)$ for a similar reason; $m_1(A \cup B \cup C) = 0.1$ to $B \cup C$. Finally, one gets with BCR2-based conditioning:

$$\begin{aligned}
m_{BCR2}(B|B \cup C) &= 0.14 \\
m_{BCR2}(C|B \cup C) &= 0.28 \\
m_{BCR2}(B \cup C|B \cup C) &= 0.24 \\
m_{BCR2}(B \cap A|B \cup C) &= 0.14 \\
m_{BCR2}((B \cap A) \cup (C \cap A) \cup (B \cap C)|B \cup C) &= 0.10 \\
m_{BCR2}(B \cup (C \cap A)|B \cup C) &= 0.10
\end{aligned}$$

c) In BCR3, instead of k-largest D_1 elements, we consider k-smallest ones. $m(D_2) = m_1(A) = 0.2$ is exactly distributed as in BCR2. But $m(D_3)$ is, in each case, redistributed to the k-smallest D_1 element, which is $A \cap B \cap C$ (1-smallest herein). Hence:

$$\begin{aligned}
m_{BCR3}(B|B \cup C) &= 0.14 \\
m_{BCR3}(C|B \cup C) &= 0.28 \\
m_{BCR3}(B \cup C|B \cup C) &= 0.14 \\
m_{BCR3}(B \cap A|B \cup C) &= 0.14 \\
m_{BCR3}(A \cap B \cap C|B \cup C) &= 0.30
\end{aligned}$$

d) In BCR4, we use k-median.

- $A \cup B$ includes the following D_1 elements:

$$A \cap B \cap C, B \cap C, B \cap A, C \cap A, (B \cap C) \cup (B \cap A), \quad \underbrace{|}_{\text{median herein}}$$

$$(B \cap C) \cup (C \cap A), (B \cap A) \cup (C \cap A), (B \cap C) \cup (B \cap A) \cup (C \cap A), B, B \cup (A \cap C)$$

  Hence we take the whole class: $(B \cap C) \cup (B \cap A)$, $(B \cap C) \cup (C \cap A)$, $(B \cap A) \cup (C \cap A)$, i.e. 3-medians; each one receiving 1/3 of $0.1 = m_1(A \cup B)$.

- $A \cup (B \cap C)$ includes the following D_1 elements:

  $$A \cap B \cap C, B \cap C, B \cap A, C \cap A, \underbrace{|}_{\text{median herein}}$$

  $$(B\cap C)\cup(B\cap A), (B\cap C)\cup(C\cap A), (B\cap A)\cup(C\cap A), (B\cap C)\cup(B\cap A)\cup(C\cap A)$$

  Hence we take the left and right (to the median) classes: $B \cap C$, $B \cap A$, $C \cap A$, $(B \cap C) \cup (B \cap A)$, $(B \cap C) \cup (C \cap A)$, $(B \cap A) \cup (C \cap A)$, i.e. 6-medians, each ones receiving 1/6 of $0.1 = m_1(A \cup (B \cap C))$.
- $A \cup B \cup C$ includes all D_1 elements, hence the 3-medians are $(B \cap C) \cup (B \cap A)$, $(B\cap C)\cup(C\cap A)$ and $(B\cap A)\cup(C\cap A)$; each one receiving 1/3 of $0.1 = m_1(A\cup B\cup C)$

Totalizing, one finally gets:

$$\begin{aligned}
&m_{BCR4}(B|B \cup C) = 42/300\\
&m_{BCR4}(C|B \cup C) = 84/300\\
&m_{BCR4}(B \cup C|B \cup C) = 42/300\\
&m_{BCR4}(B \cap A|B \cup C) = 47/300\\
&m_{BCR4}(B \cap C|B \cup C) = 5/300\\
&m_{BCR4}(C \cap A|B \cup C) = 5/300\\
&m_{BCR4}((B \cap C) \cup (B \cap A)|B \cup C) = 25/300\\
&m_{BCR4}((B \cap C) \cup (C \cap A)|B \cup C) = 25/300\\
&m_{BCR4}((B \cap A) \cup (C \cap A)|B \cup C) = 25/300
\end{aligned}$$

e) In BCR5, we compute k-average, i.e. the k-average of DSm cardinals of the D_1 elements included in each $W \in D_3$.

- For $A \cup B$, using the results got in BCR4 above:

  $$\sum_{X\in D_1, X\subset A\cup B} Card_{DSm}(X) = 1 + 3 \cdot (2) + 3 \cdot (3) + 4 + 4 + 5 = 29$$

  The average DSm cardinal per element is $29/10 = 2.9 \approx 3$. Hence $(B\cap C)\cup(B\cap A)$, $(B \cap C) \cup (C \cap A)$, $(B \cap A) \cup (C \cap A)$, i.e. the 3-average elements, receive each 1/3 of $0.1 = m_1(A \cup B)$.
- For $A \cup (B \cap C)$, one has

  $$\sum_{X\in D_1, X\subset A\cup(B\cap C)} Card_{DSm}(X) = 1 + 3 \cdot (2) + 3 \cdot (3) + 4 = 20$$

  The average DSm cardinal per element is $20/8 = 2.5 \approx 3$. Hence again $(B \cap C) \cup (B \cap A)$, $(B \cap C) \cup (C \cap A)$, $(B \cap A) \cup (C \cap A)$, i.e. the 3-average elements, receive each 1/3 of $0.1 = m_1(A \cup (B \cap C))$.
- For $A \cup B \cup C$, one has

  $$\sum_{X\in D_1, X\subset A\cup B\cup C} Card_{DSm}(X) = 1 + 3 \cdot (2) + 3 \cdot (3) + 4 + 2 \cdot (4) + 2 \cdot (5) + 6 = 44$$

The average DSm cardinal per element is $44/13 = 3.38 \approx 3$. Hence $(B \cap C) \cup (B \cap A)$, $(B \cap C) \cup (C \cap A)$, $(B \cap A) \cup (C \cap A)$, i.e. the 3-average elements, receive each 1/3 of $0.1 = m_1(A \cup B \cup C)$.

Totalizing, one finally gets:

$$\begin{aligned}
&m_{BCR5}(B|B \cup C) = 42/300 \\
&m_{BCR5}(C|B \cup C) = 84/300 \\
&m_{BCR5}(B \cup C|B \cup C) = 42/300 \\
&m_{BCR5}(B \cap A|B \cup C) = 42/300 \\
&m_{BCR5}((B \cap C) \cup (B \cap A)|B \cup C) = 30/300 \\
&m_{BCR5}((B \cap C) \cup (C \cap A)|B \cup C) = 30/300 \\
&m_{BCR5}((B \cap A) \cup (C \cap A)|B \cup C) = 30/300
\end{aligned}$$

f) In BCR6, the k-average is replaced by uniform redistribution of D_3 elements' masses to all D_1 elements included in each $W \in D_3$.

- The mass $m_1(A \cup B) = 0.1$ is equally split among each D_1 element included in $A \cup B$ (see the list of them in BCR4 above), hence 1/10 of 0.1 to each.
- Similarly, $m_1(A \cup (B \cap C)) = 0.1$ is equally split among each D_1 element included in $A \cup (B \cap C)$, hence 1/8 of 0.1 to each.
- And, in the same way, $m_1(A \cup B \cup C) = 0.1$ is equally split among each D_1 element included in $A \cup B \cup C$, hence 1/13 of 0.1 to each.

Totalizing, one finally gets:

$$\begin{aligned}
&m_{BCR6}(B|B \cup C) = 820/5200 \\
&m_{BCR6}(C|B \cup C) = 1996/5200 \\
&m_{BCR6}(B \cup C|B \cup C) = 768/5200 \\
&m_{BCR6}(B \cap A|B \cup C) = 885/5200 \\
&m_{BCR6}(A \cap B \cap C|B \cup C) = 157/5200 \\
&m_{BCR6}(B \cap C|B \cup C) = 157/5200 \\
&m_{BCR6}(C \cap A|B \cup C) = 157/5200 \\
&m_{BCR6}((B \cap C) \cup (B \cap A)|B \cup C) = 157/5200 \\
&m_{BCR6}((B \cap C) \cup (C \cap A)|B \cup C) = 157/5200 \\
&m_{BCR6}((B \cap A) \cup (C \cap A)|B \cup C) = 157/5200 \\
&m_{BCR6}((B \cap C) \cup (B \cap A) \cup (C \cap A)|B \cup C) = 157/5200 \\
&m_{BCR6}(B \cup (C \cap A)|B \cup C) = 92/5200 \\
&m_{BCR6}(C \cup (B \cap A)|B \cup C) = 40/5200
\end{aligned}$$

g) In BCR7, $m(D_2)$ is also indiscriminately redistributed, but $m(D_3)$ is redistributed in a different more refined way.

- The mass $m_1(A \cup B) = 0.1$ is transferred to B and $B \cap A$ since these are the only D_1 elements included in $A \cup B$ whose masses are non-zero, proportionally to their corresponding masses, i.e.

$$\frac{x_B}{0.1} = \frac{w_{B\cap A}}{0.1} = \frac{0.1}{0.2} = 0.5$$

  whence $x_B = 0.05$ and $w_{B\cap A} = 0.05$.
- $m_1(A \cup (B \cap C)) = 0.1$ is transferred to $B \cap A$ only since no other D_1 element with non-zero mass is included in $A \cup (B \cap C)$.
- $m_1(A \cup B \cup C) = 0.1$ is similarly transferred to B, C, $B \cap A$, $B \cup C$, i.e.

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B\cup C}}{0.1} = \frac{w_{B\cap A}}{0.1} = \frac{0.1}{0.5} = 0.2$$

  whence $x_B = 0.02$, $y_C = 0.04$, $z_{B\cup C} = 0.02$ and $w_{B\cap A} = 0.02$.

Totalizing, one finally gets:

$$\begin{aligned}
m_{BCR7}(B|B \cup C) &= 0.21\\
m_{BCR7}(C|B \cup C) &= 0.32\\
m_{BCR7}(B \cup C|B \cup C) &= 0.16\\
m_{BCR7}(B \cap A|B \cup C) &= 0.31
\end{aligned}$$

h) In BCR8-11, since there is no $W \in D_3$ such that the sum of masses of D_1 elements included in W be zero, i.e. $s(W) \neq 0$, we can not deal with "k-elements", hence the results are identical to BCR7.

i) In BCR12, $m(D_2)$ is redistributed differently. $m_1(A) = 0.2$ is transferred to $(A \cap B) \cup (A \cap C)$ since this is the 1-largest D_1 element included in A. $m(D_3)$ is transferred exactly as in BCR2. Finally, one gets:

$$\begin{aligned}
m_{BCR12}(B|B \cup C) &= 0.1\\
m_{BCR12}(C|B \cup C) &= 0.2\\
m_{BCR12}(B \cup C|B \cup C) &= 0.2\\
m_{BCR12}(B \cap A|B \cup C) &= 0.1\\
m_{BCR12}((B \cap A) \cup (C \cap A) \cup (B \cap C)|B \cup C) &= 0.1\\
m_{BCR12}((A \cap B) \cup (A \cap C)|B \cup C) &= 0.1\\
m_{BCR12}(B \cup (C \cap A)|B \cup C) &= 0.1
\end{aligned}$$

j) In BCR13, $m(D_2)$ is redistributed to the 1-smallest, i.e. to $A \cap B \cap C$ and $m(D_3)$ is redistributed as in BCR3. Therefore one gets:

$$\begin{aligned}
m_{BCR13}(B|B \cup C) &= 0.1\\
m_{BCR13}(C|B \cup C) &= 0.2\\
m_{BCR13}(B \cup C|B \cup C) &= 0.1\\
m_{BCR13}(B \cap A|B \cup C) &= 0.1\\
m_{BCR13}(A \cap B \cap C|B \cup C) &= 0.5
\end{aligned}$$

k) In BCR14, $m_1(A) = 0.2$, where $A \in D_2$, is redistributed to the k-medians of $A \cap B \cap C$, $B \cap A$, $C \cap A$, $(B \cap A) \cup (C \cap A)$ which are included in A and belong to D_1. The 2-medians are $B \cap A$, $C \cap A$, hence each receives 1/2 of 0.2. $m(D_3)$ is redistributed as in BCR4. Therefore one gets:

$$
\begin{aligned}
&m_{BCR14}(B|B \cup C) = 30/300\\
&m_{BCR14}(C|B \cup C) = 60/300\\
&m_{BCR14}(B \cup C|B \cup C) = 30/300\\
&m_{BCR14}(B \cap A|B \cup C) = 65/300\\
&m_{BCR14}(B \cap C|B \cup C) = 5/300\\
&m_{BCR14}(C \cap A|B \cup C) = 35/300\\
&m_{BCR14}((B \cap C) \cup (B \cap A)|B \cup C) = 25/300\\
&m_{BCR14}((B \cap C) \cup (C \cap A)|B \cup C) = 25/300\\
&m_{BCR14}((B \cap A) \cup (C \cap A)|B \cup C) = 25/300
\end{aligned}
$$

l) In BCR15, $m_1(A) = 0.2$, where $A \in D_2$, is redistributed to the k-averages.

$$
\frac{1}{4}\cdot[Card_{DSm}(A\cap B\cap C)+Card_{DSm}(B\cap C)+Card_{DSm}(C\cap A)+Card_{DSm}((B\cap A)\cup(C\cap A))] = \frac{1+2+2+3}{4} = 2
$$

Hence each of $B\cap C$, $C\cap A$ receives 1/2 of 2; $m(D_3)$ is redistributed as in BCR5. Therefore one gets:

$$
\begin{aligned}
&m_{BCR15}(B|B \cup C) = 3/30\\
&m_{BCR15}(C|B \cup C) = 6/30\\
&m_{BCR15}(B \cup C|B \cup C) = 3/30\\
&m_{BCR15}(B \cap A|B \cup C) = 6/30\\
&m_{BCR15}(C \cap A|B \cup C) = 3/30\\
&m_{BCR15}((B \cap C) \cup (B \cap A)|B \cup C) = 3/30\\
&m_{BCR15}((B \cap C) \cup (C \cap A)|B \cup C) = 3/30\\
&m_{BCR15}((B \cap A) \cup (C \cap A)|B \cup C) = 3/30
\end{aligned}
$$

m) In BCR16, $m_1(A) = 0.2$, where $A \in D_2$, is uniformly transferred to all D_1 elements included in A, i.e. to $A \cap B \cap C$, $B \cap A$, $C \cap A$, $(B \cap A) \cup (C \cap A)$, hence each one receives

1/4 of 0.2. $m(D_3)$ is redistributed as in BCR6. Therefore one gets:

$$m_{BCR16}(B|B \cup C) = 612/5200$$
$$m_{BCR16}(C|B \cup C) = 1080/5200$$
$$m_{BCR16}(B \cup C|B \cup C) = 560/5200$$
$$m_{BCR16}(B \cap A|B \cup C) = 937/5200$$
$$m_{BCR16}(A \cap B \cap C|B \cup C) = 417/5200$$
$$m_{BCR16}(B \cap C|B \cup C) = 157/5200$$
$$m_{BCR16}(C \cap A|B \cup C) = 417/5200$$
$$m_{BCR16}((B \cap C) \cup (B \cap A)|B \cup C) = 157/5200$$
$$m_{BCR16}((B \cap C) \cup (C \cap A)|B \cup C) = 157/5200$$
$$m_{BCR16}((B \cap A) \cup (C \cap A)|B \cup C) = 417/5200$$
$$m_{BCR16}((B \cap C) \cup (B \cap A) \cup (C \cap A)|B \cup C) = 157/5200$$
$$m_{BCR16}(B \cup (C \cap A)|B \cup C) = 92/5200$$
$$m_{BCR16}(C \cup (B \cap A)|B \cup C) = 40/5200$$

n) In BCR17, $m_1(A) = 0.2$, where $A \in D_2$, is transferred to $B \cap A$ since $B \cap A \subset A$ and $m_1(B \cap A) > 0$. No other D_1 element with non-zero mass is included in A. $m(D_3)$ is redistributed as in BCR7. Therefore one gets:

$$m_{BCR17}(B|B \cup C) = 0.17$$
$$m_{BCR17}(C|B \cup C) = 0.24$$
$$m_{BCR17}(B \cup C|B \cup C) = 0.12$$
$$m_{BCR17}(B \cap A|B \cup C) = 0.47$$

o) BCR18-21 give the same result as BCR17 since no k-elements occur in these cases.

p) SCR does not work for free DSm models. But we can use the extended (from the power set 2^Θ to the hyper-power set D^Θ) Dempster's rule (see Daniel's Chapter [1]) in order to combine $m_1(.)$ with $m_2(B \cup C) = 1$, because the truth is in $B \cup C$, as in Shafer's conditioning rule. But since we have a free DSm model, no transfer is needed, hence Dempster's rule is reduced to DSm Classic rule (DSmC), which is a generalization of conjunctive rule. One gets:

$$m_{DSmC}(B|B \cup C) = 0.1$$
$$m_{DSmC}(C|B \cup C) = 0.2$$
$$m_{DSmC}(B \cup C|B \cup C) = 0.2$$
$$m_{DSmC}(B \cap A|B \cup C) = 0.1$$
$$m_{DSmC}((A \cap B) \cup (A \cap C)|B \cup C) = 0.2$$
$$m_{DSmC}(B \cup (A \cap C)|B \cup C) = 0.1$$
$$m_{DSmC}((A \cap B) \cup (B \cap C) \cup (C \cap A)|B \cup C) = 0.1$$

In the free DSm model, if the truth is in A, BCR12 gives the same result as $m_1(.)$ fusioned with $m_2(A) = 1$ using the classic DSm rule.

9.4.2 Example no. 2 (Shafer's model with non-Bayesian bba)

Let's consider $\Theta = \{A, B, C\}$ with Shafer's model and the following prior bba:

$$m_1(A) = 0.2 \qquad m_1(B) = 0.1 \qquad m_1(C) = 0.2$$

$$m_1(A \cup B) = 0.1 \qquad m_1(B \cup C) = 0.1 \qquad m_1(A \cup B \cup C) = 0.3$$

Let's assume as conditioning constraint that the truth is in $B \cup C$. D^Θ is decomposed into

$$D_1 = \{B, C, B \cup C\}$$

$$D_2 = \{A\}$$

$$D_3 = \{A \cup B, A \cup C, A \cup B \cup C\}$$

The Venn Diagram corresponding to Shafer's model for this example is given in Figure 9.2 below.

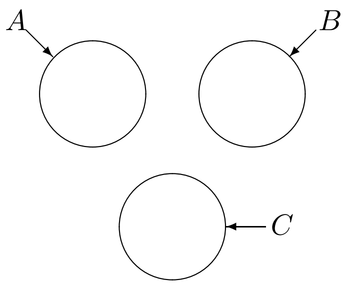

Figure 9.2: Venn Diagram for the 3D Shafer's model

a) In BCR1, $m(D_2 \cup D_3) = m_1(A) + m_1(A \cup B) + m_1(A \cup B \cup C) = 0.6$ is redistributed to B, C, $B \cup C$, proportionally to their corresponding masses 0.1, 0.2, 0.1 respectively, i.e.

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{0.6}{0.4} = 1.5$$

whence $x_B = 0.15$, $y_C = 0.30$, $z_{B \cup C} = 0.15$ are added to the original masses of B, C, $B \cup C$ respectively. Finally, one gets with BCR1-based conditioning:

$$m_{BCR1}(B|B \cup C) = 0.25$$
$$m_{BCR1}(C|B \cup C) = 0.50$$
$$m_{BCR1}(B \cup C|B \cup C) = 0.25$$

b) In BCR2, $m(D_2) = m_1(A) = 0.2$ and is indiscriminately distributed to B, C and $B \cup C$ proportionally to their corresponding masses, i.e.

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B \cup C}}{0.1} = \frac{0.2}{0.4} = 0.5$$

whence $x_B = 0.05$, $y_C = 0.10$, and $z_{B \cup C} = 0.05$.

For D_3, $m_1(A\cup B) = 0.1$ is transferred to B (1-largest), $m_1(A\cup B\cup C) = 0.3$ is transferred to $A\cup B$. Finally, one gets with BCR2-based conditioning:

$$m_{BCR2}(B|B\cup C) = 0.25$$
$$m_{BCR2}(C|B\cup C) = 0.30$$
$$m_{BCR2}(B\cup C|B\cup C) = 0.45$$

c) In BCR3 for D_3, $m_1(A\cup B) = 0.1$ is transferred to B (1-smallest), $m_1(A\cup B\cup C) = 0.3$ is transferred to B, C (2-smallest). Finally, one gets with BCR3-based conditioning:

$$m_{BCR3}(B|B\cup C) = 0.40$$
$$m_{BCR3}(C|B\cup C) = 0.45$$
$$m_{BCR3}(B\cup C|B\cup C) = 0.15$$

d) In BCR4 for D_3, $m_1(A\cup B) = 0.1$ is transferred to B (1-median), $m_1(A\cup B\cup C) = 0.3$ is transferred to B, C (2-medians). Finally, one gets same result as with BCR3, i.e.

$$m_{BCR4}(B|B\cup C) = 0.40$$
$$m_{BCR4}(C|B\cup C) = 0.45$$
$$m_{BCR4}(B\cup C|B\cup C) = 0.15$$

e) In BCR5 for D_3, $m_1(A\cup B) = 0.1$ is transferred to B. Let's compute

$$\frac{1}{3}\cdot [Card_{DSm}(B) + Card_{DSm}(C) + Card_{DSm}(B\cup C)] = \frac{1+1+2}{3} \approx 1$$

Hence 2-averages are B and C. So with BCR5, one gets same result as with BCR3, i.e.

$$m_{BCR5}(B|B\cup C) = 0.40$$
$$m_{BCR5}(C|B\cup C) = 0.45$$
$$m_{BCR5}(B\cup C|B\cup C) = 0.15$$

f) In BCR6 for D_3, $m_1(A\cup B) = 0.1$ is transferred to B (the only D_1 element included in $A\cup B$), $m_1(A\cup B\cup C) = 0.3$ is transferred to B, C, $B\cup C$, each one receiving 1/3 of 0.3. Finally, one gets

$$m_{BCR6}(B|B\cup C) = 0.35$$
$$m_{BCR6}(C|B\cup C) = 0.40$$
$$m_{BCR6}(B\cup C|B\cup C) = 0.25$$

g) In BCR7 for D_3, $m_1(A\cup B) = 0.1$ is transferred to B since $B\subset A\cup B$ and $m(B) > 0$; $m_1(A\cup B\cup C) = 0.3$ is transferred to B, C, $B\cup C$ proportionally to their corresponding masses:

$$\frac{x_B}{0.1} = \frac{y_C}{0.2} = \frac{z_{B\cup C}}{0.1} = \frac{0.3}{0.4} = 0.75$$

whence $x_B = 0.075$, $y_C = 0.15$, and $z_{B\cup C} = 0.075$. Finally, one gets

$$m_{BCR7}(B|B\cup C) = 0.325$$
$$m_{BCR7}(C|B\cup C) = 0.450$$
$$m_{BCR7}(B\cup C|B\cup C) = 0.225$$

h) BCR8-11 give the same result as BCR7 in this example, since there is no case of k-elements.

i) In BCR12: For D_2 for all BCR12-21, $m_1(A) = 0.2$ is redistributed to B, C, $B \cup C$ as in BCR2. $m(D_3)$ is redistributed as in BCR2. The result is the same as in BCR2.

j) BCR13-15 give the same result as in BCR3.

k) BCR16 gives the same result as in BCR6.

l) BCR17-21: For D_3, $m_1(A \cup B) = 0.1$ is transferred to B (no case of k-elements herein); $m_1(A \cup B \cup C) = 0.3$ is transferred to B, C, $B \cup C$ proportionally to their corresponding masses as in BCR7. Therefore one gets same result as in BCR7, i.e.

$$m_{BCR17}(B|B \cup C) = 0.325$$
$$m_{BCR17}(C|B \cup C) = 0.450$$
$$m_{BCR17}(B \cup C|B \cup C) = 0.225$$

m) BCR22, 23, 24, 25, 26 give the same results as BCR7, 8, 9, 10, 11 respectively since D_2 is indiscriminately redistributed to D_1 elements.

n) BCR27, 28, 29, 30, 31 give the same results as BCR2, 3, 4, 5, 6 respectively for the same reason as previously.

o) If one applies the SCR, i.e. one combines with Dempster's rule $m_1(.)$ with $m_2(B \cup C) = 1$, because the truth is in $B \cup C$ as Glenn Shafer proposes, one gets:

$$m_{SCR}(B|B \cup C) = 0.25$$
$$m_{SCR}(C|B \cup C) = 0.25$$
$$m_{SCR}(B \cup C|B \cup C) = 0.50$$

9.4.3 Example no. 3 (Shafer's model with Bayesian bba)

Let's consider $\Theta = \{A, B, C, D\}$ with Shafer's model and the following prior Bayesian bba:

$$m_1(A) = 0.4 \qquad m_1(B) = 0.1 \qquad m_1(C) = 0.2 \qquad m_1(D) = 0.3$$

Let's assume that one finds out that the truth is in $C \cup D$. From formulas of BCRs conditioning rules one gets the same result for all the BCRs in such example according to the following table

	A	B	C	D
$m_1(.)$	0.4	0.1	0.2	0.3
$m_{BCR1-31}(.\vert C \cup D)$	0	0	0.40	0.60

Table 9.1: Conditioning results based on BCRs given the truth is in $C \cup D$.

Let's examine the conditional bba obtained directly from the *fusion* of the prior bba $m_1(.)$ with the belief assignment focused only on $C \cup D$, say $m_2(C \cup D) = 1$ using three main rules of combination (Dempster's rule, DSmH and PCR5). After elementary derivations, one gets final results given in Table 9.2. In the Bayesian case, all BCRs and Shafer's conditioning rule (with Dempster's rule) give the same result.

	A	B	C	D	$C \cup D$	$A \cup C \cup D$	$B \cup C \cup D$
$m_{DS}(.\vert C \cup D)$	0	0	0.40	0.60	0	0	0
$m_{DSmH}(.\vert C \cup D)$	0	0	0.20	0.30	0	0.40	0.10
$m_{PCR5}(.\vert C \cup D)$	0.114286	0.009091	0.20	0.30	0.376623	0	0

Table 9.2: Conditioning results based on Dempster's, DSmH and PCR5 fusion rules.

9.5 Classification of the BCRs

Let's note:
D_2^u = Redistribution of the whole D_2 is done undifferentiated to D_1
D_3^u = Redistribution of the whole D_3 is done undifferentiated to D_1
D_2^p = Redistribution of D_2 is particularly done from each $Z \in D_2$ to specific elements in D_1
D_3^p = Redistribution of D_3 is particularly done from each $W \in D_3$ to specific elements in D_1
$D_2^s = D_2$ is split into two disjoint subsets: one whose elements have the property that $s(W) \neq 0$, an another one such that its elements have $s(W) = 0$. Each subset is differently redistributed to D_1
$D_3^s = D_3$ is similarly split into two disjoint subsets, that are redistributed as in D_2^s.

Thus, we can organize and classify the BCRs as follows:

Ways of redistribution	Belief Conditioning Rule	Specific Elements
D_2^u, D_3^u	$BCR1$	
D_2^u, D_3^p	$\boxed{\text{BCR2}}$	k – largest
	$BCR3$	k – smallest
	$BCR4$	k – median
	$BCR5$	k – average
	$BCR6$	uniform distribution
D_2^u, D_3^s	$\boxed{\text{BCR7}}$	k – largest
	$BCR8$	k – smallest
	$BCR9$	k – median
	$BCR10$	k – average
	$BCR11$	uniform distribution
D_2^p, D_3^p	$\boxed{\text{BCR12}}$	k – largest
	$BCR13$	k – smallest
	$BCR14$	k – median
	$BCR15$	k – average
	$BCR16$	uniform distribution
D_2^s, D_3^s	$\boxed{\text{BCR17}}$	k – largest
	$BCR18$	k – smallest
	$BCR19$	k – median
	$BCR20$	k – average
	$BCR21$	uniform distribution

Table 9.3: Classification of Belief Conditioning Rules

Other belief conditioning rules could also be defined according to Table 9.4. But in our opinions, the most detailed and exact transfer is done by BCR17. So, we suggest to use preferentially BCR17 for a pessimistic/prudent view on conditioning problem and a more refined redistribution of conflicting masses, or BCR12 for a very pessimistic/prudent view and less refined redistribution. If the Shafer's models holds for the frame under consideration, BCR12-21 will coincide with BCR2-11.

Ways of redistribution	Belief Conditioning Rule	Specific Elements
D_2^p, D_3^s	$\boxed{\text{BCR22}}$	$k-$ largest
	$BCR23$	$k-$ smallest
	$BCR24$	$k-$ median
	$BCR25$	$k-$ average
	$BCR26$	uniform distribution
D_2^s, D_3^p	$\boxed{\text{BCR27}}$	$k-$ largest
	$BCR28$	$k-$ smallest
	$BCR29$	$k-$ median
	$BCR30$	$k-$ average
	$BCR31$	uniform distribution

Table 9.4: More Belief Conditioning Rules

In summary, the best among these BCR1-31, that we recommend to use, are: BCR17 for a pessimistic/prudent view on conditioning problem and a more refined redistribution of conflicting masses, or BCR12 for a very pessimistic/prudent view and less refined redistribution.

BCR17 does the most refined redistribution of all BCR1-31, i.e.
- the mass $m(W)$ of each element W in $D_2 \cup D_3$ is transferred to those $X \in D_1$ elements which are included in W if any proportionally with respect to their non-empty masses;
- if no such X exists, the mass $m(W)$ is transferred in a pessimistic/prudent way to the k-largest elements from D_1 which are included in W (in equal parts) if any;
- if neither this way is possible, then $m(W)$ is indiscriminately distributed to all $X \in D_1$ proportionally with respect to their nonzero masses.

BCR12 does the most pessimistic/prudent redistribution of all BCR1-31, i.e.:
- the mass $m(W)$ of each W in $D_2 \cup D_3$ is transferred in a pessimistic/prudent way to the k-largest elements X from D_1 which are included in W (in equal parts) if any;
- if this way is not possible, then $m(W)$ is indiscriminately distributed to all X from D_1 proportionally with respect their nonzero masses.

BCR12 is simpler than BCR17. BCR12 can be regarded as a generalization of SCR from the power set to the hyper-power set in the free DSm free model (all intersections non-empty). In this case the result of BCR12 is equal to that of $m_1(.)$ combined with $m_2(A) = 1$, when the truth is in A, using the DSm Classic fusion rule.

9.6 Properties for all BCRs

1. For any $X \notin \mathcal{P}_{\mathcal{D}}(A) = D_1$, one has $m_{BCR}(X|A) = 0$ by definition.

2. One has:

$$\sum_{X \in \mathcal{P}_{\mathcal{D}}(A)} m_{BCR}(X|A) = 1$$

   This can be proven from the fact that $\sum_{X \in D^\Theta} m(X) = 1.$ and $D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$, where D_1, D_2 and D_3 have no element in common two by two. Since all masses of all elements from D_2 and D_3 are transferred to the non-empty elements of D_1using BCRs, no mass is lost neither gained, hence the total sum of masses remains equal to 1.

3. Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and $A = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ be the total ignorance. Then, $m_{BCR1-31}(X|A) = m(X)$ for all X in D^Θ, because $D^\Theta \setminus \{\emptyset\}$ coincides with D_1. Hence there is no mass to be transferred from D_2 or D_3 to D_1 since D_2 and D_3 do not exist (are empty).

4. This property reduces all BRCs to the Bayesian formula: $m_{BCR}(X|A) = m(X \cap A)/m(A)$ for the trivial Bayesian case when focal elements are only singletons (no unions, neither intersections) and the truth is in one singleton only.

   Proof: Let's consider $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, $n \geq 2$, and all θ_i not empty, and $D^\Theta \equiv \Theta$. Let's have a bba $m(.) : D^\Theta \mapsto [0, 1]$. Without loss of generality, suppose the truth is in θ_1 where $m(\theta_1) > 0$. Then $m_{BCR}(\theta_1|\theta_1) = 1$ and $m_{BCR}(X|\theta_1) = 0$ for all X different from θ_1. Then $1 = m_{BCR}(\theta_1|\theta_1) = m(\theta_1 \cap \theta_1)/m(\theta_1) = 1$, and for $i \neq 1$, we have $0 = m_{BCR}(\theta_i|\theta_1) = m(\emptyset)/m(\theta_1) = 0$.

5. In the Shafer's model, and a Bayesian bba $m(.)$, all BCR1-31 coincide with SCR. In this case the conditioning with BCRs and fusioning with Dempster's rule commute.

   Proof: In a general case we can prove it as follows: Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, $n \geq 2$, and without loss of generality let's suppose the truth is in $T = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_p$, for $1 \leq p \leq n$. Let's consider two Bayesian masses $m_1(.)$ and $m_2(.)$. Then we can consider all other elements $\theta_{p+1}, \ldots, \theta_n$ as empty sets and in consequence the sum of their masses as the mass of the empty set (as in Smets' open world). BCRs work now exactly as (or we can say it is reduced to) Dempster's rule redistributing this empty set mass to the elements in T proportionally with their nonzero corresponding mass. $D_1 = \{\theta_1, \theta_2, \ldots, \theta_p\}$, $D_2 = \{\theta_{p+1}, \ldots, \theta_n\}$, D_3 does not exist. And redistributing in $m_1(.|T)$ this empty sets' mass to non-empty sets θ_1, θ_2, $\ldots$, θ_p using BCRs is equivalent to combining $m_1(.)$ with $m_S(\theta_1 \cup \theta_2 \cup \ldots \cup \theta_p) = 1$. Similarly for $m_2(.|T)$. Since Dempter's fusion rule and Shafer's conditioning rule commute and BCRs are reduced to Dempster's rule in a Shafer's model and Bayesian case, then BCRs commute with Dempster's fusion rule in this case. QED

6. In the free DSm model, BCR12 can be regarded as a generalization of SCR from the power set to the hyper-power set. The result of BCR12 conditioning of a mass $m_1(.)$, when the truth is in A, is equal to that of fusioning $m_1(.)$ with $m_2(A) = 1$, using the DSm Classic Rule.

9.7 Open question on conditioning versus fusion

It is not too difficult to verify that fusion rules and conditioning rules do not commute in general, except in Dempster-Shafer Theory because Shafer's fusion and conditioning rules are based on the same operator[2] (Dempster's rule), which make derivations very simple and appealing.

We however think that things may be much more complex in reality than what has been proposed up to now if we follow our interpretation of belief conditioning and do not see the belief conditioning as just a simple fusion of the prior bba with a bba focused on the conditioning event where the truth is (subjectively) supposed to be. From our belief conditioning interpretation, we make a strong difference between the fusion of several sources of evidences (i.e. combination of bba's) and the conditioning of a given belief assignment according some extra knowledge (carrying some objective/absolute truth on a given subset) on the model itself. In our opinion, the conditioning must be interpreted as a revision of bba according to new integrity constraint on the truth of the space of the solutions. Based on this new idea on conditioning, we are face to a new and very important open question which can be stated as follows[3]:

Let's consider two prior bba's $m_1(.)$ and $m_2(.)$ provided by two (cognitively) independent sources of evidences defined on D^Θ for a given model $\mathcal{M}$ (free, hybrid or Shafer's model) and then let's assume that the truth is known to be later on in a subset $A \in D^\Theta$, how to compute the combined conditional belief?

There are basically two possible answers to this question depending on the order the fusion and the conditioning are carried out. Let's denote by $\oplus$ the generic symbol for fusion operator (PCR5, DSmH or whatever) and by $Cond(.)$ the generic symbol for conditioning operator (typically BCRs).

1. Answer 1 (Fusion followed by conditioning (FC)):

$$m_{FC}(.|A) = Cond(m_1(.) \oplus m_2(.)) \tag{9.26}$$

2. Answer 2 (Conditioning followed by the fusion (CF)):

$$m_{CF}(.|A) = \underbrace{Cond(m_1(.))}_{m_1(.|A)} \oplus \underbrace{Cond(m_2(.))}_{m_2(.|A)} \tag{9.27}$$

Since in general[4] the conditioning and the fusion do not commute, $m_{FC}(.|A) \neq m_{CF}(.|A)$, the fundamental open question arises: How to justify the choice for one answer with respect to the other one (or maybe with respect to some other answers if any) to compute the combined conditional bba from $m_1(.)$, $m_2(.)$ and any conditioning subset A?

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[2]Proof of commutation between the Shafer's conditioning rule and Dempster's rule: Let $m_1(.)$ be a bba and $m_S(A) = 1$. Then, because Dempster's rule, denoted $\oplus$, is associative we have $(m_1 \oplus m_S) \oplus (m_2 \oplus m_S) = m_1 \oplus (m_S \oplus m_2) \oplus m_S$ and because it is commutative we get $m_1 \oplus (m_2 \oplus m_S) \oplus m_S$ and again because it is associative we have: $(m_1 \oplus m_2) \oplus (m_S \oplus m_S)$; hence, since $m_S \oplus m_S = m_S$, it is equal to: $(m_1 \oplus m_2) \oplus m_S = m_1 \oplus m_2 \oplus m_S$, QED.

[3]The question can be extended for more than two sources actually.

[4]Because none of the new fusion and conditioning rules developed up to now satisfies the commutativity, but Dempster's rule.

The only argumentation (maybe) for justifying the choice of $m_{FC}(.|A)$ or $m_{CF}(.|A)$ is only imposed by the possible temporal/sequential processing of sources and extra knowledge one receives, i.e. if one gets first $m_1(.)$ and $m_2(.)$ and later one knows that the truth is in A then $m_{FC}(.|A)$ seems intuitively suitable, but if one gets first $m_1(.)$ and A, and later $m_2(.)$, then $m_{CF}(.|A)$ looks in better agreement with the chronology of information one has received in that case. If we make abstraction of temporal processing, then this fundamental and very difficult question remains unfortunately totally open.

9.7.1 Examples of non commutation of BCR with fusion

9.7.1.1 Example no. 1 (Shafer's model and Bayesian bba's)

Let's consider $\Theta = \{A, B, C\}$ with Shafer's model and the following prior Bayesian bba's

$$m_1(A) = 0.2 \qquad m_1(B) = 0.6 \qquad m_1(C) = 0.2$$
$$m_2(A) = 0.1 \qquad m_2(B) = 0.4 \qquad m_2(C) = 0.5$$

Let's suppose one finds out the truth is in $A \cup B$ and let's examine the results $m_{CF}(.|A \cup B)$ and $m_{FC}(.|A \cup B)$ obtained from either the conditioning followed by the fusion, or the fusion followed by the conditioning.

- Case 1 : BCRs-based Conditioning followed by the PCR5-based Fusion

  Using BCRs for conditioning, the mass $m_1(C) = 0.2$ is redistributed to A and B proportionally to the masses 0.2 and 0.6 respectively; thus $x/0.2 = y/0.6 = 0.2/(0.2+0.6) = 1/4$ and therefore $x = 0.2 \cdot (1/4) = 0.05$ is added to $m_1(A)$, while $y = 0.6 \cdot (1/4) = 0.15$ is added to $m_1(B)$. Hence, one finally gets

  $$m_1(A|A \cup B) = 0.25 \qquad m_1(B|A \cup B) = 0.75 \qquad m_1(C|A \cup B) = 0$$

  Similarly, the conditioning of $m_2(.)$ using the BCRs, will provide

  $$m_2(A|A \cup B) = 0.2 \qquad m_2(B|A \cup B) = 0.8 \qquad m_2(C|A \cup B) = 0$$

  If one combines $m_1(.|A \cup B)$ and $m_2(.|A \cup B)$ with PCR5 fusion rule, one gets[5]

  $$m_{C_{BCRs}F_{PCR5}}(A|A \cup B) = 0.129198 \quad m_{C_{BCRs}F_{PCR5}}(B|A \cup B) = 0.870802$$

- Case 2 : PCR5-based Fusion followed by the BCRs-based Conditioning

  If one combines first $m_1(.)$ and $m_2(.)$ with PCR5 fusion rule, one gets

  $$m_{PCR5}(A) = 0.090476 \quad m_{PCR5}(B) = 0.561731 \quad m_{PCR5}(C) = 0.347793$$

  and if one applies any of BCR rules for conditioning the combined prior $m_{PCR5}(.)$, one finally gets

  $$m_{F_{PCR5}C_{BCRs}}(A|A \cup B) = 0.138723 \qquad m_{F_{PCR5}C_{BCRs}}(B|A \cup B) = 0.861277$$

[5] We specify explicitly in notations $m_{CF}(.)$ and $m_{FC}(.)$ the type of the conditioning and fusion rules used for convenience, i.e $m_{C_{BCRs}F_{PCR5}}(.)$ means that the conditioning is based on BCRs and the fusion is based on PCR5.

From cases 1 and 2, one has proved that there exists at least one example for which PCR5 fusion and BCRs conditioning do not commute since

$$m_{F_{PCR5}C_{BCRs}}(.|A \cup B) \neq m_{C_{BCRs}F_{PCR5}}(.|A \cup B).$$

- Case 3 : BCRs-based Conditioning followed by Dempster's rule-based Fusion

  If we consider the same masses $m_1(.)$ and $m_2(.)$ and if we apply the BCRs to each of them, one gets

$$m_1(A|A \cup B) = 0.25 \qquad m_1(B|A \cup B) = 0.75 \qquad m_1(C|A \cup B) = 0$$
$$m_2(A|A \cup B) = 0.20 \qquad m_2(B|A \cup B) = 0.80 \qquad m_2(C|A \cup B) = 0$$

  then if one combines them with Dempster's rule, one finally gets

$$m_{C_{BCRs}F_{DS}}(A|A \cup B) = 0.076923 \quad m_{C_{BCRs}F_{DS}}(B|A \cup B) = 0.923077$$

- Case 4 : Dempster's rule based Fusion followed by BCRs-based Conditioning

  If we apply first the fusion of $m_1(.)$ with $m_2(.)$ with Dempster's rule of combination, one gets

$$m_{DS}(A) = 0.055555 \quad m_{DS}(B) = 0.666667 \quad m_{DS}(C) = 0.277778$$

  and if one applies BCRs for conditioning the prior $m_{DS}(.)$, one finally gets

$$m_{F_{DS}C_{BCRs}}(A|A \cup B) = 0.076923 \qquad m_{F_{DS}C_{BCRs}}(B|A \cup B) = 0.923077$$

  From cases 3 and 4, we see that all BCRs (i.e. BCR1-BCR31) commute with Dempster's fusion rule in a Shafer's model and Bayesian case since:

$$m_{F_{DS}C_{BCRs}}(.|A \cup B) = m_{C_{BCRs}F_{DS}}(.|A \cup B).$$

  But this is a trivial result because in this specific case (Shafer's model with Bayesian bba's), we know (cf Property 5 in Section 9.6) that BCRs coincide with SCR and already know that SCR commutes with Dempter's fusion rule.

9.7.1.2 Example no. 2 (Shafer's model and non Bayesian bba's)

Let's consider $\Theta = \{A, B, C\}$ with Shafer's model and the following prior non Bayesian bba's

$$m_1(A) = 0.3 \qquad m_1(B) = 0.1 \qquad m_1(C) = 0.2 \qquad m_1(A \cup B) = 0.1 \qquad m_1(B \cup C) = 0.3$$

$$m_2(A) = 0.1 \qquad m_2(B) = 0.2 \qquad m_2(C) = 0.3 \qquad m_2(A \cup B) = 0.2 \qquad m_2(B \cup C) = 0.2$$

Let's suppose one finds out the truth is in $B \cup C$ and let's examine the results $m_{CF}(.|B \cup C)$ and $m_{FC}(.|B \cup C)$ obtained from either the conditioning followed by the fusion, or the fusion followed by the conditioning. In this second example we only provide results for BCR12 and BCR17 since we consider them as the most appealing BCR rules. We decompose D^Θ into $D_1 = \{B, C, B \cup C\}$, $D_2 = \{A\}$ and $D_3 = \{A \cup B\}$.

- Case 1 : BCR12/BCR17-based Conditioning followed by the PCR5-based Fusion

  Using BCR12 or BCR17 for conditioning $m_1(.)$ and $m_2(.)$, one gets herein the same result with both BCRs for each conditional bba, i.e.

$$m_1(B|B \cup C) = 0.25 \qquad m_1(C|B \cup C) = 0.30 \qquad m_1(B \cup C|B \cup C) = 0.45$$

$$m_2(B|B \cup C) = 15/35 \qquad m_2(C|B \cup C) = 12/35 \qquad m_2(B \cup C|B \cup C) = 8/35$$

  If one combines $m_1(.|B \cup C)$ and $m_2(.|B \cup C)$ with PCR5 fusion rule, one gets

$$m_{C_{BCR17}F_{PCR5}}(.|B \cup C) = m_{C_{BCR12}F_{PCR5}}(.|B \cup C)$$

  with

$$m_{C_{BCR12}F_{PCR5}}(B|B \cup C) = 0.446229$$
$$m_{C_{BCR12}F_{PCR5}}(C|B \cup C) = 0.450914$$
$$m_{C_{BCR12}F_{PCR5}}(B \cup C|B \cup C) = 0.102857$$

- Case 2 : PCR5-based Fusion followed by BCR12/BCR17-based Conditioning

  If one combines first $m_1(.)$ and $m_2(.)$ with PCR5 fusion rule, one gets

$$m_{PCR5}(A) = 0.236167 \quad m_{PCR5}(B) = 0.276500 \quad m_{PCR5}(C) = 0.333333$$

$$m_{PCR5}(A \cup B) = 0.047500 \quad m_{PCR5}(B \cup C) = 0.141612$$

  and if one applies any of BCR12 or BCR17 rules for conditioning the (combined) prior $m_{PCR5}(.)$, one finally gets the same final result with BCR12 and BCR17, i.e.

$$m_{F_{PCR5}C_{BCR17}}(.|B \cup C) = m_{F_{PCR5}C_{BCR12}}(.|B \cup C)$$

  with

$$m_{F_{PCR5}C_{BCR12}}(B|B \cup C) = 0.415159$$
$$m_{F_{PCR5}C_{BCR12}}(C|B \cup C) = 0.443229$$
$$m_{F_{PCR5}C_{BCR12}}(B \cup C|B \cup C) = 0.141612$$

From cases 1 and 2, one has proved that there exists at least one example for which PCR5 fusion and BCR12/17 conditioning rules do not commute since

$$m_{F_{PCR5}C_{BCR12/17}}(.|B \cup C) \neq m_{C_{BCR12/17}F_{PCR5}}(.|B \cup C).$$

- Case 3 : BCR12/BCR17-based Conditioning followed by Dempster's rule-based Fusion

  If we consider the same masses $m_1(.)$ and $m_2(.)$ and if we apply the BCR12 or BCR17 to each of them, one gets same result, i.e.

$$m_1(B|B \cup C) = 0.25 \qquad m_1(C|B \cup C) = 0.30 \qquad m_1(B \cup C|B \cup C) = 0.45$$

$$m_2(B|B \cup C) = 15/35 \qquad m_2(C|B \cup C) = 12/35 \qquad m_2(B \cup C|B \cup C) = 8/35$$

then if one combines them with Dempster's rule, one finally gets

$$m_{C_{BCR12}F_{DS}}(B|B \cup C) = \frac{125}{275}$$

$$m_{C_{BCR12}F_{DS}}(C|B \cup C) = \frac{114}{275}$$

$$m_{C_{BCR12}F_{DS}}(C|B \cup C) = \frac{36}{275}$$

and same result for $m_{C_{BCR17}F_{DS}}(.)$.

- Case 4 : Dempster's rule based Fusion followed by BCR12/BCR17-based Conditioning

  If we apply first the fusion of $m_1(.)$ with $m_2(.)$ with Dempster's rule of combination, one gets

$$m_{DS}(A) = 10/59 \quad m_{DS}(B) = 22/59 \quad m_{DS}(C) = 19/59$$

$$m_{DS}(A \cup B) = 2/59 \quad m_{DS}(B \cup C) = 6/59$$

  and if one applies BCR12 (or BCR17) for conditioning the prior $m_{DS}(.)$, one finally gets (same result is obtained with BCR17)

$$m_{F_{DS}C_{BCR12}}(B|B \cup C) = \frac{1348}{2773}$$

$$m_{F_{DS}C_{BCR12}}(C|B \cup C) = \frac{1083}{2773}$$

$$m_{F_{DS}C_{BCR12}}(B \cup C|B \cup C) = \frac{342}{2773}$$

  In BCR12, $m_{DS}(A) = 10/59$ is distributed to B, C, $B \cup C$ proportionally to their masses, i.e.

$$\frac{x_B}{22/59} = \frac{y_C}{19/59} = \frac{z_{B\cup C}}{6/59} = \frac{10/59}{47/59} = \frac{10}{47}$$

  whence $x_B = (22/59) \cdot (10/47) = 220/2773$, $y_C = (19/59) \cdot (10/47) = 190/2773$ and $z_{B\cup C} = (6/59) \cdot (10/47) = 60/2773$, and $m_{DS}(A \cup B) = 2/59$ is distributed to B only, since B is the 1-largest.

  In BCR17, $m_{DS}(A) = 10/59$ is similarly distributed to B, C, $B \cup C$ and $m_{DS}(A \cup B)$ is also distributed to B only, since $B \subset A$ and $m_{DS}(B) > 0$ and B is the only element with such properties. Herein BCR12 and BCR17 give the same result.

Therefore from cases 3 and 4, we see that BCR12 (and BCR17) don't commute with Dempster's rule for Shafer's model and a non-Bayesian bba since

$$m_{C_{BCR12}F_{DS}}(.|B \cup C) \neq m_{F_{DS}C_{BCR12}}(.|B \cup C).$$

- Case 5 : SCR-based Conditioning followed by Dempster's rule based Fusion

  If we consider the masses $m_1(.)$ and $m_2(.)$ and if we apply the SCR to each of them for conditioning, one gets

$$m_1(B|B \cup C) = 2/7 \qquad m_1(C|B \cup C) = 2/7 \qquad m_1(B \cup C|B \cup C) = 3/7$$

$$m_2(B|B \cup C) = 4/9 \qquad m_2(C|B \cup C) = 3/9 \qquad m_2(B \cup C|B \cup C) = 2/9$$

  then if one combines them with Dempster's rule, one finally gets

$$m_{C_{SCR}F_{DS}}(B|B \cup C) = \frac{24}{49} \quad m_{C_{SCR}F_{DS}}(C|B \cup C) = \frac{19}{49} \quad m_{C_{SCR}F_{DS}}(C|B \cup C) = \frac{6}{49}$$

- Case 6 : Dempster's rule based Fusion followed by the SCR-based Conditioning

  If we apply first the fusion of $m_1(.)$ with $m_2(.)$ with Dempster's rule of combination, one gets

$$m_{DS}(A) = 10/59 \quad m_{DS}(B) = 22/59 \quad m_{DS}(C) = 19/59$$

$$m_{DS}(A \cup B) = 2/59 \quad m_{DS}(B \cup C) = 6/59$$

  and if one applies SCR for conditioning the prior $m_{DS}(.)$, one finally gets

$$m_{F_{DS}C_{SCR}}(B|B \cup C) = \frac{24}{49} \quad m_{F_{DS}C_{BCRs}}(C|B \cup C) = \frac{19}{49} \quad m_{F_{DS}C_{BCRs}}(B \cup C|B \cup C) = \frac{6}{49}$$

From cases 5 and 6, we verify that SCR commutes with Demspter's rule for Shafer's model and non-Bayesian bba[6] because

$$m_{C_{SCR}F_{DS}}(.|B \cup C) = m_{F_{DS}C_{SCR}}(.|B \cup C).$$

9.8 Conclusion

We have proposed in this chapter several new Belief Conditioning Rules (BCRs) in order to adjust a given prior bba $m(.)$ with respect to the new conditioning information that have come in. The BCRs depend on the model of D^{Θ}. Several examples were presented that compared these BCRs among themselves and as well with Shafer's Conditioning Rule (SCD). Except for SCD, in general the BCRs do not commute with the fusion rules, and the sequence in which they should be combined depends on the chronology of information received.

9.9 References

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Chapter 10

Fusion of qualitative beliefs

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Abstract: *This chapter introduces the notion of qualitative belief assignment to model beliefs of human experts expressed in natural language (with linguistic labels). We show how qualitative beliefs can be efficiently combined using an extension of Dezert-Smarandache Theory (DSmT) of plausible and paradoxical quantitative reasoning to qualitative reasoning. We propose a new arithmetic on linguistic labels which allows a direct extension of classical or hybrid DSm fusion rules. An approximate qualitative PCR5 rule is also proposed jointly with a Qualitative Average Operator. We also show how crisp or interval mappings can be used to deal indirectly with linguistic labels. A very simple example is provided to illustrate our qualitative fusion rules.*

10.1 A brief historic of previous works

Since fifteen years qualitative methods for reasoning under uncertainty developed in Artificial Intelligence are attracting more and more people of Information Fusion community, specially those working in the development of modern multi-source[1] systems for defense. Their aim is to propose solutions for processing and combining qualitative information to take into account efficiently information provided by human sources (or *semi-intelligent* expert systems) and usually expressed in natural language rather than direct quantitative information. George Polya was one of the first mathematicians to attempt a formal characterization of qualitative human reasoning in 1954 [26, 27], then followed by Lofti Zadeh's works [40]- [45].

The interest of qualitative reasoning methods is to help in decision-making for situations in which the precise numerical methods are not appropriate (whenever the information/input are not directly expressed in numbers). Several formalisms for qualitative reasoning have been proposed as extensions on the frames of probability, possibility and/or evidence theories [2, 5,

[1]Where both computers, sensors and human experts are involved in the loop.

9, 12, 37, 39, 42, 45]. The limitations of numerical techniques are discussed in [22]. We browse here few main approaches. A detailed presentation of theses techniques can be found in [24].

In [34], Wellman proposes a general characterization of *qualitative probability* to relax precision in representation and reasoning within the probabilistic framework. His basic idea was to develop Qualitative Probabilistic Networks (QPN) based on a Qualitative Probability Language (QPL) defined by a set of numerical underlying probability distributions. The major interest of QPL is to specify the partial rankings among degrees of belief rather than assessing their magnitudes on a cardinal scale. Such method cannot be considered as truly qualitative in our opinion, since it rather belongs to the family of imprecise probability [33] and probability bounds analysis (PBA) methods [11].

Some advances have been done by Darwiche in [5] for a symbolic generalization of Probability Theory; more precisely, Darwiche proposes a support (symbolic and/or numeric) structure which contains all information able to represent and conditionalize the state of belief. Darwiche shows that Probability Theory fits within his new support structure framework as several other theories, but Demspter-Shafer Theory doesn't fit in. Based on Demspter-Shafer Theory [29] (DST), Wong and Lingras [38] propose a method for generating a (numerical) basic belief functions from preference relations between each pair of propositions be specified qualitatively. The algorithm proposed doesn't provide however a unique solution and doesn't check the consistency of qualitative preference relations. Bryson and al. [4, 16] propose a procedure called Qualitative Discriminant Procedure (QDP) that involves qualitative scoring, imprecise pairwise comparisons between pairs of propositions and an optimization algorithm to generate consistent imprecise quantitative belief function to combine. Very recently, Ben Yaglane in [1] has reformulated the problem of generation of quantitative (consistent) belief functions from qualitative preference relations as a more general optimization problem under additional non linear constraints in order to minimize different uncertainty measures (Bezdek's entropy, Dubois & Prade non-specificity, etc).

In [18, 19], Parsons proposes a qualitative Dempster-Shafer Theory, called Qualitative Evidence Theory (QET), by using techniques from qualitative reasoning [2]. Parsons' idea is to use *qualitative belief assignments* (qba), denoted here $qm(.)$ assumed to be only 0 or $+$, where $+$ means some unknown value between 0 and 1. Parsons proposes, using operation tables, a very simple arithmetic for qualitative addition $+$ and multiplication $\times$ operators. The combination of two (or more) qba's then actually follows the classical conjunctive consensus operator based on his qualitative multiplication table. Because of impossibility of qualitative normalization, Parsons uses the un-normalized version of Dempster's rule by committing a *qualitative mass* to the empty set following the open-world approach of Smets [32]. This approach cannot deal however with truly closed-world problems because there is no issue to transfer the conflicting qualitative mass or to normalize the qualitative belief assignments in the spirit of DST. An improved version of QET has been proposed [18] for using refined linguistic quantifiers as suggested by Dubois & Prade in [10]. The fusion of refined qualitative belief masses follows the un-normalized Dempster's rule based on an underlying numerical interval arithmetic associated with linguistic quantifiers. Actually, this refined QTE fits directly within DSmT framework since it corresponds to imprecise (quantitative) DSmC fusion rule [6, 30].

From 1995, Parsons seems to have switched back to qualitative probabilistic reasoning [23] and started to develop Qualitative Probabilistic Reasoner (QPR). Recently, Parsons discussed about the flaw discovered in QPR and gave some issues with new open questions [25].

In Zadeh's paradigm of computing with words (CW) [42]- [45] the combination of qualitative/vague information expressed in natural language is done essentially in three steps: 1) a translation of qualitative information into fuzzy membership functions, 2) a fuzzy combination of fuzzy membership functions; 3) a retranslation of fuzzy (quantitative) result into natural language. All these steps cannot be uniquely accomplished since they depend on the fuzzy operators chosen. A possible issue for the third step is proposed in [39].

In this chapter, we propose a simple arithmetic of linguistic labels which allows a direct extension of classical (quantitative) combination rules proposed in the DSmT framework into their qualitative counterpart. Qualitative beliefs assignments are well adapted for manipulated information expressed in natural language and usually reported by human expert or AI-based expert systems. In other words, we propose here a new method for computing directly with words (CW) and combining directly qualitative information Computing with words, more precisely computing with linguistic labels, is usually more vague, less precise than computing with numbers, but it is expected to offer a better robustness and flexibility for combining uncertain and conflicting human reports than computing with numbers because in most of cases human experts are less efficient to provide (and to justify) precise quantitative beliefs than qualitative beliefs. Before extending the quantitative DSmT-based combination rules to their qualitative counterparts, it will be necessary to define few but new important operators on linguistic labels and what is a qualitative belief assignment. Then we will show though simple examples how the combination of qualitative beliefs can be obtained in the DSmT framework.

10.2 Qualitative Operators

We propose in this section a general arithmetic for computing with words (or linguistic labels). Computing with words (CW) and qualitative information is more vague, less precise than computing with numbers, but it offers the advantage of robustness if done correctly since :

" It would be a great mistake to suppose that vague knowledge must be false. On the contrary, a vague belief has a much better chance of being true than a precise one, because there are more possible facts that would verify it." – Bertrand Russell [28].

So let's consider a finite frame $\Theta = \{\theta_1, \dots, \theta_n\}$ of n (exhaustive) elements θ_i, $i = 1, 2, \dots, n$, with an associated model $\mathcal{M}(\Theta)$ on Θ (either Shafer's model $\mathcal{M}^0(\Theta)$, free-DSm model $\mathcal{M}^f(\Theta)$, or more general any Hybrid-DSm model [30]). A model $\mathcal{M}(\Theta)$ is defined by the set of integrity constraints on elements of Θ (if any); Shafer's model $\mathcal{M}^0(\Theta)$ assumes all elements of Θ truly exclusive, while free-DSm model $\mathcal{M}^f(\Theta)$ assumes no exclusivity constraints between elements of the frame Θ.

Let's define a finite set of linguistic labels $\tilde{L} = \{L_1, L_2, \dots, L_m\}$ where $m \geq 2$ is an integer. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \dots \prec L_m$. To work on a close linguistic set under linguistic addition and multiplication operators, we extends $\tilde{L}$ with two extreme values L_0 and L_{m+1} where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value, in such a way that

$$L_0 \prec L_1 \prec L_2 \prec \dots \prec L_m \prec L_{m+1}$$

where $\prec$ means inferior to, or less (in quality) than, or smaller (in quality) than, etc. hence a relation of order from a qualitative point of view. But if we make a correspondence between qualitative labels and quantitative values on the scale $[0, 1]$, then $L_{\min} = L_0$ would correspond to the numerical value 0, while $L_{\max} = L_{m+1}$ would correspond to the numerical value 1, and each L_i would belong to $[0, 1]$, i. e.

$$L_{\min} = L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = L_{\max}$$

From now on, we work on extended ordered set L of qualitative values

$$L = \{L_0, \tilde{L}, L_{m+1}\} = \{L_0, L_1, L_2, \ldots, L_m, L_{m+1}\}$$

The qualitative addition and multiplication operators are respectively defined in the following way:

- Addition :

$$L_i + L_j = \begin{cases} L_{i+j}, & \text{if } i + j < m + 1, \\ L_{m+1}, & \text{if } i + j \geq m + 1. \end{cases} \tag{10.1}$$

- Multiplication :

$$L_i \times L_j = L_{\min\{i,j\}} \tag{10.2}$$

These two operators are well-defined, commutative, associative, and unitary. Addition of labels is a unitary operation since $L_0 = L_{\min}$ is the unitary element, i.e. $L_i + L_0 = L_0 + L_i = L_{i+0} = L_i$ for all $0 \leq i \leq m + 1$. Multiplication of labels is also a unitary operation since $L_{m+1} = L_{\max}$ is the unitary element, i.e. $L_i \times L_{m+1} = L_{m+1} \times L_i = L_{\min\{i,m+1\}} = L_i$ for $0 \leq i \leq m+1$. L_0 is the unit element for addition, while L_{m+1} is the unit element for multiplication. L is closed under $+$ and $\times$. The mathematical structure formed by $(L, +, \times)$ is a commutative bisemigroup with different unitary elements for each operation. We recall that a bisemigroup is a set S endowed with two associative binary operations such that S is closed under both operations.

If L is not an exhaustive set of qualitative labels, then other labels may exist in between the initial ones, so we can work with labels and numbers - since a refinement of L is possible. When mapping from L to crisp numbers or intervals, $L_0 = 0$ and $L_{m+1} = 1$, while $0 < L_i < 1$, for all i, as crisp numbers, or L_i included in $[0, 1]$ as intervals/subsets.

For example, L_1, L_2, L_3 and L_4 may represent the following qualitative values: $L_1 \triangleq$ very poor, $L_2 \triangleq$ poor, $L_3 \triangleq$ good and $L_4 \triangleq$ very good where $\triangleq$ symbol means "by definition".

We think it is better to define the multiplication $\times$ of $L_i \times L_j$ by $L_{\min\{i,j\}}$ because multiplying two numbers a and b in $[0, 1]$ one gets a result which is less than each of them, the product is not bigger than both of them as Bolanos et al. did in [3] by approximating $L_i \times L_j = L_{i+j} > \max\{L_i, L_j\}$. While for the addition it is the opposite: adding two numbers in the interval $[0, 1]$ the sum should be bigger than both of them, not smaller as in [3] case where $L_i + L_j = \min\{L_i, L_j\} < \max\{L_i, L_j\}$.

10.2.1 Qualitative Belief Assignment

We define a qualitative belief assignment (qba), and we call it *qualitative belief mass* or *q-mass* for short, a mapping function

$$qm(.) : G^{\Theta} \mapsto L$$

where G^Θ corresponds the space of propositions generated with $\cap$ and $\cup$ operators and elements of Θ taking into account the integrity constraints of the model. For example if Shafer's model is chosen for Θ, then G^Θ is nothing but the classical power set 2^Θ [29], whereas if free DSm model is adopted G^Θ will correspond to Dedekind's lattice (hyper-power set) D^Θ [30]. Note that in this qualitative framework, there is no way to define normalized $qm(.)$, but qualitative quasi-normalization is still possible as seen further. Using the qualitative operations defined previously we can easily extend the combination rules from quantitative to qualitative. In the sequel we will consider $s \geq 2$ qualitative belief assignments $qm_1(.), \ldots, qm_s(.)$ defined over the same space G^Θ and provided by s independent sources $S_1, \ldots, S_s$ of evidence.

Important note: The addition and multiplication operators used in all qualitative fusion formulas in next sections correspond to *qualitative addition* and *qualitative multiplication* operators defined in (10.1) and (10.2) and must not be confused with classical addition and multiplication operators for numbers.

10.2.2 Qualitative Conjunctive Rule (qCR)

The qualitative Conjunctive Rule (qCR) of $s \geq 2$ sources is defined similarly to the quantitative conjunctive consensus rule, i.e.

$$qm_{qCR}(X) = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \tag{10.3}$$

The total qualitative conflicting mass is given by

$$K_{1\ldots s} = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta \\ X_1 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} qm_i(X_i)$$

10.2.3 Qualitative DSm Classic rule (q-DSmC)

The qualitative DSm Classic rule (q-DSmC) for $s \geq 2$ is defined similarly to DSm Classic fusion rule (DSmC) as follows : $qm_{qDSmC}(\emptyset) = L_0$ and for all $X \in D^\Theta \setminus \{\emptyset\}$,

$$qm_{qDSmC}(X) = \sum_{\substack{X_1,,\ldots,X_s \in D^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \tag{10.4}$$

10.2.4 Qualitative DSm Hybrid rule (q-DSmH)

The qualitative DSm Hybrid rule (q-DSmH) is defined similarly to quantitative DSm hybrid rule [30] as follows: $qm_{qDSmH}(\emptyset) = L_0$ and for all $X \in G^\Theta \setminus \{\emptyset\}$

$$qm_{qDSmH}(X) \triangleq \phi(X) \cdot \Big[qS_1(X) + qS_2(X) + qS_3(X) \Big] \tag{10.5}$$

where $\phi(X)$ is the *characteristic non-emptiness function* of a set X, i.e. $\phi(X) = L_{m+1}$ if $X \notin \boldsymbol{\emptyset}$ and $\phi(X) = L_0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\boldsymbol{\emptyset}_{\mathcal{M}}, \emptyset\}$. $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $qS_1(X) \equiv qm_{qDSmC}(X)$, $qS_2(X)$, $qS_3(X)$ are defined by

$$qS_1(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s \in D^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \tag{10.6}$$

$$qS_2(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_s \in \boldsymbol{\emptyset} \\ [\mathcal{U}=X] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (X = I_t)]}} \prod_{i=1}^{s} qm_i(X_i) \tag{10.7}$$

$$qS_3(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ X_1 \cup X_2 \cup \ldots \cup X_s = X \\ X_1 \cap X_2 \cap \ldots \cap X_s \in \boldsymbol{\emptyset}}} \prod_{i=1}^{s} qm_i(X_i) \tag{10.8}$$

where all sets are in canonical form and where $\mathcal{U} \triangleq u(X_1) \cup \ldots \cup u(X_s)$ where $u(X)$ is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \ldots \cup \theta_n$ is the total ignorance. $qS_1(X)$ is nothing but the qDSmC rule for s independent sources based on $\mathcal{M}^f(\Theta)$; $qS_2(X)$ is the qualitative mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $qS_3(X)$ transfers the sum of relatively empty sets directly onto the (canonical) disjunctive form of non-empty sets. qDSmH generalizes qDSmC works for any models (free DSm model, Shafer's model or any hybrid models) when manipulating qualitative belief assignments.

10.3 Qualitative Average Operator

The Qualitative Average Operator (QAO) is an extension of Murphy's numerical average operator [15]. But here we define two types of QAO's:

a) A pessimistic (cautious) one :

$$QAO_p(L_i, L_j) = L_{\lfloor \frac{i+j}{2} \rfloor} \tag{10.9}$$

where $\lfloor x \rfloor$ means the lower integer part of x, i.e. the greatest integer less than or equal to x;

a) An optimistic one :

$$QAO_o(L_i, L_j) = L_{\lceil \frac{i+j}{2} \rceil} \tag{10.10}$$

where $\lceil x \rceil$ means the upper integer part of x, i.e. the smallest integer greater than or equal to x.

QAO can be generalized for $s \geq 2$ qualitative sources.

10.4 Qualitative PCR5 rule (q-PCR5)

In classical (i.e. quantitative) DSmT framework, the Proportional Conflict Redistribution rule no. 5 (PCR5) defined in Chapter 1 has been proven to provide very good and coherent results for combining (quantitative) belief masses, see [7, 31] and Chapters 2 and 13 in this volume for discussions and justifications. When dealing with qualitative beliefs and using Dempster-Shafer Theory (DST), we unfortunately cannot normalize, since it is not possible to divide linguistic labels by linguistic labels. Previous authors have used the un-normalized Dempster's rule, which actually is equivalent to the Conjunctive Rule in Shafer's model and respectively to DSm conjunctive rule in hybrid and free DSm models. Following the idea of (quantitative) PCR5 fusion rule (1.32), we can however use a rough approximation for a qualitative version of PCR5 (denoted qPCR5) as it will be presented in next example, but we did not succeed so far to get a general formula for qualitative PCR5 fusion rule (q-PCR5) because the division of labels could not be defined.

10.5 A simple example

Let's consider the following set of ordered linguistic labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5\}$ (for example, L_1, L_2, L_3 and L_4 may represent the values: $L_1 \triangleq$ *very poor*, $L_2 \triangleq$ *poor*, $L_3 \triangleq$ *good* and $L_4 \triangleq$ *very good*, where $\triangleq$ symbol means *by definition*), then addition and multiplication tables are:

$+$	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_1	L_2	L_3	L_4	L_5
L_1	L_1	L_2	L_3	L_4	L_5	L_5
L_2	L_2	L_3	L_4	L_5	L_5	L_5
L_3	L_3	L_4	L_5	L_5	L_5	L_5
L_4	L_4	L_5	L_5	L_5	L_5	L_5
L_5	L_5	L_5	L_5	L_5	L_5	L_5

Table 10.1: Addition table

$\times$	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_0	L_0	L_0	L_0	L_0
L_1	L_0	L_1	L_1	L_1	L_1	L_1
L_2	L_0	L_1	L_2	L_2	L_2	L_2
L_3	L_0	L_1	L_2	L_3	L_3	L_3
L_4	L_0	L_1	L_2	L_3	L_4	L_4
L_5	L_0	L_1	L_2	L_3	L_4	L_5

Table 10.2: Multiplication table

The tables for QAO_p and QAO_o operators are:

QAO_p	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_0	L_1	L_1	L_2	L_2
L_1	L_0	L_1	L_1	L_2	L_2	L_3
L_2	L_1	L_1	L_2	L_2	L_3	L_3
L_3	L_1	L_2	L_2	L_3	L_3	L_4
L_4	L_2	L_2	L_3	L_3	L_4	L_4
L_5	L_2	L_3	L_3	L_4	L_4	L_5

Table 10.3: Table for QAO_p

QAO_o	L_0	L_1	L_2	L_3	L_4	L_5
L_0	L_0	L_1	L_1	L_2	L_2	L_3
L_1	L_1	L_1	L_2	L_2	L_3	L_3
L_2	L_1	L_2	L_2	L_3	L_3	L_4
L_3	L_2	L_2	L_3	L_3	L_4	L_4
L_4	L_2	L_3	L_3	L_4	L_4	L_5
L_5	L_3	L_3	L_4	L_4	L_5	L_5

Table 10.4: Table for QAO_o

Let's consider now a simple two-source case with a 2D frame $\Theta = \{\theta_1, \theta_2\}$, Shafer's model for Θ, and qba's expressed as follows:

$$qm_1(\theta_1) = L_1, \quad qm_1(\theta_2) = L_3, \quad qm_1(\theta_1 \cup \theta_2) = L_1$$

$$qm_2(\theta_1) = L_2, \quad qm_2(\theta_2) = L_1, \quad qm_2(\theta_1 \cup \theta_2) = L_2$$

10.5.1 Qualitative Fusion of qba's

- **Fusion with (qCR)**: According to qCR combination rule (10.3), one gets the result in Table 10.5, since

$$\begin{aligned}
qm_{qCR}(\theta_1) &= qm_1(\theta_1)qm_2(\theta_1) + qm_1(\theta_1)qm_2(\theta_1 \cup \theta_2) \\
&\quad + qm_2(\theta_1)qm_1(\theta_1 \cup \theta_2) \\
&= (L_1 \times L_2) + (L_1 \times L_2) + (L_2 \times L_1) \\
&= L_1 + L_1 + L_1 = L_{1+1+1} = L_3
\end{aligned}$$

$$\begin{aligned}
qm_{qCR}(\theta_2) &= qm_1(\theta_2)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1 \cup \theta_2) \\
&\quad + qm_2(\theta_2)qm_1(\theta_1 \cup \theta_2) \\
&= (L_3 \times L_1) + (L_3 \times L_2) + (L_1 \times L_1) \\
&= L_1 + L_2 + L_1 = L_{1+2+1} = L_4
\end{aligned}$$

$$qm_{qCR}(\theta_1 \cup \theta_2) = qm_1(\theta_1 \cup \theta_2)qm_2(\theta_1 \cup \theta_2) = L_1 \times L_2 = L_1$$

$$\begin{aligned} qm_{qCR}(\emptyset) \triangleq K_{12} &= qm_1(\theta_1)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1) \\ &= (L_1 \times L_1) + (L_2 \times L_3) = L_1 + L_2 = L_3 \end{aligned}$$

In summary, one gets

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qCR}(.)$	L_3	L_4	L_1	L_3	L_0

Table 10.5: Fusion with qCR

- **Fusion with (qDSmC)**: If we accepts the free-DSm model instead Shafer's model, according to qDSmC combination rule (10.4), one gets the result in Table 10.6,

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qDSmC}(.)$	L_3	L_4	L_1	L_0	L_3

Table 10.6: Fusion with qDSmC

- **Fusion with (qDSmH)**: Working with Shafer's model for Θ, according to qDSmH combination rule (10.5), one gets the result in Table 10.7.

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qDSmH}(.)$	L_3	L_4	L_4	L_0	L_0

Table 10.7: Fusion with qDSmC

since $qm_{qDSmH}(\theta_1 \cup \theta_2) = L_1 + L_3 = L_4$.

- **Fusion with QAO**: Working with Shafer's model for Θ, according to QAO combination rules (10.9) and (10.10), one gets the result in Table 10.8.

- **Fusion with (qPCR5)**: Following PCR5 method, we propose to transfer the qualitative partial masses

  a) $qm_1(\theta_1)qm_2(\theta_2) = L_1 \times L_1 = L_1$ to θ_1 and θ_2 in equal parts (i.e. proportionally to L_1 and L_1 respectively, but $L_1 = L_1$); hence $\frac{1}{2}L_1$ should go to each of them.

	θ_1	θ_2	$\theta_1 \cup \theta_2$
$qm_1(.)$	L_1	L_3	L_1
$qm_2(.)$	L_2	L_1	L_2
$qm_{QAO_p}(.)$	L_1	L_2	L_1
$qm_{QAO_o}(.)$	L_2	L_2	L_2

Table 10.8: Fusion of qba's with QAO's

b) $qm_2(\theta_1)qm_1(\theta_2) = L_2 \times L_3 = L_2$ to θ_1 and θ_2 proportionally to L_2 and L_3 respectively; but since we are not able to do an exact proportionalization of labels, we approximate through transferring $\frac{1}{3}L_2$ to θ_1 and $\frac{2}{3}L_2$ to θ_2.

The transfer $(1/3)L_2$ to θ_1 and $(2/3)L_2$ to θ_2 is not arbitrary, but it is an approximation since the transfer was done proportionally to L_2 and L_3, and L_2 is smaller than L_3; we mention that it is not possible to do an exact transferring. Nobody in the literature has done so far normalization of labels, and we tried to do a quasi-normalization [i.e. an approximation].

Summing a) and b) we get: $\frac{1}{2}L_1 + \frac{1}{3}L_2 \approx L_1$, which represents the partial conflicting qualitative mass transferred to θ_1, and $\frac{1}{2}L_1 + \frac{2}{3}L_2 \approx L_2$, which represents the partial conflicting qualitative mass transferred to θ_2. Here we have mixed qualitative and quantitative information.

Hence we will finally get:

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm_{qPCR5}(.)$	L_4	L_5	L_1	L_0	L_0

Table 10.9: Fusion with qPCR5

For the reason that we can not do a normalization (neither previous authors on qualitative fusion rules did), we propose for the first time the possibility of *quasi-normalization* (which is an approximation of the normalization), i.e. instead of dividing each qualitative mass by a coefficient of normalization, we *subtract* from each qualitative mass a qualitative coefficient (label) of quasi-normalization in order to adjust the sum of masses.

Subtraction on L is defined in a similar way to the addition:

$$L_i - L_j = \begin{cases} L_{i-j}, & \text{if } i \geq j; \\ L_0, & \text{if } i < j; \end{cases} \tag{10.11}$$

L is closed under subtraction as well.

The subtraction can be used for quasi-normalization only, i. e. moving the final label result 1-2 steps/labels up or down. It is not used together with addition or multiplication.

The increment in the sum of fusioned qualitative masses is due to the fact that multiplication on L is approximated by a larger number, because multiplying any two numbers a, b in the interval $[0, 1]$, the product is less than each of them, or we have approximated the product $a \times b = \min\{a, b\}$.

Using the quasi-normalization (subtracting L_1), one gets with qDSmH and qPCR5, the following *quasi-normalized* masses (we use $\star$ symbol to specify the quasi-normalization):

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\emptyset$	$\theta_1 \cap \theta_2$
$qm_1(.)$	L_1	L_3	L_1		
$qm_2(.)$	L_2	L_1	L_2		
$qm^{\star}_{qDSmH}(.)$	L_2	L_3	L_3	L_0	L_0
$qm^{\star}_{qPCR5}(.)$	L_3	L_4	L_0	L_0	L_0

Table 10.10: Fusion with quasi-normalization

10.5.2 Fusion with a crisp mapping

If we consider the labels as equidistant, then we can divide the whole interval [0,1] into five equal parts, hence mapping the linguistic labels L_i onto crisp numbers as follows:

$$L_0 \mapsto 0, L_1 \mapsto 0.2, L_2 \mapsto 0.4, L_3 \mapsto 0.6, L_4 \mapsto 0.8, L_5 \mapsto 1$$

Then the qba's $qm_1(.)$ and $qm_2(.)$ reduce to classical (precise) quantitative belief masses $m_1(.)$ and $m_2(.)$. In our simple example, one gets

$$m_1(\theta_1) = 0.2 \qquad m_1(\theta_2) = 0.6 \qquad m_1(\theta_1 \cup \theta_2) = 0.2$$

$$m_2(\theta_1) = 0.4 \qquad m_2(\theta_2) = 0.2 \qquad m_2(\theta_1 \cup \theta_2) = 0.4$$

We can apply any classical (quantitative) fusion rules. For example, with quantitative Conjunctive Rule, Dempster-Shafer (DS), DSmC, DSmH, PCR5 and Murphy's (Average Operator - AO) rules, one gets the results in Tables 10.11 and 10.12.

	θ_1	θ_2	$\theta_1 \cup \theta_2$
$m_1(.)$	0.2	0.6	0.2
$m_2(.)$	0.4	0.2	0.4
$m_{CR}(.)$	0.24	0.40	0.08
$m_{DSmC}(.)$	0.24	0.40	0.08
$m_{DS}(.)$	≈ 0.333	≈ 0.555	≈ 0.112
$m_{DSmH}(.)$	0.24	0.40	0.36
$m_{PCR5}(.)$	0.356	0.564	0.080
$m_{AO}(.)$	0.3	0.4	0.3

Table 10.11: Fusion through a crisp mapping

Important remark: The mapping of linguistic labels L_i into crisp numbers $x_i \in [0, 1]$ is a very difficult problem in general since the crisp mapping must generate from qualitative belief masses $qm_i(.)$, $i = 1, \ldots, s$, a set of complete normalized precise quantitative belief masses

	$\emptyset$	$\theta_1 \cap \theta_2$
$m_1(.)$		
$m_2(.)$		
$m_{CR}(.)$	0.28	0
$m_{DSmC}(.)$	0	0.28
$m_{DS}(.)$	0	0
$m_{DSmH}(.)$	0	0
$m_{PCR5}(.)$	0	0
$m_{AO}(.)$	0	0

Table 10.12: Fusion through a crisp mapping (cont'd)

$m_i(.)$, $i = 1, \dots, s$ (i.e. a set of crisp numbers in $[0, 1]$ such $\sum_{X \in G^\Theta} m_i(X) = 1$, $\forall i = 1, \dots, s$). According to [35, 36], such direct crisp mapping function can be found/built only if the qba's satisfy a given set of constraints. Generally a crisp mapping function and qba's generate for at least one of sources to combine either a paraconsistent (quantitative) belief assignments (if the sum of quantitative masses is greater than one) or an incomplete belief assignment (if the sum of masses is less than one). An issue would be in such cases to make a (quantitative) normalization of all paraconsistent and incomplete belief assignments drawn from crisp mapping and qba's before combining them with a quantitative fusion rule. The normalization of paraconsistent and incomplete bba's reflects somehow the difference in the interpretations of labels used by the sources (i.e. each source carries its own (mental/internal) representation of the linguistic label he/she uses when committing qualitative beliefs on any given proposition). It is possible to approximate the labels by crisp numbers of by subunitary subsets (in imprecise information), but the accuracy is arguable.

10.5.3 Fusion with an interval mapping

An other issue to avoid the direct manipulation of qualitative belief masses, is to try to assign intervals assign intervals or more general subunitary subsets to linguistic labels in order to model the vagueness of labels into numbers. We call this process, the *interval mapping* of qba's. This approach is less precise than the crisp mapping approach but is a quite good compromise between qualitative belief fusion and (precise) quantitative belief fusion.

In our simple example, we can easily check that the following interval mapping

$$L_0 \mapsto [0, 0.1), L_1 \mapsto [0.1, 0.3), L_2 \mapsto [0.3, 0.5), L_3 \mapsto [0.5, 0.7), L_4 \mapsto [0.7, 0.9), L_5 \mapsto [0.9, 1]$$

allows us to build two set of admissible[2] imprecise (quantitative) belief masses:

$$m_1^I(\theta_1) = [0.1, 0.3) \qquad m_2^I(\theta_1) = [0.3, 0.5)$$

$$m_1^I(\theta_2) = [0.5, 0.7) \qquad m_2^I(\theta_2) = [0.1, 0.3)$$

$$m_1^I(\theta_1 \cup \theta_2) = [0.1, 0.3) \qquad m_2^I(\theta_1 \cup \theta_2) = [0.3, 0.5)$$

[2] Admissibility condition means that we can pick up at least one number in each interval of an imprecise belief mass in such a way that the sum of these numbers is one (see [30] for details and examples). For example, $m_1^I(.)$ is admissible since there exist $0.22 \in [0.1, 0.3)$, $0.55 \in [0.5, 0.7)$, and $0.23 \in [0.1, 0.3)$ such that $0.22+0.55+0.23 = 1$.

These two admissible imprecise belief assignments can then be combined with (imprecise) combination rules proposed in [30] and based on the following operators for interval calculus: If $\mathcal{X}_1$,$\mathcal{X}_2$,..., $\mathcal{X}_n$ are real sets, then their sum is:

$$\boxed{\sum}_{k=1,\ldots,n} \mathcal{X}_k = \{x \mid x = \sum_{k=1,\ldots,n} x_k, x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n\}$$

while their product is:

$$\boxed{\prod}_{k=1,\ldots,n} \mathcal{X}_k = \{x \mid x = \prod_{k=1,\ldots,n} x_k, x_1 \in \mathcal{X}_1, \ldots, x_n \in \mathcal{X}_n\}$$

The results obtained with an interval mapping for the different (quantitative) rules of combination are summarized in Tables 10.13 and 10.14.

	θ_1	θ_2	$\theta_1 \cup \theta_2$
$m_1^I(.)$	$[0.1, 0.3)$	$[0.5, 0.7)$	$[0.1, 0.3)$
$m_2^I(.)$	$[0.3, 0.5)$	$[0.1, 0.3)$	$[0.3, 0.5)$
$m_{CR}^I(.)$	$[0.09, 0.45)$	$[0.21, 0.65)$	$[0.03, 0.15)$
$m_{DSmC}^I(.)$	$[0.09, 0.45)$	$[0.21, 0.65)$	$[0.03, 0.15)$
$m_{DSmH}^I(.)$	$[0.09, 0.45)$	$[0.21, 0.65)$	$[0.19, 0.59)$
$m_{PCR5}^I(.)$	$[0.15125, 0.640833)$	$[0.30875, 0.899167)$	$[0.03, 0.15)$
$m_{AO}^I(.)$	$[0.2, 0.4)$	$[0.3, 0.5)$	$[0.2, 0.4)$

Table 10.13: Fusion Results with interval mapping

	$\emptyset$	$\theta_1 \cap \theta_2$
$m_1^I(.)$		
$m_2^I(.)$		
$m_{CR}^I(.)$	$[0.16, 0.44)$	0
$m_{DSmC}^I(.)$	0	$[0.16, 0.44)$
$m_{DSmH}^I(.)$	0	0
$m_{PCR5}^I(.)$	0	0
$m_{AO}^I(.)$	0	0

Table 10.14: Fusion Results with interval mapping (cont'd)

10.6 Conclusion

We have extended in this chapter the use of DSmT from quantitative to qualitative belief assignments. In order to apply the fusion rules to qualitative information, we defined the $+$, $\times$, and even $-$ operators working on the set of linguistic labels. Tables of qualitative calculations are presented and examples using the corresponding qualitative-type Conjunctive, DSm Classic, DSm Hybrid, PCR5 rules, and qualitative-type Average Operator. We also mixed the qualitative and quantitative information in an attempt to refine the set of linguistic labels for a better accuracy. Since a normalization is not possible because the division of labels does not work, we introduced a quasi-normalization (i.e. approximation of the normalization). Then mappings were designed from qualitative to (crisp or interval) quantitative belief assignments.

10.7 References

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Part II

Applications of DSmT

Chapter 11

Generalized proportional conflict redistribution rule applied to Sonar imagery and Radar targets classification

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Abstract: *In this chapter, we present two applications in information fusion in order to evaluate the generalized proportional conflict redistribution rule presented in chapter [7]. Most of the time the combination rules are evaluated only on simple examples. We study here different combination rules and compare them in terms of decision on real data. Indeed, in real applications, we need a reliable decision and it is the final results that matter. Two applications are presented here: a fusion of human experts opinions on the kind of underwater sediments depicted on a sonar image and a classifier fusion for radar targets recognition.*

11.1 Introduction

We have presented and discussed some combination rules in the chapter [7]. Our study was essentially on the redistribution of conflict rules. We have proposed a new proportional conflict redistribution rule. We have seen that the decision can be different following the rule. Most of the time the combination rules are evaluated only on simple examples. In this chapter, we study different combination rules and compare them in terms of decision on real data. Indeed, in real applications, we need a reliable decision and it is the final results that matter. Hence, for a given application, the best combination rule is the rule giving the best results. For the decision step, different functions such as credibility, plausibility and pignistic probability [1, 9, 13] are usually used.

In this chapter, we present the advantages of the DSmT for the modelization of real applications and also for the combination step. First, the principles of the DST and DSmT are recalled. We present the formalization of the belief function models, different rules of combination and decision. One combination rule (PCR5) proposed by [12] for two experts is mathematically one of the best for the proportional redistribution of the conflict applicable in the context of the DST and the DSmT. We compare here an extension of this rule for more than two experts, the PCR6 rule presented in the chapter [7], and other rules using the same data model.

Two applications are presented here: a fusion of human experts opinions on the kind of underwater sediments depicted on a sonar image and a classifier fusion for radar targets recognition.

The first application relates to the seabed characterization, for instance in order to help the navigation of Autonomous Underwater Vehicles or provide data to sedimentologists. The sonar images are obtained with many imperfections due to instrumentations measuring a huge number of physical data (geometry of the device, coordinates of the ship, movements of the sonar, etc.). In this kind of applications, the reality is unknown. If human experts have to classify sonar images they can not provide with certainty the kind of sediment on the image. Thus, for instance, in order to train an automatic classification algorithm, we must take into account this difference and the uncertainty of each expert. We propose in this chapter how to solve this human experts fusion.

The second application allows to really compare the combination rules. We present an application of classifiers fusion in order to extract the information for the automatic target recognition. The real data are provided by measures in the anechoic chamber of ENSIETA (Brest, France) obtained by illuminating 10 scale reduced (1:48) targets of planes. Hence, all the experimentations are controlled and the reality is known. The results of the fusion of three classifiers are studied in terms of good-classification rates.

This chapter is organized as follow: In the first section, we recall combination rules presented in the chapter [7] that we compare them in this chapter. The section 11.3 proposes a way of fusing human expert's opinions in uncertain environments such as the underwater environment. This environment is described with sonar images which are the most appropriate in such environment. The last section presents the results of classifiers fusion in an application of radar targets recognition.

11.2 Backgrounds on combination rules

We recall here the combination rules presented and discussed in chapter [7] and compare them on two real applications in the following sections. For more details on the theory, see chapter [7].

In the context of the DST, the non-normalized conjunctive rule is one of the most used rule and is given by [13] for all $X \in 2^\Theta$ by:

$$m_c(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j), \tag{11.1}$$

where $Y_j \in 2^\Theta$ is a response of the expert j, and $m_j(Y_j)$ is the associated basic belief assignment.

In this chapter, we focus on rules where the conflict is redistributed. With the rule given by Dubois and Prade [3], a mixed conjunctive and disjunctive rule, the conflict is redistributed on partial ignorance. This rule is given for all $X \in 2^\Theta$, $X \neq \emptyset$ by:

$$m_{\mathrm{DP}}(X) = \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) \; + \sum_{\substack{Y_1 \cup \ldots \cup Y_M = X \\ Y_1 \cap \ldots \cap Y_M = \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \tag{11.2}$$

where $Y_j \in 2^\Theta$ is a response of the expert j, and $m_j(Y_j)$ the associated basic belief assignment.

In the context of the DSmT, the non-normalized conjunctive rule can be used for all $X \in D^\Theta$ and $Y \in D^\Theta$. The mixed rule given by the equation (11.2) has been rewritten in [10], and called DSmH, for all $X \in D^\Theta$, $X \not\equiv \emptyset$ [1] by:

$$\begin{aligned} m_H(X) = & \sum_{Y_1 \cap \ldots \cap Y_M = X} \prod_{j=1}^{M} m_j(Y_j) \; + \sum_{\substack{Y_1 \cup \ldots \cup Y_M = X \\ Y_1 \cap \ldots \cap Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) \; + \\ & \sum_{\substack{\{u(Y_1) \cup \ldots \cup u(Y_M) = X\} \\ Y_1, \ldots, Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j) \; + \sum_{\substack{\{u(Y_1) \cup \ldots \cup u(Y_M) \equiv \emptyset \text{ and } X = \Theta\} \\ Y_1, \ldots, Y_M \equiv \emptyset}} \prod_{j=1}^{M} m_j(Y_j), \end{aligned} \tag{11.3}$$

where $Y_j \in D^\Theta$ is a response of the expert j, $m_j(Y_j)$ the associated basic belief assignment, and $u(Y)$ is the function giving the union of the terms that compose Y [11]. For example if $Y = (A \cap B) \cup (A \cap C)$, $u(Y) = A \cup B \cup C$.

If we want to take the decision only on the elements in Θ, some rules propose to redistribute the conflict proportionally on these elements. The most accomplished is the PCR5 given in [12]. The equation for M experts, for $X \in D^\Theta$, $X \not\equiv \emptyset$ is given in [2] by:

$$\begin{aligned} & m_{\mathrm{PCR5}}(X) = m_{\mathrm{c}}(X) + \\ & \sum_{i=1}^{M} m_i(X) \sum_{\substack{(Y_{\sigma_i(1)}, \ldots, Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1} \\ \bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset}} \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \mathbb{1}_{j>i}\right) \prod_{Y_{\sigma_i(j)} = X} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{\sum_{Z \in \{X, Y_{\sigma_i(1)}, \ldots, Y_{\sigma_i(M-1)}\}} \prod_{Y_{\sigma_i(j)} = Z} \left(m_{\sigma_i(j)}(Y_{\sigma_i(j)}) . T(X{=}Z, m_i(X))\right)}, \end{aligned} \tag{11.4}$$

where σ_i counts from 1 to M avoiding i:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j + 1 & \text{if } j \geq i, \end{cases} \tag{11.5}$$

and:

$$\begin{cases} T(B, x) = x & \text{if } B \text{ is true}, \\ T(B, x) = 1 & \text{if } B \text{ is false}, \end{cases} \tag{11.6}$$

[1]The notation $X \not\equiv \emptyset$ means that $X \neq \emptyset$ and following the chosen model in D^Θ, X is not one of the elements of D^Θ defined as $\emptyset$. For example, if $\Theta = \{A, B, C\}$, we can define a model for which the expert can provide a mass on $A \cap B$ and not on $A \cap C$, so $A \cap B \not\equiv \emptyset$ and $A \cap C \equiv \emptyset$

We have proposed another proportional conflict redistribution rule PCR6 rule in the chapter [7], for M experts, for $X \in D^\Theta$, $X \neq \emptyset$:

$$\begin{aligned} m_{\mathrm{PCR6}}(X) &= m_{\mathrm{c}}(X) + \\ & \sum_{i=1}^{M} m_i(X)^2 \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right), \end{aligned} \tag{11.7}$$

where σ is defined like in (11.5).

As Y_i is a focal element of expert i, $m_i(X) + \sum_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$; the belief function m_c is the conjunctive consensus rule given by the equation (11.1). The PCR6 and PCR5 rules are exactly the same in the case of 2 experts.

We have also proposed two more general rules given by:

$$\begin{aligned} & m_{\mathrm{PCRf}}(X) = m_{\mathrm{c}}(X) + \\ & \textstyle\sum_{i=1}^{M} m_i(X) f(m_i(X)) \displaystyle\sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \left(\frac{\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{f(m_i(X)) + \sum_{j=1}^{M-1} f(m_{\sigma_i(j)}(Y_{\sigma_i(j)}))} \right), \end{aligned} \tag{11.8}$$

with the same notations that in the equation (11.7), and f is an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$.

The second generalized rule is given by:

$$\begin{aligned} & m_{\mathrm{PCRg}}(X) = m_{\mathrm{c}}(X) + \sum_{i=1}^{M} \sum_{\substack{\bigcap_{k=1}^{M-1} Y_{\sigma_i(k)} \cap X \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}) \in (D^\Theta)^{M-1}}} \\ & m_i(X) \frac{\left(\prod_{j=1}^{M-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \right) \left(\prod_{Y_{\sigma_i(j)} = X} \mathbb{1}_{j>i} \right) g\left(m_i(X) + \sum_{Y_{\sigma_i(j)} = X} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \right)}{\sum_{Z \in \{X, Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(M-1)}\}} g\left(\sum_{Y_{\sigma_i(j)} = Z} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) + m_i(X) \mathbb{1}_{X=Z} \right)}, \end{aligned} \tag{11.9}$$

with the same notations that in the equation (11.7), and g is an increasing function defined by the mapping of $[0,1]$ onto $\mathbb{R}^+$.

In this chapter, we choose $f(x) = g(x) = x^\alpha$, with $\alpha \in \mathbb{R}^+$.

11.3 Experts fusion in Sonar imagery

Seabed characterization serves many useful purposes, *e.g.* help the navigation of Autonomous Underwater Vehicles or provide data to sedimentologists. In such sonar applications, seabed images are obtained with many imperfections [5]. Indeed, in order to build images, a huge number of physical data (geometry of the device, coordinates of the ship, movements of the sonar, etc.) has to be taken into account, but these data are polluted with a large amount of noises caused by instrumentations. In addition, there are some interferences due to the signal traveling on multiple paths (reflection on the bottom or surface), due to speckle, and due to fauna and flora. Therefore, sonar images have a lot of imperfections such as imprecision and uncertainty; thus sediment classification on sonar images is a difficult problem. In this kind of applications, the reality is unknown and different experts can propose different classifications of the image. Figure 11.1 exhibits the differences between the interpretation and the certainty of two sonar experts trying to differentiate the type of sediment (rock, cobbles, sand, ripple, silt) or shadow when the information is invisible. Each color corresponds to a kind of sediment and the associated certainty of the expert for this sediment expressed in term of sure, moderately sure and not sure. Thus, in order to train an automatic classification algorithm, we must take into account this difference and the uncertainty of each expert. Indeed, image classification is generally done on a local part of the image (pixel, or most of the time on small tiles of *e.g.* 16×16 or 32×32 pixels). For example, how a tile of rock labeled as *not sure* must be taken into account in the learning step of the classifier and how to take into account this tile if another expert says that it is sand? Another problem is: how should we consider a tile with more than one sediment?

In this case, the space of discernment Θ represents the different kind of sediments on sonar images, such as rock, sand, silt, cobble, ripple or shadow (that means no sediment information). The experts give their perception and belief according to their certainty. For instance, the expert can be moderately sure of his choice when he labels one part of the image as belonging to a certain class, and be totally doubtful on another part of the image. Moreover, on a considered tile, more than one sediment can be present.

Consequently we have to take into account all these aspects of the applications. In order to simplify, we consider only two classes in the following: the rock referred as A, and the sand, referred as B. The proposed models can be easily extended, but their study is easier to understand with only two classes.

Hence, on certain tiles, A and B can be present for one or more experts. The belief functions have to take into account the certainty given by the experts (referred respectively as c_A and c_B, two numbers in $[0, 1]$) as well as the proportion of the kind of sediment in the tile X (referred as p_A and p_B, also two numbers in $[0, 1]$). We have two interpretations of "the expert believes A": it can mean that the expert thinks that there is A on X and not B, or it can mean that the expert thinks that there is A on X and it can also have B but he does not say anything about it. The first interpretation yields that hypotheses A and B are exclusive and with the second they are not exclusive. We only study the first case: A and B are exclusive. But on the tile X, the expert can also provide A and B, in this case the two propositions "the expert believes A" and "the expert believes A and B" are not exclusive.

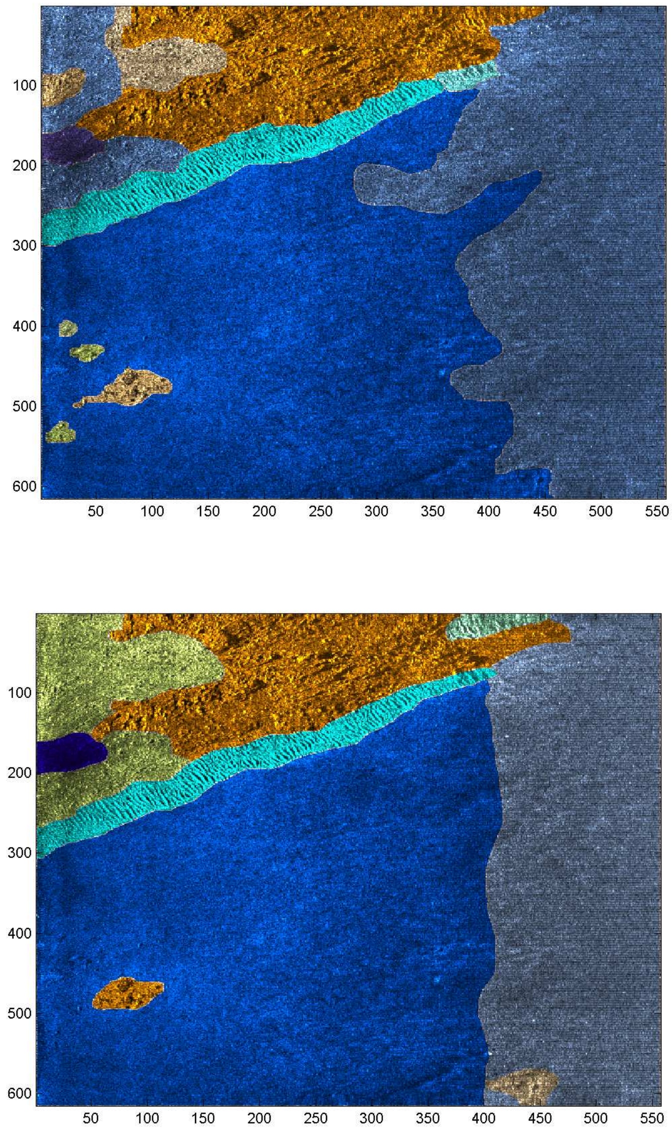

Figure 11.1: Segmentation given by two experts.

11.3.1 Models

We have proposed five models and studied these models for the fusion of two experts [6]. We present here the three last models for two experts and two classes. In this case the conjunctive rule (11.1), the mixed rule (11.2) and the DSmH (11.3) are similar. We give the obtained results on a real database for the fusion of three experts in sonar.

Model M_3 In our application, A, B and C cannot be considered exclusive on X. In order to propose a model following the DST, we have to study exclusive classes only. Hence, in our application, we can consider a space of discernment of three exclusive classes $\Theta = \{A \cap B^c, B \cap A^c, A \cap B\} = \{A', B', C'\}$, following the notations given on the figure 11.2.

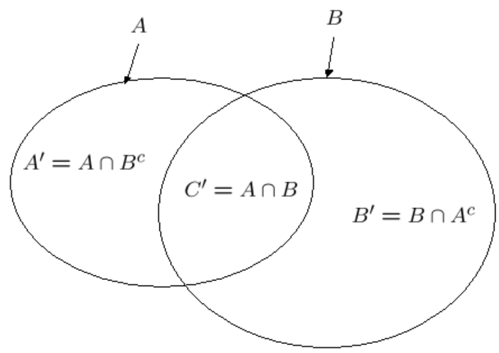

Figure 11.2: Notation of the intersection of two classes A and B.

Hence, we can propose a new model M_3 given by:

$$
\begin{array}{l}
\text{if the expert says } A\text{:}\\
\begin{cases} m(A' \cup C') = c_A, \\ m(A' \cup B' \cup C') = 1 - c_A, \end{cases}\\
\\
\text{if the expert says } B\text{:}\\
\begin{cases} m(B' \cup C') = c_B, \\ m(A' \cup B' \cup C') = 1 - c_B, \end{cases}\\
\\
\text{if the expert says } A \text{ and } B\text{:}\\
\begin{cases} m(C') = p_A.c_A + p_B.c_B, \\ m(A' \cup B' \cup C') = 1 - (p_A.c_A + p_B.c_B). \end{cases}
\end{array}
\tag{11.10}
$$

Note that $A' \cup B' \cup C' = A \cup B$. On our numerical example we obtain:

	$A' \cup C'$	$B' \cup C'$	C'	$A' \cup B' \cup C'$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

Hence, the conjunctive rule, the credibility, the plausibility and the pignistic probability are given by:

element	m_c	bel	pl	betP
$\emptyset$	0	0	0	$-$
$A' = A \cap B^c$	0	0	0.5	0.2167
$B' = B \cap A^c$	0	0	0.2	0.0667
$A' \cup B' = (A \cap B^c) \cup (B \cap A^c)$	0	0	0.5	0.2833
$C' = A \cap B$	0.5	0.5	1	0.7167
$A' \cup C' = A$	0.3	0.8	1	0.9333
$B' \cup C' = B$	0	0.5	1	0.7833
$A' \cup B' \cup C' = A \cup B$	0.2	1	1	1

where

$$m_c(C') = m_c(A \cap B) = 0.2 + 0.3 = 0.5. \tag{11.11}$$

In this example, with this model M_3 the decision will be A with the maximum of the pignistic probability. But the decision could *a priori* be taken also on $C' = A \cap B$ because $m_c(C')$ is the highest. We have seen that if we want to take the decision on $A \cap B$, we must considered the maximum of the masses because of inclusion relations of the credibility, plausibility and pignistic probability.

Model M_4 In the context of the DSmT, we can write $C = A \cap B$ and easily propose a fourth model M_4, without any consideration on the exclusivity of the classes, given by:

$$\begin{array}{l} \text{if the expert says } A\text{:} \\ \begin{cases} m(A) = c_A, \\ m(A \cup B) = 1 - c_A, \end{cases} \\ \\ \text{if the expert says } B\text{:} \\ \begin{cases} m(B) = c_B, \\ m(A \cup B) = 1 - c_B, \end{cases} \\ \\ \text{if the expert says } A \text{ and } B\text{:} \\ \begin{cases} m(A \cap B) = p_A.c_A + p_B.c_B, \\ m(A \cup B) = 1 - (p_A.c_A + p_B.c_B). \end{cases} \end{array} \tag{11.12}$$

This model M_4 allows to represent our problem without adding an artificial class C. Thus, the model M_4 based on the DSmT gives:

	A	B	$A \cap B$	$A \cup B$
m_1	0.6	0	0	0.4
m_2	0	0	0.5	0.5

The obtained mass m_c with the conjunctive rule yields:

$$\begin{aligned} m_c(A) &= 0.30, \\ m_c(B) &= 0, \\ m_c(A \cap B) &= m_1(A)m_2(A \cap B) + m_1(A \cup B)m_2(A \cap B) \\ &= 0.30 + 0.20 = 0.5, \\ m_c(A \cup B) &= 0.20. \end{aligned} \tag{11.13}$$

These results are exactly similar to the model M_3. These two models do not present ambiguity and show that the mass on $A \cap B$ (rock and sand) is the highest.

The generalized credibility, the generalized plausibility and the generalized pignistic probability are given by:

element	m_c	Bel	Pl	GPT
$\emptyset$	0	0	0	–
A	0.3	0.8	1	0.9333
B	0	0.5	0.7	0.7833
$A \cap B$	0.5	0.5	1	0.7167
$A \cup B$	0.2	1	1	1

Like the model M_3, on this example, the decision will be A with the maximum of pignistic probability criteria. But here also the maximum of m_c is reached for $A \cap B = C'$.

If we want to consider only the kind of possible sediments A and B and do not allow their conjunction, we can use a proportional conflict redistribution rule such as the PCR rule:

$$\begin{array}{l} m_{PCR}(A) = 0.30 + 0.5 = 0.8, \\ m_{PCR}(B) = 0, \\ m_{PCR}(A \cup B) = 0.20. \end{array} \tag{11.14}$$

The credibility, the plausibility and the pignistic probability are given by:

element	m_{PCR}	bel	pl	betP
$\emptyset$	0	0	0	$-$
A	0.8	0.8	1	0.9
B	0	0	0.2	0.1
$A \cup B$	0.2	1	1	1

On this numerical example, the decision will be the same as the conjunctive rule, here the maximum of pignistic probability is reached for A (rock). In the next section, we see that is not always the case.

Model M_5 Another model is M_5 which can be used in both the DST and the DSmT. It considers only one belief function according to the proportion, given by:

$$\begin{cases} m(A) = p_A.c_A, \\ m(B) = p_B.c_B, \\ m(A \cup B) = 1 - (p_A.c_A + p_B.c_B). \end{cases} \tag{11.15}$$

If for one expert, the tile contains only A, $p_A = 1$, and $m(B) = 0$. If for another expert, the tile contains A and B, we take into account the certainty and proportion of the two sediments but not only on one focal element. Consequently, we have simply:

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0.3	0.2	0.5

In the DST context, the conjunctive rule, the credibility, the plausibility and the pignistic probability are given by:

element	m_c	bel	pl	betP
$\emptyset$	0.12	0	0	$-$
A	0.6	0.6	0.8	0.7955
B	0.08	0.08	0.28	0.2045
$A \cup B$	0.2	0.88	0.88	1

In this case we can not decide $A \cap B$, because the conflict is on $\emptyset$.

In the DSmT context, the conjunctive rule, the generalized credibility, the generalized plausibility and the generalized pignistic probability are given by:

element	m_c	Bel	Pl	GPT
$\emptyset$	0	0	0	–
A	0.6	0.72	0.92	0.8933
B	0.08	0.2	0.4	0.6333
$A \cap B$	0.12	0.12	1	0.5267
$A \cup B$	0.2	1	1	1

The decision with the maximum of pignistic probability criteria is still A.

The PCR rule provides:

element	m_{PCR}	bel	pl	betP
$\emptyset$	0	0	0	–
A	0.69	0.69	0.89	0.79
B	0.11	0.11	0.31	0.21
$A \cup B$	0.2	1	1	1

where

$$m_{PCR}(A) = 0.60 + 0.09 = 0.69,$$
$$m_{PCR}(B) = 0.08 + 0.03 = 0.11.$$

With this model and example of the PCR rule, the decision will be also A, and we do not have difference between the conjunctive rules in the DST and DSmT.

11.3.2 Experimentation

Database Our database contains 42 sonar images provided by the GESMA (Groupe d'Etudes Sous-Marines de l'Atlantique). These images were obtained with a Klein 5400 lateral sonar with a resolution of 20 to 30 cm in azimuth and 3 cm in range. The sea-bottom depth was between 15 m and 40 m.

Three experts have manually segmented these images, giving the kind of sediment (rock, cobble, sand, silt, ripple (horizontal, vertical or at 45 degrees)), shadow or other (typically ships) parts on images, helped by the manual segmentation interface presented in figure 11.3. All sediments are given with a certainty level (sure, moderately sure or not sure). Hence, each pixel of every image is labeled as being either a certain type of sediment or a shadow or other.

The three experts provide respectively, 30338, 31061, and 31173 homogeneous tiles, 8069, 7527, and 7539 tiles with two sediments, 575, 402, and 283 tiles with three sediments, 14, 7, and 2 tiles with four, and 1, 0, and 0 tile for five sediments, and 0 for more.

Results We note A = rock, B = cobble, C = sand, D = silt, E = ripple, F = shadow and G = other, hence we have seven classes and $\Theta = \{A, B, C, D, E, F, G\}$. We applied the

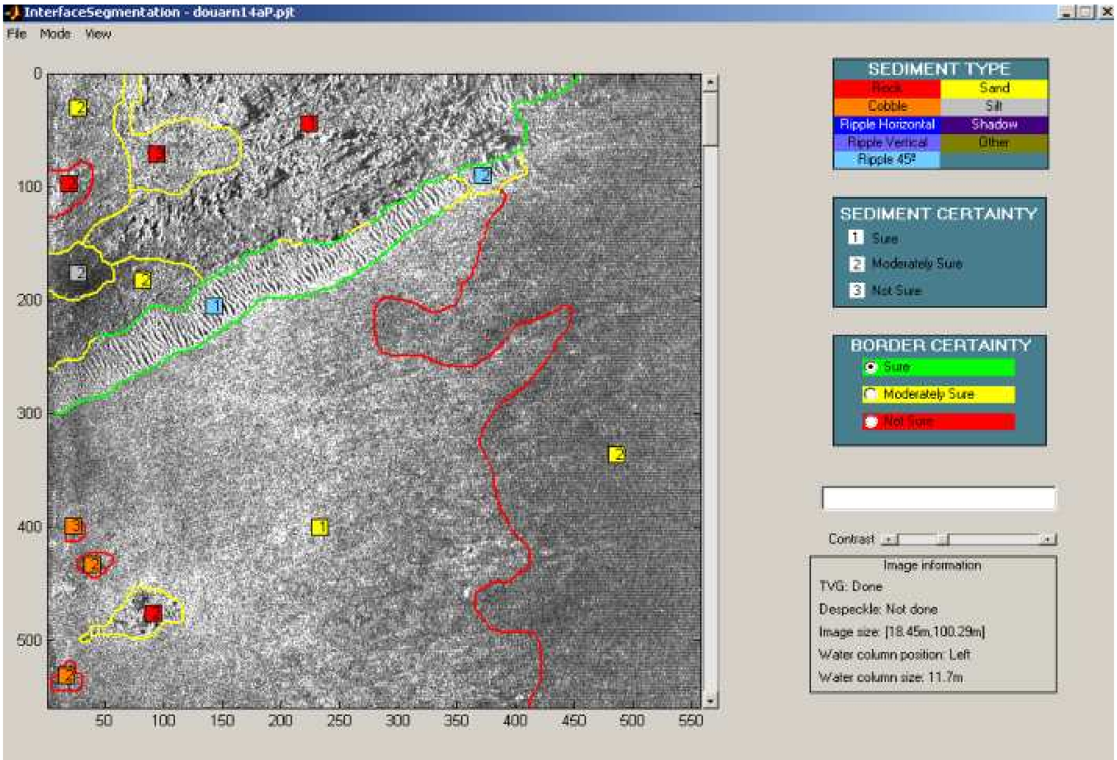

Figure 11.3: Manual Segmentation Interface.

generalized model M_5 on tiles of size 32×32 given by:

$$\begin{cases} m(A) = p_{A1}.c_1 + p_{A2}.c_2 + p_{A3}.c_3, \text{ for rock,} \\ m(B) = p_{B1}.c_1 + p_{B2}.c_2 + p_{B3}.c_3, \text{ for cobble,} \\ m(C) = p_{C1}.c_1 + p_{C2}.c_2 + p_{C3}.c_3, \text{ for ripple,} \\ m(D) = p_{D1}.c_1 + p_{D2}.c_2 + p_{D3}.c_3, \text{ for sand,} \\ m(E) = p_{E1}.c_1 + p_{E2}.c_2 + p_{E3}.c_3, \text{ for silt,} \\ m(F) = p_{F1}.c_1 + p_{F2}.c_2 + p_{F3}.c_3, \text{ for shadow,} \\ m(G) = p_{G1}.c_1 + p_{G2}.c_2 + p_{G3}.c_3, \text{ for other,} \\ m(\Theta) = 1 - (m(A) + m(B) + m(C) + m(D) + m(E) + m(F) + m(G)), \end{cases} \tag{11.16}$$

where c_1, c_2 and c_3 are the weights associated to the certitude respectively: "sure", "moderately sure" and "not sure". The chosen weights are here: $c_1 = 2/3$, $c_2 = 1/2$ and $c_3 = 1/3$. Indeed we have to consider the cases when the same kind of sediment (but with different certainties) is present on the same tile. The proportion of each sediment in the tile associated to these weights is noted, for instance for A: p_{A1}, p_{A2} and p_{A3}.

The total conflict between the three experts is 0.2244. This conflict comes essentially from the difference of opinion of the experts and not from the tiles with more than one sediment. Indeed, we have a weak *auto-conflict* (conflict coming from the combination of the same expert three times). The values of the auto-conflict for the three experts are: 0.0496, 0.0474, and 0.0414. We note a difference of decision between the three combination rules given by the equations (11.7) for the PCR6, (11.2) for the mixed rule and (11.1) for the conjunctive rule.

The proportion of tiles with a different decision is 0.11% between the mixed rule and the conjunctive rule, 0.66% between

the PCR6 and the mixed rule, and 0.73% between the PCR6 and the conjunctive rule.

These results show that there is a difference of decision according to the combination rules with the same model. However, we can not know what is the best decision, and so what is the most precise rule among the experimented ones, because on this application no ground truth is known. We compare these same rules in another application, where the reality is completely known.

11.4 Classifiers fusion in Radar target recognition

Several types of classifiers have been developed in order to extract the information for the automatic target recognition (ATR). We have proposed different approaches of information fusion in order to outperform three radar target classifiers [4]. We present here the results reached by the fusion of three classifiers with the conjunctive rule, the DSmH, the PCR5 and the PCR6.

11.4.1 Classifiers

The three classifiers used here are the same as in [4]. The first one is a fuzzy K-nearest neighbors classifier, the second one is a multilayer perceptron (MLP) that is a feed forward fully connected neural network. And the third one is the SART (Supervised ART) classifier [8] that uses the principle of prototype generation like the ART neural network, but unlike this one, the prototypes are generated in a supervised manner.

11.4.2 Database

The database is the same than in [4]. The real data were obtained in the anechoic chamber of ENSIETA (Brest, France) using the experimental setup shown on figure 11.4. We have considered 10 scale reduced (1:48) targets (Mirage, F14, Rafale, Tornado, Harrier, Apache, DC3, F16, Jaguar and F117).

Each target is illuminated in the acquisition phase with a frequency stepped signal. The data snapshot contains 32 frequency steps, uniformly distributed over the band $B = [11650\text{MHZ}, 17850MHz]$, which results in a frequency increment of $\Delta f = 200\text{MHz}$. Consequently, the slant range resolution and ambiguity window are given by:

$$\Delta R_s = c/(2B) \simeq 2.4m,\ W_s = c/(2\Delta f) = 0.75m. \tag{11.17}$$

The complex signature obtained from a backscattered snapshot is coherently integrated via FFT in order to achieve the slant range profile corresponding to a given aspect of a given target. For each of the 10 targets 150 range profiles are thus generated corresponding to 150 angular positions, from -50 degrees to 69.50 degrees, with an angular increment of 0.50 degrees.

The database is randomly divided in a training set (for the three supervised classifiers) and test set (for the evaluation). When all the range profiles are available, the training set is formed by randomly selecting 2/3 of them, the others being considered as the test set.

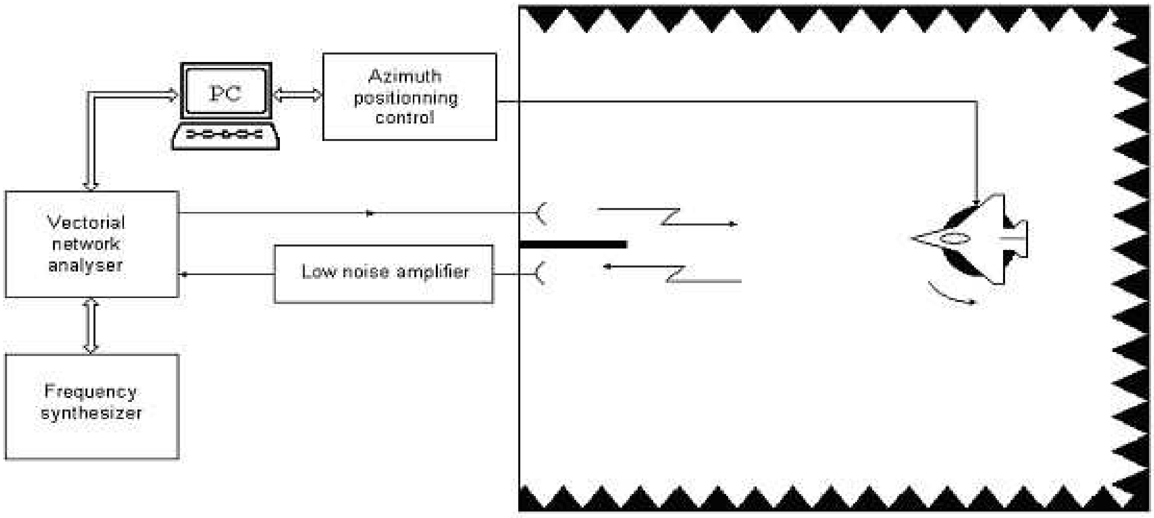

Figure 11.4: Experimental setup.

11.4.3 Model

The numerical outputs of the classifiers for each target and each classifier, normalized between 0 and 1, define the input masses. In order to keep only the most credible classes we consider the two highest values of these outputs referred as o_{ij} for the j^{th} classifier and the target i. Hence, we obtain only three focal elements (two targets and the ignorance Θ).

The classifier does not provide equivalent belief in mean. For example, the fuzzy K-nearest neighbors classifier easily provides a belief of 1 for a target, whereas the two other classifiers always provide a not null belief on the second target and on ignorance. In order to balance the classifiers, we weight each belief function by an adaptive threshold given by:

$$f_j = \frac{0.8}{mean(o_{ij})} \cdot \frac{0.8}{mean(b_{ij})}, \tag{11.18}$$

where $mean(o_{ij})$ is the mean of the belief of the two targets on all the previous considered signals for the classifier j, $mean(b_{ij})$ is the similar mean on $b_{ij} = f_j.o_{ij}$. First, f_j is initialized to 1. Hence, the mean of belief on the singletons tends toward 0.8 for each classifier, and 0.2 on Θ.

Moreover, if the belief assignment on Θ for a given signal and classifier is less than 0.001, we keep the maximum of the mass and force the other in order to reach 0.001 on the ignorance and so avoid total conflict with the conjunctive rule.

11.4.4 Results

We have conducted the division of the database into training database and test database, 800 times in order to estimate better the good-classification rates. We have obtained a total conflict of 0.4176. The auto-conflict, reached by the combination of the same classifier three times, is 0.1570 for the fuzzy K-nearest neighbor, 0.4055 for the SART and 0.3613 for the multilayer perceptron. The auto-conflict for the fuzzy K-nearest neighbor is weak because it happens

Rule	Conj.	DP	PCR$f_{\sqrt{x}}$	PCR$g_{\sqrt{x}}$	PCR6	PCRg_{x^2}	PCRf_{x^2}	PCR5
Conj.	0	0.68	1.53	1.60	2.02	2.53	2.77	2.83
DP	0.68	0	0.94	1.04	1.47	2.01	2.27	2.37
PCR$f_{\sqrt{x}}$	1.53	0.94	0	0.23	0.61	1.15	1.49	1.67
PCR$g_{\sqrt{x}}$	1.60	1.04	0.23	0	0.44	0.99	1.29	1.46
PCR6	2.04	1.47	0.61	0.44	0	0.55	0.88	1.08
PCRg_{x^2}	2.53	2.01	1.15	0.99	0.55	0	0.39	0.71
PCRf_{x^2}	2.77	2.27	1.49	1.29	0.88	0.39	0	0.51
PCR5	2.83	2.37	1.67	1.46	1.08	0.71	0.51	0

Table 11.1: Proportion of targets with a different decision (%)

Rule	%	Confidence Interval
Conjunctive	89.83	[89.75 : 89.91]
DP	89.99	[89.90 : 90.08]
PCR$f_{x^{0.3}}$	90.100	[90.001 : 90.200]
PCR$f_{\sqrt{x}}$	**90.114**	[90.015 : 90.213]
PCR$f_{x^{0.7}}$	90.105	[90.006 : 90.204]
PCR$g_{\sqrt{x}}$	90.08	[89.98 : 90.18]
PCR6	90.05	[89.97 : 90.13]
PCRg_{x^2}	90.00	[89.91 : 90.10]
PCRf_{x^2}	89.94	[89.83 : 90.04]
PCR5	89.85	[89.75 : 89.85]

Table 11.2: Good-classification rates (%)

many times that the mass is only on one class (and ignorance), whereas there are two classes with a non-null mass for the SART and the multilayer perceptron. Hence, the fuzzy K-nearest neighbor reduces the total conflict during the combination. The total conflict here is higher than in the previous application, but it comes here from the modelization essentially and not from a difference of opinion given by the classifiers.

The proportion of targets with a different decision is given in percentage, in the table 11.1. These percentages are more important for this application than the previous application on sonar images. Hence the conjunctive rule and the mixed rule are very similar. In terms of similarity, we can give this order: conjunctive rule, the mixed rule (DP), PCR6f and PCR6g with a concave mapping, PCR6, PCR6f and PCR6g with a convex mapping, and PCR5.

The final decision is taken with the maximum of the pignistic probabilities. Hence, the results reached by the generalized PCR are significantly better than the conjunctive rule and the PCR5, and better than the mixed rule (DP). The conjunctive rule and the PCR5 give the worst classification rates on these data (there is no significantly difference), whereas they have a high proportion of targets with a different decision.

The best classification rate (see table 11.2) is obtained with PCR$f_{\sqrt{x}}$, but is not significantly better than the results obtained with the other versions PCRf, using a different concave mapping.

11.5 Conclusion

In this chapter, we have proposed a study of the combination rules compared in terms of decision. The generalized proportional conflict redistribution (PCR6) rule (presented in the chapter [7]) has been evaluated. We have shown on real data that there is a difference of decision following the choice of the combination rule. This difference can be very small in percentage but leads to significant difference in good-classification rates. Moreover, a high proportion with a different decision does not lead to a high difference in terms of good-classification rates. The last application shows that we can achieve better good-classification rates with the generalized PCR6 than with the conjunctive rule, the DSmH (i.e. the mixed DP rule), or PCR5.

The first presented application shows that the modelization on D^Θ can resolve easily some problems. If the application needs a decision step and if we want to consider the conjunctions of the elements of the discernment space, we have to take the decision directly on the masses (and not on the credibilities, plausibilities or pignistic probabilities). Indeed, these functions are increasing and can not give a decision on the conjunctions of elements. In real applications, most of the time, there is no ambiguity and we can take the decision, else we have to propose a new decision function that can reach a decision on conjunctions and also on singletons.

The conjunctions of elements can be considered (and so D^Θ) in many applications, especially in image processing, where an expert can provide elements with more than one class. In estimation applications, where intervals are considered, encroaching intervals (with no empty intersection) can provide a better modelization.

11.6 References

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Chapter 12

Multitarget Tracking in Clutter based on Generalized Data Association: Performance Evaluation of Fusion Rules

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Abstract: *The objective of this chapter is to present and compare different fusion rules which can be used for Generalized Data Association (GDA) for multitarget tracking (MTT) in clutter. Most of tracking methods including Target Identification (ID) or attribute information are based on classical tracking algorithms such PDAF, JPDAF, MHT, IMM, etc. and either on the Bayesian estimation and prediction of target ID, or on fusion of target class belief assignments through the Dempster-Shafer Theory (DST) and Dempster's rule of combination. The main purpose of this study is to pursue our previous works on the development of a new GDA-MTT based on Dezert-Smarandache Theory (DSmT) but compare it also with standard fusion rules (Dempster's, Dubois & Prade's, Yager's) and with the new fusion rules: Proportional Conflict Redistribution rule No.5(PCR5), fusion rule based on T-Conorm and T-Norm Fuzzy Operators(TCN rule) and the Symmetric Adaptive Combination (SAC) rule. The goal is to assess the efficiency of all these different fusion rules for the applied GDA-MTT in critical, highly conflicting situation. This evaluation is based on a Monte Carlo simulation for a particular difficult maneuvering MTT problem in clutter.*

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12.1 Introduction

The idea of incorporating Target Identification (ID) information or target attribute measurements into classical (i.e. kinematics-based) tracking filters to improve multitarget tracking systems is not new and many approaches have been proposed in the literature over the last fifteen years. For example, in [14, 15, 21] an improved PDAF (Probabilistic Data Association Filter) had been developed for autonomous navigation systems based on Target Class ID and ID Confusion matrix, and also on another version based on imprecise attribute measurements combined within Dempster's rule. At the same time Lerro in [20] developed the AI-PDAF (Amplitude Information PDAF). Since the nineties many improved versions of classical tracking algorithms like IMM, JPDAF, IMM-PDAF, MHT, etc. including attribute information have been proposed (see [12] and [6] for a recent overview). Recent contributions have been done by Blasch and al. in [7–10, 31] for Group Target Tracking and classification. In last two years efforts have been done also by Hwang and al. in [17–19]. We recently discovered that the Hwang's MTIM (Multiple-target Tracking and Identity Management) algorithm is very close to our Generalized Data Association GDA-MTT. The difference between MTIM and GDA-MTT lies fundamentally in the Attribute Data Association procedure. MTIM is based on MAJPDA (Modified Approximated JPDA) coupled with RMIMM (Residual-mean Interacting Multiple Model) algorithm while the GDA-MTT is based on GNN (Global Nearest Neighbour) approach for data association incorporating both kinematics and attribute measurements (with more sophisticated fusion rules dealing with fuzzy, imprecise and potentially highly conflicting target attribute measurements), coupled with standard IMM-CMKF(Converted Measurement Kalman Filter) [1, 5, 23]. The last recent attempt for solving the GDA-MTT problem was proposed by Bar-Shalom and al. in [6] and expressed as a multiframe assignment problem where the multiframe association likelihood was developed to include the target classification results based on the confusion matrix that specifies the prior accuracy of the target classifier. Such multiframe s-D assignment algorithm should theoretically provide performances close to the optimality for MTT systems but remains computationally greedy. The purpose of this chapter is to compare the performances of several fusion rules usable into our new GDA-MTT algorithm based on a difficult MTT scenario with eleven closely spaced and maneuvering in some regions targets, belonging only to two classes within clutter and with only 2D kinematical measurements and attribute measurement.

This chapter is organized as follows. In section 12.2 we present our approach for GDA-MTT algorithm emphasizing only on the new developments in comparison with our previous GDA-MTT algorithm, developed in [26, 29]. In our previous works, we proved the efficiency of GDA-MTT (in term of Track Purity Performance) based on the DSm Hybrid rule of combination over the GDA-MTT based on Dempster's rule but also over the KDA-MTT (Kinematics-only-based Data Association) trackers on simple two targets scenarios (with and without clutter). In section 12.3 we remind the main fusion rules we investigate for our new GDA-MTT algorithm. Most of these rules are well-known in the literature [24, 26], but the PCR5, TCN and SAC rules presented here, which are really new ones, were recently proposed in [16, 27, 28, 30]. Due to space limitations, we assume that the reader is familiar with basics on Target Tracking [2–5, 11, 12], on DST [25] and on DSmT [26] for fusion of uncertain, imprecise and possibly highly conflicting information. Section 12.4 presents and compares several Monte Carlo results for different versions of our GDA-MTT algorithm based on the fusion rules proposed in section 12.3 for a particular MTT scenario. Conclusion is given in section 12.5.

12.2 General principle of GDA-MTT

Classical target tracking algorithms consist mainly in two basic steps: *data association* to associate proper measurements (usually kinematics measurement $\mathbf{z}(k)$ representing either position, distance, angle, velocity, acceleration, etc.) with correct targets and *track filtering* to estimate and predict the state of targets once data association has been performed. The first step is very important for the quality of tracking performance since its goal is to associate correctly (or at least as best as possible) observations to existing tracks (or eventually new born targets). The data association problem is very difficult to solve in dense multi-target and cluttered environment. To eliminate unlikely (kinematics-based) observation-to-track pairings, the classical validation test is carried on the Mahalanobis distance $d^2(i,j) \triangleq (\mathbf{z}_j(k) - \hat{\mathbf{z}}_i(k|k-1))'\mathbf{S}^{-1}(k)(\mathbf{z}_j(k) - \hat{\mathbf{z}}_i(k|k-1)) \leq \gamma$ computed from the measurement $\mathbf{z}_j(k)$ and its prediction $\hat{\mathbf{z}}_i(k|k-1)$ computed by the tracker of target i (see [2] for details). Once all the validated measurements have been defined for the surveillance region, a clustering procedure defines the clusters of the tracks with shared observations. Further the decision about observation-to-track associations within the given cluster with n existed tracks and m received measurements is considered. The Converted Measurement Kalman Filter coupled with a classical IMM (Interacting Multiple Model) for maneuvering target tracking is used to update the targets' state vectors.

This new GDA-MTT improves data association process by adding attribute measurements (like amplitude information or RCS (radar cross section)), or eventually as in [6] Target ID decision coupled with confusion matrix, to classical kinematical measurements to increase the performance of the MTT system. When attribute data are available, the generalized (kinematics and attribute) likelihood ratios are used to improve the assignment. The GNN approach is used in order to make a decision for data association. Our new GDA approach consists in choosing a set of assignments $\{\chi_{ij}\}$, for $i = 1, \dots n$ and $j = 1, \dots, m$, that assures maximum of the total generalized likelihood ratio sum by solving the classical assignment problem $\min \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\chi_{ij}$ using the extended Munkres algorithm [13] and where $a_{ij} = -\log(LR_{gen}(i,j))$ with $LR_{gen}(i,j) = LR_k(i,j)LR_a(i,j)$, where $LR_k(i,j)$ and $LR_a(i,j)$ are kinematics and attribute likelihood ratios respectively, and

$$\chi_{ij} = \begin{cases} 1 & \text{if measurement } j \text{ is assigned to track } i \\ 0 & \text{otherwise} \end{cases}$$

and where the elements a_{ij} of the assignment matrix $\mathbf{A} = [a_{ij}]$ take the following values [22]:

$$a_{ij} = \begin{cases} \infty & \text{if } d_{ij}^2 > \gamma \\ -\log(LR_{\mathrm{k}}(i,j)LR_{\mathrm{a}}(i,j)) & \text{if } d_{ij}^2 \leq \gamma \end{cases}$$

The solution of the assignment matrix is the one that minimizes the sum of the chosen elements. We solve the assignment problem by realizing the extension of Munkres algorithm, given in [13]. As a result one obtains the optimal measurements to tracks association. Once the optimal assignment is found, i.e. the (what we feel) correct association is available, then standard tracking filter is used depending on the dynamics of the target under tracking. We will not recall classical tracking filtering methods here which can be found in many standard textbooks [5, 12].

12.2.1 Kinematics Likelihood Ratios for GDA

The kinematics likelihood ratios $LR_{\mathrm{k}}(i,j)$ involved into a_{ij} are quite easy to obtain because they are based on classical statistical models for spatial distribution of false alarms and for correct measurements [5]. $LR_{\mathrm{k}}(i,j)$ is evaluated as $LR_{\mathrm{k}}(i,j) = LF_{\mathrm{true}}(i,j)/LF_{\mathrm{false}}$ where LF_{true} is the likelihood function that the measurement j originated from target (track) i and LF_{false} the likelihood function that the measurement j originated from false alarm. At any given time k, LF_{true} is defined[1] as $LF_{\mathrm{true}} = \sum_{l=1}^{r} \mu_l(k) LF_l(k)$ where r is the number of the models (in our case of two nested models $r = 2$ is used for CMKF-IMM), $\mu_l(k)$ is the probability (weight) of the model l for the scan k, $LF_l(k)$ is the likelihood function that the measurement j is originated from target (track) i according to the model l, i.e. $LF_l(k) = \frac{1}{2\pi\sqrt{|\mathbf{S}_l^i(k)|}} e^{-d_l^2(i,j)/2}$. LF_{false} is defined as $LF_{\mathrm{false}} = P_{fa}/V_c$, where P_{fa} is the false alarm probability and V_c is the resolution cell volume chosen as in [6] as $V_c = \prod_{i=1}^{n_z} \sqrt{12 R_{ii}}$. In our case, $n_z = 2$ is the measurement vector size and R_{ii} are sensor error standard deviations for azimuth β and distance D measurements.

12.2.2 Attribute Likelihood Ratios for GDA

The major difficulty to implement GDA-MTT depends on the correct derivation of coefficients a_{ij}, and more specifically the attribute likelihood ratios $LR_{\mathrm{a}}(i,j)$ for correct association between measurement j and target i based only on attribute information. When attribute data are available and their quality is sufficient, the attribute likelihood ratio helps a lot to improve MTT performance. In our case, the target type information is utilized from RCS attribute measurement through fuzzification interface proposed in [29]. A particular confusion matrix is constructed to model the sensor's classification capability. This work presents different possible issues to evaluate $LR_{\mathrm{a}}(i,j)$ depending on the nature of the attribute information and the fusion rules used to predict and to update each of them. The specific attribute likelihood ratios are derived within both DSmT and DST frameworks.

12.2.2.1 Modeling the Classifier

The way of constructing the confusion matrix is based on some underlying decision-making process based on specific attribute features measurements. In this particular case, it is based on the fuzzification interface, described in our previous work [26, 29]. Through Monte Carlo simulations, the confusion matrix for two different average values of RCS is obtained, in terms of the first frame of hypotheses $\Theta_1 = \{(\mathrm{S})\mathrm{mall}, (\mathrm{B})\mathrm{ig}\}$. Based on the fuzzy rules, described in [29], defining the correspondence between RCS values and the respective targets' types, the final confusion matrix $\mathbf{T} = [t_{ij}]$ in terms of the second frame of hypotheses $\Theta_2 = \{(\mathrm{F})\mathrm{ighter}, (\mathrm{C})\mathrm{argo}\}$ is constructed. Their elements t_{ij} represent the probability to declare that the target type is i when its real type is j. Thus the target's type probability mass vector for classifier output is the j-th column of the confusion matrix $\mathbf{T}$. When false alarms arise, their mass vector consists in an equal distribution of masses among the two classes of targets.

12.2.2.2 Attribute Likelihood Ratio within DSmT

The approach for deriving $LR_{\mathrm{a}}(i,j)$ within DSmT is based on relative variations of pignistic probabilities [26] for the target type hypotheses, H_j ($j = 1$ for Fighter, $j = 2$ for Cargo)

[1] where indexes i and j have been omitted here for LF notation convenience.

included in the frame Θ_2 conditioned by the correct assignment. These pignistic probabilities are derived after the fusion between the generalized basic belief assignments of the track's old attribute state history and the new attribute/ID observation, obtained within the particular fusion rule. It is proven [26] that this approach outperforms most of the well-known ones for attribute data association. It is defined as:

$$\delta_i(P^*) = \frac{\mid \Delta_i(P^*|Z) - \Delta_i(P^*|\hat{Z} = T_i) \mid}{\Delta_i(P^*|\hat{Z} = T_i)} \tag{12.1}$$

where

$$\begin{cases} \Delta_i(P^*|Z) = \sum_{j=1}^{2} \frac{|P^*_{T_i Z}(H_j) - P^*_{T_i}(H_j)|}{P^*_{T_i}(H_j)} \\ \\ \Delta_i(P^*|\hat{Z} = T_i) = \sum_{j=1}^{2} \frac{|P^*_{T_i Z = T_i}(H_j) - P^*_{T_i}(H_j)|}{P^*_{T_i}(H_j)} \end{cases}$$

i.e. $\Delta_i(P^*|\hat{Z} = T_i)$ is obtained by forcing the attribute observation mass vector to be the same as the attribute mass vector of the considered real target, i.e. $m_Z(.) = m_{T_i}(.)$. The decision for the right association relies on the minimum of expression (12.1). Because the generalized likelihood ratio LR_{gen} is looking for the maximum value, we define the final form of the attribute likelihood ratio to be inversely proportional to the $\delta_i(P^*)$ with i defining the number of the track, i.e. $LR_{\mathrm{a}}(i, j) = 1/\delta_i(P^*)$.

12.2.2.3 Attribute Likelihood Ratio within DST

$LR_{\mathrm{a}}(i, j)$ within DST is defined from the derived attribute likelihood function proposed in [3, 12]. If one considers the observation-to-track fusion process using Dempster's rule, the degree of conflict k_{ij} is computed as the assignment of mass committed to the conflict, i.e. $m(\emptyset)$. The larger this assignment is, the less likely is the correctness of observation j to track i assignment. Then, the reasonable choice for the attribute likelihood function is $LHF_{i,j} = 1 - k_{ij}$. The attribute likelihood function for the possibility that a given observation j originated from the false alarm is computed as $LHF_{fa,j} = 1 - k_{fa,j}$. Finally the attribute likelihood ratio to be used in GDA is obtained as $LR_{\mathrm{a}}(i, j) = LHF_{i,j}/LHF_{fa,j}$.

12.3 Fusion rules proposed for GDA-MTT

Imprecise, uncertain and even contradicting information or data are characteristics of the real world situations and must be incorporated into modern MTT systems to provide a complete and accurate model of the monitored problem. On the other hand, the conflict and paradoxes' management in collected knowledge is a major problem especially during the fusion of many information sources. Indeed the conflict increases with the number of sources or with the number of processed scans in MTT. Hence a reliable issue for processing and/or reassigning the conflicting probability masses is required. Such a situation involves also some decision-making procedures based on specific data bases to achieve proper knowledge extraction for a better understanding of the overall monitored problem. It is important and valuable to achieve hierarchical extraction of relevant information and to improve the decision accuracy such that highly accurate decisions can be made progressively. There are many valuable fusion rules in the literature to deal with imperfect information based on different mathematical models and

on different methods for transferring the conflicting mass onto admissible hypotheses of the frame of the problem. DST [24, 25] was the first theory for combining uncertain information expressed as basic belief assignments with Dempster's rule.
Recently, DSmT [26] was developed to overcome the limitations of DST (mainly due to the well-known inconsistency of Dempster's rule for highly conflicting fusion problem and the limitations of the Shafer's model itself) and for combining uncertain, imprecise and possibly highly conflicting sources of information for static or dynamic fusion applications. DSmT is actually a natural extension of DST. The major differences between these two theories is on the nature of the hypotheses of the frame Θ on which are defined the basic belief assignments (bba) $m(.)$, i.e. either on the power set 2^Θ for DST or on the hyper-power set (Dedekind's lattice., i.e. the lattice closed by $\cap$ and $\cup$ set operators) D^Θ for DSmT. Let's consider a frame $\Theta = \{\theta_1, \ldots, \theta_n\}$ of finite number of hypotheses assumed for simplicity to be exhaustive. Let's denote G^Θ the classical power set of Θ (if we assume Shafer's model with all exclusivity constraints between elements of Θ) or denote G^Θ the hyper-power set D^Θ (if we adopt DSmT and we know that some elements can't be refined because of their intrinsic fuzzy and continuous nature). A basic belief assignment $m(.)$ is then defined as $m : G^\Theta \to [0,1]$ with $m(\emptyset) = 0$ and $\sum_{X \in G^\Theta} m(X) = 1$. The differences between DST and DSmT lie in the model of the frame Θ one wants to deal with but also in the rules of combination to apply. Recently in [30] the authors propose to connect the combination rules for information fusion with particular fuzzy operators, focusing on the T-norm based Conjunctive rule as an analog of the ordinary conjunctive rule of combination. It is especially because the conjunctive rule is appropriate for identification problems, restricting the set of hypotheses one is looking for. A new fusion rule, called Symmetric Adaptive Combination (SAC) rule, has been recently proposed in [16] which is an adaptive mixing between the disjunctive and conjunctive rule.

The main fusion rules we have investigated in this work, already presented in details in Chapter 1 of this volume and in [26], are: Dempster's rule, Yager's rule, Dubois & Prade's rule, Hybrid DSm fusion rule, and PCR5 fusion rule. Moreover the two following fusion rules have been also tested and analyzed in this work:

- **T-Conorm-Norm fusion rule**

  The TCN (T-Conorm-Norm) rule represents a new class of combination rules based on specified fuzzy T-Conorm/T-Norm operators. It does not belong to the general Weighted Operator Class. This rule takes its source from the T-norm and T-conorm operators in fuzzy logics, where the AND logic operator corresponds in information fusion to the conjunctive rule and the OR logic operator corresponds to the disjunctive rule. The general principle of the new TCN rule developed in [30] consists in the following steps :

  - **Step 1**: Defining the min T-norm conjunctive consensus: The min T-norm conjunctive consensus is based on the default min T-norm function. The way of association between the focal elements of the given two sources of information is defined as $X = X_i \cap X_j$, and the degree of association is as follows:

$$\tilde{m}(X) = \min \{m_1(X_i), m_2(X_j)\}$$

where $\tilde{m}(X)$ represents[2] the mass of belief associated to the given proposition X by using T-Norm based conjunctive rule. The TCN Combination rule in Dempster Shafer Theory framework is defined for $\forall X \in 2^{\Theta}$ by the equation:

$$\tilde{m}(X) = \sum_{\substack{X_i \cap X_j = X \\ X_i, X_j \in 2^{\Theta}}} \min\{m_1(X_i), m_2(X_j)\} \tag{12.2}$$

– **Step 2**: Distribution of the mass, assigned to the conflict

The distribution of the mass, assigned to the obtained partial conflicts follows in some degree the distribution of conflicting mass in DSmT Proportional Conflict Redistribution Rule 5 [27], but the procedure here is based on fuzzy operators. Let us denote the two bbas, associated with the information sources in a matrix form:

$$\begin{bmatrix} m_1(.) \\ m_2(.) \end{bmatrix} = \begin{bmatrix} m_1(\theta_1) & m_1(\theta_2) & m_1(\theta_1 \cup \theta_2) \\ m_2(\theta_1) & m_2(\theta_2) & m_2(\theta_1 \cup \theta_2) \end{bmatrix}$$

The general procedure for fuzzy based PCR5 conflict redistribution is as follows:

* Calculate all partial conflicting masses separately;
* If $\theta_1 \cap \theta_2 = \emptyset$, then θ_1 and θ_2 are involved in the conflict; redistribute the corresponding masses $m_{12}(\theta_1 \cap \theta_2) > 0$ involved in the particular partial conflicts to the non-empty sets θ_1 and θ_2 with respect to the maximum between $m_1(\theta_1)$ and $m_2(\theta_2)$ and with respect to the maximum between $m_1(\theta_2)$ and $m_2(\theta_1)$;
* Finally, for the given above two sources the min T-Norm conjunctive consensus yields:

$$\tilde{m}(\theta_1) = \min(m_1(\theta_1), m_2(\theta_1)) + \min(m_1(\theta_1), m_2(\theta_1 \cup \theta_2)) + \min(m_1(\theta_1 \cup \theta_2), m_2(\theta_1))$$

$$\tilde{m}(\theta_2) = \min(m_1(\theta_2), m_2(\theta_2)) + \min(m_1(\theta_2), m_2(\theta_1 \cup \theta_2)) + \min(m_1(\theta_1 \cup \theta_2), m_2(\theta_2))$$

$$\tilde{m}(\theta_1 \cup \theta_2) = \min(m_1(\theta_1 \cup \theta_2), m_2(\theta_1 \cup \theta_2))$$

* The basic belief assignment, obtained as a result of the applied TCN rule with fuzzy based Proportional Conflict Redistribution Rule 5 becomes:

$$\tilde{m}_{PCR5}(\theta_1) = \tilde{m}(\theta_1) + m_1(\theta_1) \times \frac{\min(m_1(\theta_1), m_2(\theta_2))}{\max(m_1(\theta_1), m_2(\theta_2))} + m_2(\theta_1) \times \frac{\min(m_1(\theta_2), m_2(\theta_1))}{\max(m_1(\theta_2), m_2(\theta_1))}$$

$$\tilde{m}_{PCR5}(\theta_2) = \tilde{m}(\theta_2) + m_2(\theta_2) \times \frac{\min(m_1(\theta_1), m_2(\theta_2))}{\max(m_1(\theta_1), m_2(\theta_2))} + m_1(\theta_2) \times \frac{\min(m_1(\theta_2), m_2(\theta_1))}{\max(m_1(\theta_2), m_2(\theta_1))}$$

[2]We introduce in this chapter the over-tilded notation for masses to specify that the masses of belief are obtained with fuzzy T-norm operator.

- **Step 3**: Normalization of the result:
  The final step of the TCN rule concerns the normalization procedure:

$$\tilde{m}_{PCR5}(X) = \frac{\tilde{m}_{PCR5}(X)}{\sum_{\substack{X \neq \emptyset \\ X \in 2^{\Theta}}} \tilde{m}_{PCR5}(X)} \; .$$

The nice features of the new rule could be defined as: very easy to implement, satisfying the impact of neutrality of Vacuous Belief Assignment; commutative, convergent to idempotence, reflecting majority opinion, assuring an adequate data processing in case of total conflict.

- **Symmetric Adaptive Combination rule**

The generic adaptive combination rule (ACR) is a mixing between the disjunctive and conjunctive rule and it is defined by $m_{ACR}(A) = 0$ and $\forall A \in 2^{\Theta}$ by:

$$m_{ACR}(A) = \alpha(k_{12}).m_{\cup}(A) + \beta(k_{12}).m_{\cap}(A) \; ,$$

where α and β are functions of the conflict $k_{12} = m_{\cap}(\emptyset)$ from $[0, 1]$ to $[0, +\infty]$. $m_{ACR}(.)$ must be a normalized bba(assuming a closed world) and a desirable behavior of ACR is that it should act more like the disjunctive rule whenever $k_{12} \to 1$ (at least one source is unreliable), while it should act more like the conjunctive rule, when $k_{12} \to 0$ (both sources are reliable). The three following conditions have to be satisfied by the weighting functions α and β:

- **C1**: α is increasing with $\alpha(0) = 0$ and $\alpha(1) = 1$;
- **C2**: β is decreasing with $\beta(0) = 1$ and $\beta(1) = 0$;
- **C3**: $\alpha(k_{12}) = 1 - (1 - k_{12})\beta(k_{12})$.

A symmetric AC (SAC rule) with symmetric weightings for $m_{\cap}(.)$ and $m_{\cup}(.)$ is defined by $m_{SAC}(\emptyset) = 0$ and $\forall A \in 2^{\Theta}$ by:

$$m_{SAC}(A) = \alpha_0(k_{12}).m_{\cup}(A) + \beta_0(k_{12}).m_{\cap}(A) \; ,$$

where

$$\alpha_0(k_{12}) = \frac{k_{12}}{1 - k_{12} + k_{12}^2} \; ;$$

$$\beta_0(k_{12}) = \frac{1 - k_{12}}{1 - k_{12} + k_{12}^2} \; .$$

12.4 Simulation scenario and results

12.4.1 Simulation scenario

The simulation scenario (Fig.12.1) consists of eleven air targets with only two classes. The stationary sensor is located at the origin. The sampling period is $T_{scan} = 5$ sec and measurement standard deviations are 0.3 deg and 100 m for azimuth and range respectively. The targets go from West to East in three groups with the following type order CFCFCFCFCFC (F=Fighter, C=Cargo) with constant velocity $100m/sec$. The first group consists of three targets (CFC) moving from North-West with heading 120 degrees from North. At scan number 15th the group performs a maneuver with transversal acceleration $5.2m/s^2$ and settles towards East, moving in parallel according to X axis. The second group consists of five closely spaced targets (FCFCF) moving in parallel from West to East without maneuvering. The third group consists of three targets (CFC) moving from South-West with heading 60 degrees from North. At scan number 15th the group performs a maneuver with transversal acceleration $-5.2m/s^2$ and settles towards East, moving in parallel according to X axis. The inter-distance between the targets during scans 17th - 48th (the parallel segment) is approximately 300 m. At scan number 48th the first and the third group make new maneuvers. The first one is directed to North-East and the second - to South-East. Process noise standard deviations for the two nested models for constant velocity IMM are $0.1\mathrm{m/s}^2$ and $7\mathrm{m/s}^2$ respectively. The number of false alarms (FA) follows a Poisson distribution and FA are uniformly distributed in the surveillance region.

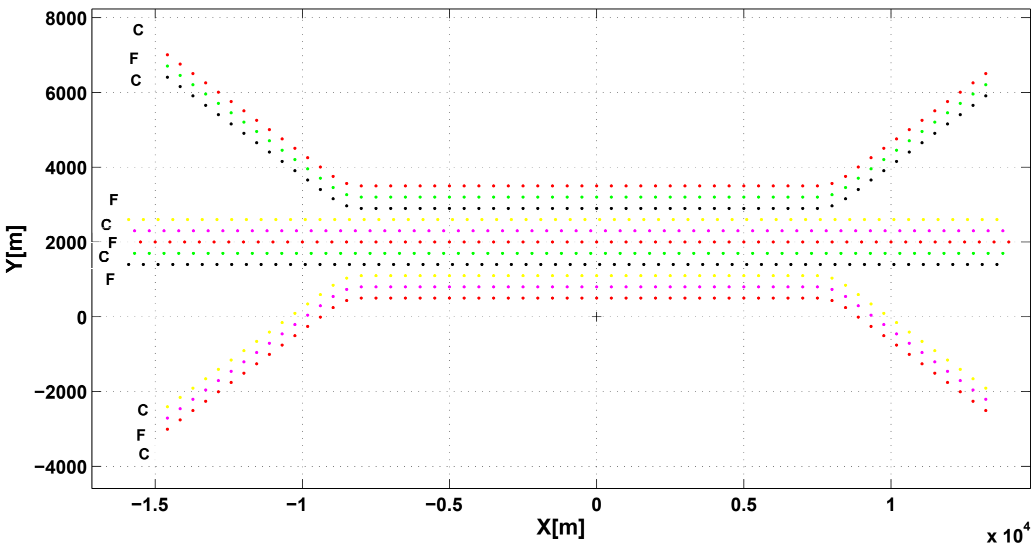

Figure 12.1: Multitarget Scenario with eleven targets

Monte Carlo simulations are made for two different average values of Radar Cross Section in order to obtain the confusion matrix in terms of the first frame of hypotheses $\Theta_1 = \{Small, Big\}$. According to the fuzzy rules in [26, 29], defining the correspondence between Radar Cross Section values and the respective targets' types, the confusion matrix in terms of the second frame of hypotheses $\Theta_2 = \{Fighter, Cargo\}$ is constructed. The two simulation cases correspond to the following parameters for the probability of target detection, the probability of false alarms and the confusion matrices:

- Case 1: $P_d = 1.0$, $P_{fa} = 0.0$, $T_1 = \begin{bmatrix} 0.995 & 0.005 \\ 0.005 & 0.995 \end{bmatrix}$

- Case 2: $P_d = 0.98$, $P_{fa} = 1.e^{-5}$, $T_2 = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$

12.4.2 Simulation results

In this section we present and discuss the simulation results for 100 Monte Carlo runs. The evaluation of fusion rules' performance is based on the criteria of tracks' purity, tracks' life, percentage of miscorrelation and variation of pignistic entropy in confirmed tracks' attribute states. Track's purity criteria examines the ratio between the number of particular performed (observation j-track i) associations (in case of detected target) over the total number of all possible associations during the tracking scenario. Track's life is evaluated as an average number of scans before track's deletion. The track deletion is performed after the a priori defined number (in our case it is assumed to be 3) of incorrect associations or missed detections. The percentage of miscorrelation examines the relative number of incorrect (observation-track) associations during the scans. The results for GDA are obtained by different fusion rules. Relying on our previous work [26, 29], where the performance of DSm Classic and DSm Hybrid rules were examined, in the present work the attention is directed to the well-known Dempster's rule, Yager's, Dubois & Prade's, and especially to PCR5 and the new TCN and SAC rules. From results presented in Tables 12.1-12.4 in next sections, it is obvious that for both cases 1 and 2 the track's purity and tracks' life in the case of KDA-MTT are significantly lower with respect to all GDA-MTT, and a higher percentage of miscorrelation is obtained with KDA-MTT than with GDA-MTT. The figures 12.2 and 12.3 show typical tracking performances for KDA-MTT and GDA-MTT systems.

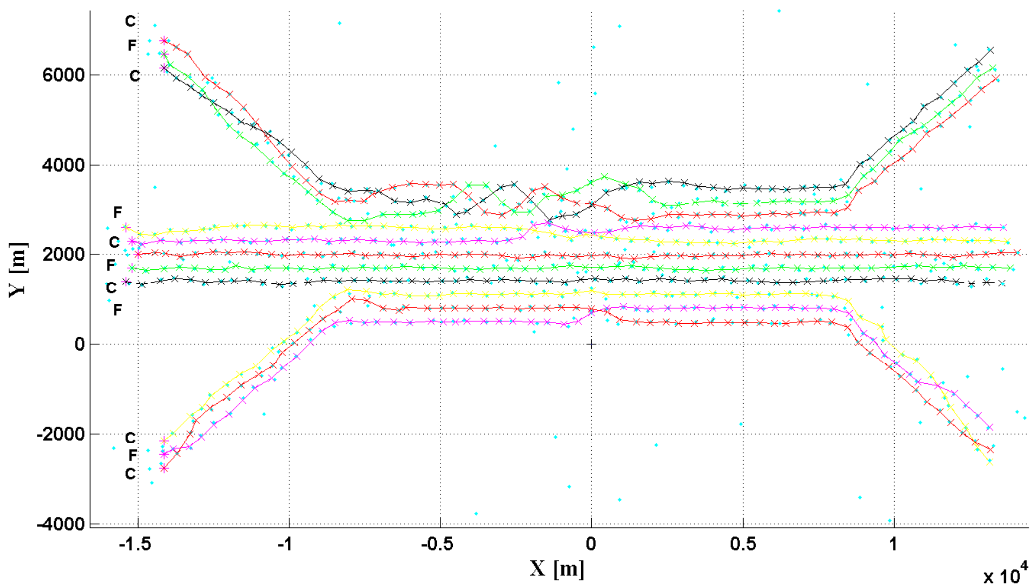

Figure 12.2: Typical performance with KDA-MTT

12.4.2.1 Simulation results for case 1

Case no. 1 is characterized by maximum probability of target detection, $P_d = 1$, probability of false alarms $P_{fa} = 0$, and well defined confusion matrix: $T_1 = \begin{bmatrix} 0.995 & 0.005 \\ 0.005 & 0.995 \end{bmatrix}$. The problem

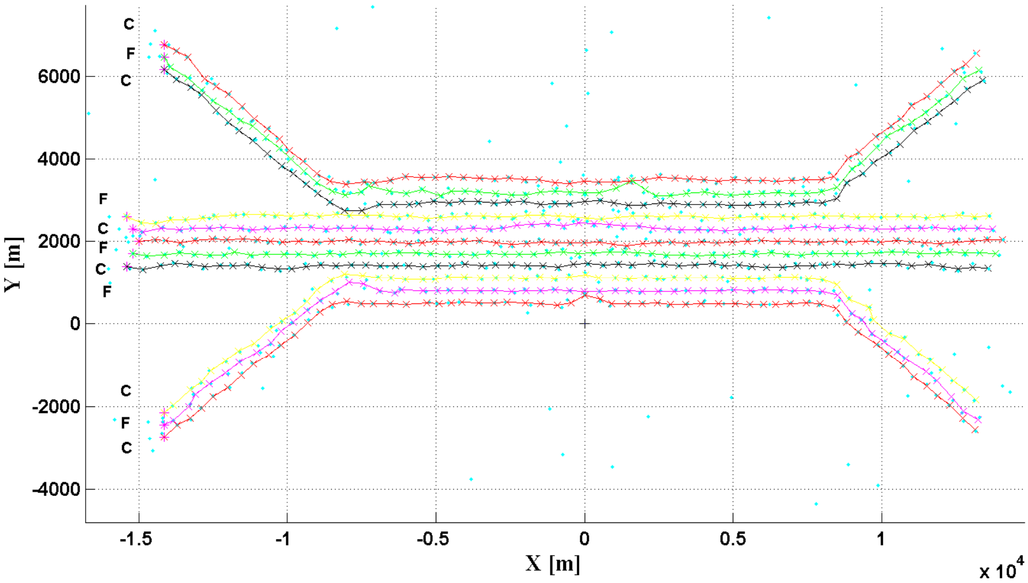

Figure 12.3: Typical performance with GDA-MTT

consists in the proximity of the targets (inter-distance of 300 m) with bad sensor distance resolution ($\sigma_D = 100m$). It results in cross-associations. The Monte Carlo results on track purity based on KDA-MTT and on GDA-MTT (based on PCR5, Dempster's (DS), Yager's (Y),Dubois & Prade's (DP) rule (DP) rules[3], DSmH) rule and the new TCN and SAC fusion rules) are given in Table 12.1. Each number of the table gives the ratio of correct measurement-target association for a given target and a given MTT algorithm and the last row of the table provides the average purity performance for all targets and for each algorithm.
One can see that the corresponding fields for results obtained via Dempster's rule of combination are empty (see Tables 12.1-12.4). There are two major reasons for this:

1. The case of increasing intrinsic conflicts between the fused bodies of evidence (generalized basic belief assignments of targets' tracks histories and new observations), yields a poor targets tracks' performance. The situation when this conflict becomes unity, is a stressful, but a real one. It is the moment, when Dempster's rule produces indefiniteness. The fusion process stops and the attribute histories of tracking tracks cannot be updated. As a result the whole tracking process corrupts. Actually in such a case there is a need of an artificial break and intervention into the real time tracking process, which could cause noncoherent results. Taking into account all these particularities, we can summarize that the fusion process within DST is not fluent and cannot be controlled without prior unjustified and artificial assumptions and some heuristic engineering tricks. As a consequence no one of the performance criteria cannot be evaluated.

2. In case when in the updated track's attribute history one of the hypotheses in the frame of

[3]Yager's rule, Dubois & Prade's (DP) rule, DSmH) rule coincide in our example because we are working with only a 2D specific classes frame Θ_2. This is normal. In general, Yager's, DP and DSmH) do no longer coincide when the cardinality of the frame becomes greater than two.

the problem is supported by unity, from this point on, Dempster's rule becomes indifferent to all observations, detected during the next scans. It means, the track's attribute history remains unchanged regardless of the new observations. It is a dangerous situation, which hides the real opportunity for producing the non-adequate results.

	KDA	PCR5	TCN	SAC	DS	DSmH/Y/DP/
T_1	0.4102	0.9971	0.9971	0.9971	-	0.9971
T_2	0.3707	0.9966	0.9769	0.9955	-	0.9953
T_3	0.4226	0.9990	0.9793	0.9979	-	0.9978
T_4	0.6198	0.9998	0.9703	0.9903	-	0.9903
T_5	0.5826	0.9997	0.9541	0.9902	-	0.9867
T_6	0.5836	1.0000	0.9743	1.0000	-	0.9964
T_7	0.6174	1.0000	0.9500	0.9900	-	1.0000
T_8	0.6774	0.9847	0.9478	0.9671	-	0.9847
T_9	0.4774	0.9426	0.9478	0.8812	-	0.9410
T_{10}	0.4241	0.9543	0.9645	0.7729	-	0.9528
T_{11}	0.4950	0.9595	0.9581	0.8238	-	0.9595
Average	0.5164	0.9848	0.9655	0.9460	-	0.9820

Table 12.1: Track's purity for KDA and GDA-MTT (case 1)

The results of the percentage of track's life duration and miscorrelation are given in Table 12.2.

Trackers	Track Life [%]	MisCor [%]
KDA-MTT	58.15	48.36
GDA$_{PCR5}$-MTT	98.75	1.52
GDA$_{TCN}$-MTT	97.03	3.45
GDA$_{SAC}$-MTT	95.23	5.40
GDA$_{DS}$-MTT	-	-
GDA$_{DSm/Y/DP}$-MTT	98.52	1.80

Table 12.2: Average Track's life and Miscorrelations (case 1)

The figure 12.4 shows the average variation of pignistic entropy in tracks' attribute histories during the scans, obtained by using different fusion rules (PCR5, TCN, SAC and DSmH/Y/DP). Looking on the results achieved according to GDA-MTT, it can be underlined that :

1. The tracks' purity, obtained by PCR5 and DSmH/Y/DP rules outperform the tracks' purity results obtained by using all other rules. In this 2D frame case based on Shafer's model DSmH/Y/DP tracks' purity results are equal which is normal. The TCN rule leads to a small (approximately 2 percent) decrease in GDA performance.

2. According to Table 12.2, the average tracks' life and the percentage of miscorrelation related to the performance of the PCR5 rule are a little bit better than the DSmH/Y/DP, and outperforms all other rules' results (approximately with 2 percent for TCN and with 3 percent for SAC rule).

3. According to the average values of pignistic entropy, associated with updated tracks' attribute histories during the consecutive scans (Fig.12.4), one can see that it is characterized with small values (for all fusion rules), in the interval $[0, 0.05]$. The entropy, obtained

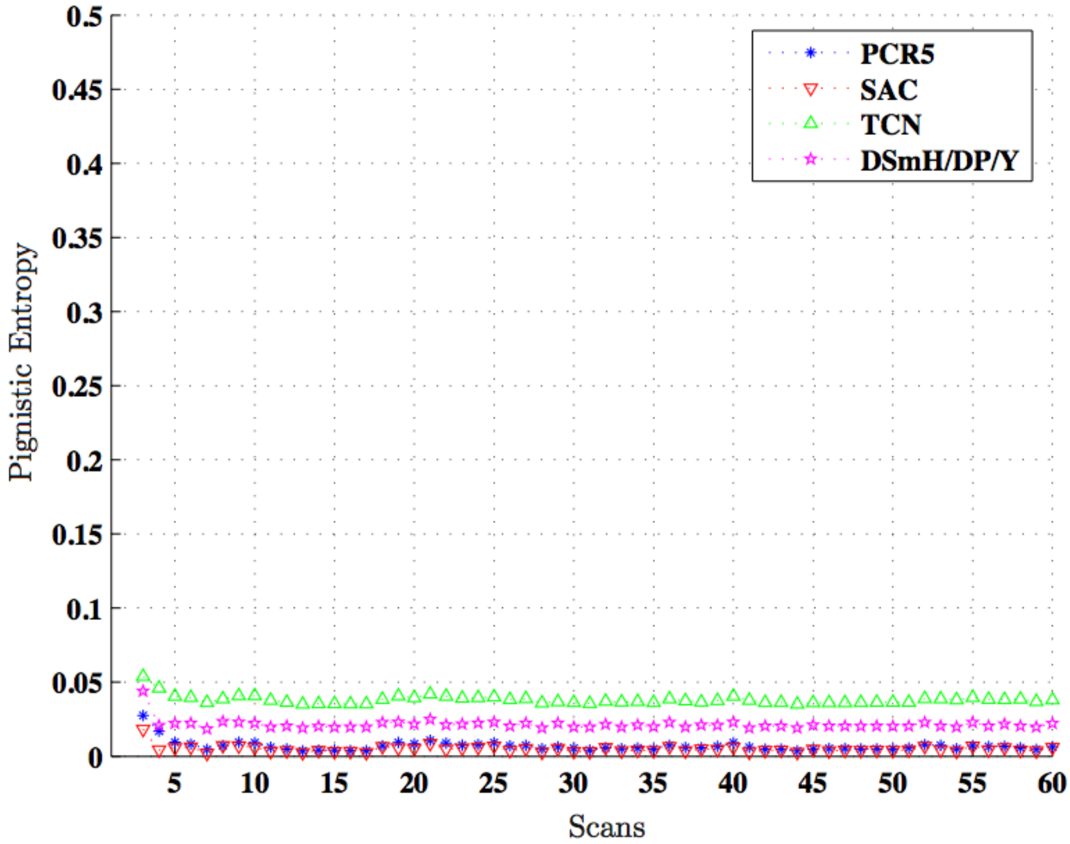

Figure 12.4: Average variation of Pignistic Entropy in tracks' attribute histories

via PCR5 and SAC rules demonstrates smallest values, approaching zero, following by DSmH/Y/DP and TCN fusion rules.

12.4.2.2 Simulation results for case 2

Case no. 2 ($P_d = 0.98$, $P_{fa} = 1.e^{-5}$, $T_2 = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$) is more difficult than case no.1 since the presence of false alarms and missed target detections significantly degrade the process of data association even in the case of GDA. But in comparison with KDA, one can see in Table 12.3 that the use of the attribute type information still helps to reduce the cross-associations and increases the track's purity performance. PCR5 rule behaves stable and keeps its best performance in this difficult case, followed by DSmH/Y/DP, SAC and TCN rules. While in case 1, the TCN performs very well (following PCR5 and DSmH/Y/DP), in case 2 it shows poor tracks' purity results, because of the fuzzy based processing and the confusion matrix's influence. The results of tracks' life duration and miscorrelation are given in Table 12.4.

The figure 12.5 shows the average variation of pignistic entropy in tracks' attribute histories during the scans, obtained by using different fusion rules (PCR5, TCN, SAC and DSmH/Y/DP) in case 2.

	KDA	PCR5	TCN	SAC	DS	DSmH/Y/DP/
T_1	0.3055	0.8138	0.3660	0.3971	-	0.6431
T_2	0.2717	0.7921	0.3219	0.3657	-	0.6252
T_3	0.3338	0.7907	0.3657	0.4024	-	0.6550
T_4	0.5114	0.8778	0.5074	0.6707	-	0.7709
T_5	0.4022	0.8164	0.4071	0.6074	-	0.7209
T_6	0.3712	0.8055	0.4588	0.6298	-	0.6922
T_7	0.4069	0.8098	0.4464	0.6043	-	0.6921
T_8	0.4545	0.8367	0.4974	0.6179	-	0.7359
T_9	0.7436	0.7436	0.4253	0.4886	-	0.6310
T_{10}	0.3040	0.7055	0.3931	0.4397	-	0.6202
T_{11}	0.3697	0.7621	0.4566	0.5086	-	0.6414
Average	0.3742	0.7958	0.4223	0.5211	-	0.6753

Table 12.3: Track's purity for KDA and GDA-MTT (case 2)

Trackers	Track Life [%]	MisCor [%]
KDA-MTT	45.87	62.58
GDA$_{PCR5}$-MTT	84.26	20.42
GDA$_{TCN}$-MTT	50.34	57.77
GDA$_{SAC}$-MTT	59.26	47.89
GDA$_{DS}$-MTT	-	-
GDA$_{DSm/Y/DP}$-MTT	73.29	32.47

Table 12.4: Average Track's life and Miscorrelations (case 2)

The variation of pignistic entropy in updated tracks' attribute histories, based on all fusion rules starts with peaks, because of the full ignorance, encountered in initial tracks' attribute states (initial tracks' histories). During the next 3-4 scans it decreases gradually and settles in the interval $[0.05 - 0.3]$. The pignistic entropies, obtained by PCR5 and SAC rules show smallest values. It means that in this more difficult case 2, PCR5 and SAC rules lead to results which are more informative in comparison with the other rules.

12.5 Conclusions

In this paper a comparison of the performances of different fusion rules is presented and compared in order to assess their efficiency for GDA for MTT in highly conflicting situations in clutter. A model of an attribute type classifier is considered on the base of particular input fuzzification interface according to the target RCS values and on fuzzy rule base according to the target type. A generalized likelihood ratio is obtained and included in the process of GDA. The classification results rely on the confusion matrix specifying the accuracy of the classifier and on the implemented fusion rules (Dempster's, Yager's, Dubois & Prade's, DSmH), PCR5, TCN and SAC). The goal was to examine their advantages and milestones and to improve association results. This work confirms the benefits of attribute utilization and shows some hidden drawbacks, when the sources of information remain in high conflict, especially in case of using Dempster's rule of combination. In clutter-free environment with maximum of target detection probability and very good classifier quality, the results, according to the performance criteria, obtained via PCR5 rule outperform the corresponding results obtained by using all

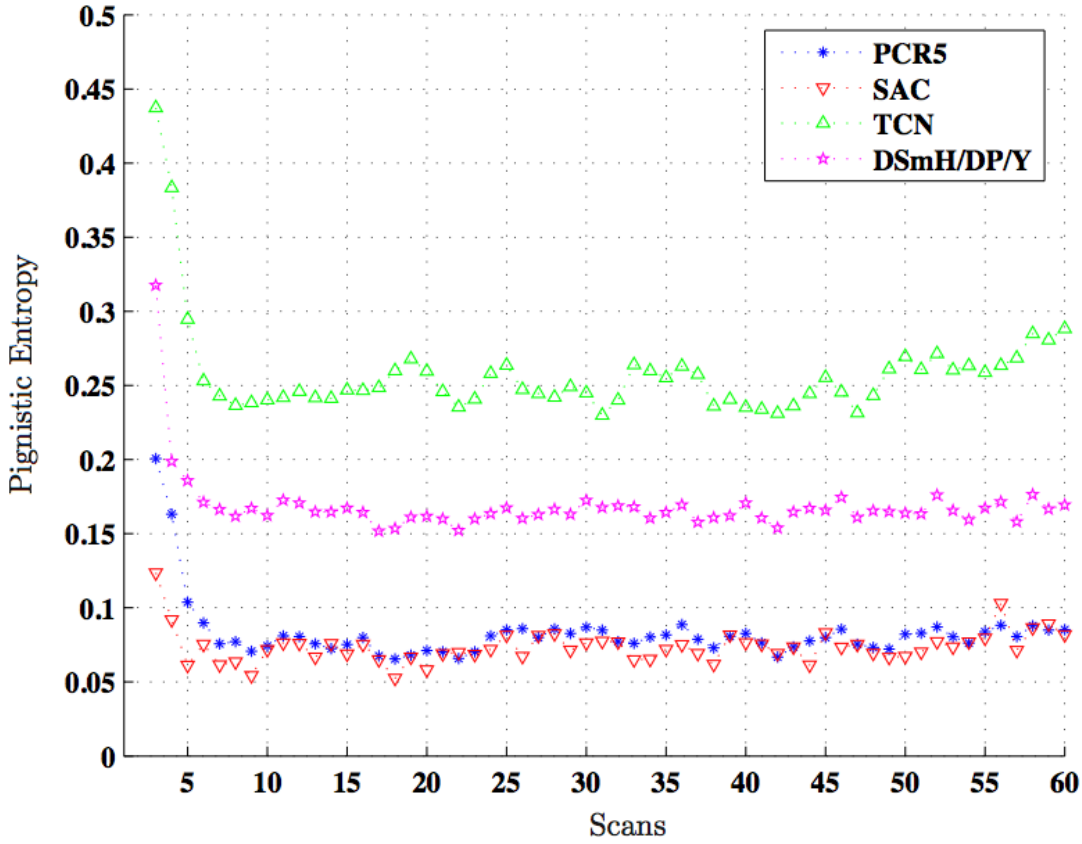

Figure 12.5: Average variation of Pignistic Entropy in tracks' attribute histories

the other combination rules tested. When tracking conditions decrease (presence of clutter, missed target detections with lower classifier quality), the PCR5 fusion rule still provides the best performances with respect to other rules tested for our new GDA-MTT algorithm. This work reveals also the real difficulty to define and to choose an unique or a multiple performance criteria for the fair evaluation of different fusion rules. Actually the choice of the fusion rule is in practice highly conditioned by the performance criteria that the system designer considers as the most important for his application. More efforts on multicriteria-based methods for performance evaluation are under investigations. Further works on GDA-MTT would be to define some precise benchmark for difficult multitarget tracking and classification scenarios and to see if the recent MITM approach (i.e. RMIMM coupled with MAJPDA) can be improved by our new generalized data association method.

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Chapter 13

Target Type Tracking with Different Fusion Rules: A Comparative Analysis

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Abstract: *We analyze the behavior of several combinational rules for temporal/sequential attribute data fusion for target type estimation. Our comparative analysis is based on: Dempster's fusion rule, Proportional Conflict Redistribution rule no. 5 (PCR5), Symmetric Adaptive Combination (SAC) rule and a new fusion rule, based on fuzzy T-conorm and T-norm operators (TCN). We show through very simple scenario and Monte-Carlo simulation, how PCR5, TCN and SAC rules allow a very efficient Target Type Tracking and reduce drastically the latency delay for correct Target Type decision with respect to Demspter's rule. For cases presenting some short Target Type switches, Demspter's rule is proved to be unable to detect the switches and thus to track correctly the Target Type changes. The approach proposed here is totally new, efficient and promising to be incorporated in real-time Generalized Data Association - Multi Target Tracking systems (GDA-MTT). The Matlab source code of simulations is freely available upon request to authors and part of this code can also be found in [5].*

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13.1 Introduction

The main purpose of information fusion is to produce reasonably aggregated, refined and/or complete granule of data obtained from a single or multiple sources with consequent reasoning process, consisting in using evidence to choose the best hypothesis, supported by it. Data Association (DA) with its main goal to partitioning observations into available tracks becomes a key function of any surveillance system. An issue to improve track maintenance performances of modern Multi Target Trackers (MTT) [1, 2], is to incorporate Generalized Data[1] Association (GDA) in tracking algorithms [15]. At each time step, GDA consists in associating current (attribute and kinematics) measurements with predicted measurements (attributes and kinematics) for each target. GDA can be actually decomposed into two parts [15]: Attribute-based Data Association (ADA) and Kinematics-based Data Association (KDA). Once ADA is obtained, the estimation of the attribute/type of each target must be updated using a proper and an efficient fusion rule. This process is called *attribute tracking* and consists in combining information collected over time from one (or more) sensor to refine the knowledge about the possible changes of the attributes of the targets. We consider here the possibility that the attributes tracked by the system can change over time, like the color of a chameleon moving in a variable environment. In some military applications, target attribute can change since for example it can be declared as neutral at a given scan and can become a foe several scans later; or like in the example considered in this chapter, a tracker can become mistaken when tracking several closely-spaced targets and thus could eventually *track* sequentially different targets observing that way a true sequence of different types of targets. In such a case, although the attribute of each target is invariant over time, at the attribute-tracking level the type of the target committed to the (hidden unresolved) track varies with time and must be tracked efficiently to help to discriminate how many different targets are hidden in the same unresolved track. Our motivation for attribute fusion is inspired from the necessity to ascertain the targets' types, information, that in consequence has an important implication for enhancing the tracking performance. Combination rules are special types of aggregation methods. To be useful, one system has to provide a way to capture, analyze and utilize through the fusion process the new available data (evidence) in order to update the current state of knowledge about the problem under consideration.

Dempster-Shafer Theory (DST) [10] is one of widely used frameworks in target tracking when one wants to deal with uncertain information and take into account attribute data and/or human-based information into modern tracking systems. DST, thanks to belief functions, is well suited for representing uncertainty and combining information, especially in case of low conflicts between the sources (bodies of evidence) with high beliefs. When the conflict increases[2] and becomes very high (close to 1), Dempster's rule yields unfortunately unexpected, or what authors feel, counter-intuitive results [11, 17]. Dempster's rule also presents difficulties in its implementation/programming because of unavoidable numerical rounding errors due to the finite precision arithmetic of our computers.
To overcome the drawbacks of Dempster's fusion rule and in the meantime extend the domain of application of the belief functions, we have proposed recently a new mathematical framework, called Dezert-Smarandache Theory (DSmT) with a new set of combination rules, among them the Proportional Conflict Redistribution no. 5 which proposes a sophisticated and efficient so-

[1]Data being kinematics and attribute.
[2]Which often occurs in Target Type Tracking problem as it will be shown in the sequel.

lution for information fusion as it will be shown further. The basic idea of DSmT is to work on Dedekind's lattice (called Hyper-Power Set) rather than on the classical power set of the frame as proposed in DST and, when needed, DSmT can also take into account the integrity constraints on elements of the frame, constraints which can also sometimes change over time with new knowledge. Hence DSmT deals with uncertain, imprecise and high conflicting information for static and dynamic fusion as well [3, 4, 11].

Recently in [16] the authors propose to connect the combination rules for information fusion with particular fuzzy operators. These rules take their source from the T-norm and T-conorm operators in fuzzy logics, where the AND logic operator corresponds in information fusion to the conjunctive rule and the OR logic operator corresponds to the disjunctive rule. While the logic operators deal with degrees of truth and false, the fusion rules deal with degrees of belief of hypotheses. In [16] the focus is on the T-norm based Conjunctive rule as an analog of the ordinary conjunctive rule of combination. It is appropriate for identification problems, restricting the set of hypotheses one is looking for. A new fusion rule, called Symmetric Adaptive Combination (SAC) rule, has been recently proposed in [7] which is an adaptive mixing between the disjunctive and conjunctive rule. This rule acts more like the disjunctive rule whenever at least one source is unreliable, while it acts more like the conjunctive rule, when both sources are reliable. In the next section we present briefly the basics of DST and DSmT. In section 13.3, we present the Target Type Tracking problem and examine four solutions to solve it; the first solution being based on Dempster's rule and the next ones based on PCR5, TCN and SAC rules. In section 13.4, we evaluate all the solutions on a very simple academic but checkable[3] example and provide a comparative analysis on Target Type Tracking performances obtained by Dempster's, PCR5, TCN and SAC rules. Concluding remarks are given in section 13.5.

13.2 Fusion Rules proposed for Target Type Tracking

13.2.1 Basics on DST and DSmT

Shafer's model, denoted here $\mathcal{M}^0(\Theta)$, in DST [10] considers $\Theta = \{\theta_1, \dots, \theta_n\}$ as a finite set of n exhaustive and exclusive elements representing the possible states of the world, i.e. solutions of the problem under consideration. Θ is called the *frame of discernment* by Shafer. In DSmT framework [11], one starts with the free DSm model $\mathcal{M}^f(\Theta)$ where $\Theta = \{\theta_1, \dots, \theta_n\}$ (called simply frame) is only assumed to be a finite set of n exhaustive elements[4]. If one includes some integrity constraints in $\mathcal{M}^f(\Theta)$, say by considering θ_1 and θ_2 truly exclusive (i.e. $\theta_1 \cap \theta_2 = \emptyset$), then the model is said *hybrid*. When we include all exclusivity constraints on elements of Θ, $\mathcal{M}^f(\Theta)$ reduces to Shafer's model $\mathcal{M}^0(\Theta)$ which can be viewed actually as a particular case of DSm hybrid model. Between the free-DSm model and the Shafer's model, there exists a wide class of fusion problems represented in term of DSm hybrid models where Θ involves both fuzzy continuous hypothesis and discrete hypothesis.

Based on Θ and Shafer's model, the *power set* of Θ, denoted 2^Θ, is defined as follows:

[3] Our Matlab source code is available upon request to help the reader to check by him/herself the validity of our results. Part of this code can be also found [5].

[4] The exclusivity assumption is not fundamental in DSmT because one wants to deal with elements which cannot be refined into precise finer exclusive elements - see [11] for discussion.

1) $\emptyset, \theta_1, \ldots, \theta_n \in 2^\Theta$.

2) If $X, Y \in 2^\Theta$, then $X \cup Y$ belong to 2^Θ.

3) No other elements belong to 2^Θ, except those obtained by using rules 1) or 2).

In DSmT and without additional assumption on Θ but the exhaustivity of its elements (which is not a crucial assumption), we define the *hyper-power set*, i.e. Dedekind's lattice, D^Θ as follows:

1') $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.

2') If $X, Y \in D^\Theta$, then $X \cap Y$ and $X \cup Y$ belong to D^Θ.

3') No other elements belong to D^Θ, except those obtained by using rules 1') or 2').

When Shafer's model $\mathcal{M}^0(\Theta)$ holds, D^Θ reduces to the classical power set 2^Θ. Without loss of generality, we denotes G^Θ the general set on which will be defined the basic belief assignments (or masses), i.e. $G^\Theta = 2^\Theta$ if Shafer's model is adopted whereas $G^\Theta = D^\Theta$ if some other (free or hybrid) DSm models are preferred depending on the nature of the problem.

From a frame Θ, we define a (general) basic belief assignment (bba) as a mapping $m(.) : G^\Theta \to [0, 1]$ associated to a given source, say s, of evidence as

$$m_s(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in G^\Theta} m_s(X) = 1 \tag{13.1}$$

$m_s(X)$ is the gbba of X committed by the source s. The elements of G having a strictly positive mass are called *focal elements* of source s. The set $\mathcal{F}$ of all focal elements is the *core* (or *kernel*) of the belief function of the source s.

The belief and plausibility of any proposition $X \in G^\Theta$ are defined[5] as:

$$\text{Bel}(X) \triangleq \sum_{\substack{Y \subseteq X \\ Y \in G^\Theta}} m(Y) \text{ and } \text{Pl}(X) \triangleq \sum_{\substack{Y \cap X \neq \emptyset \\ Y \in G^\Theta}} m(Y) \tag{13.2}$$

These definitions remain compatible with the classical Bel(.) and Pl(.) functions proposed by Shafer in [10] whenever Shafer's model is adopted for the problem under consideration since G^Θ reduces to 2^Θ.

13.2.2 Fusion rules

A wide variety of rules exists for combining basic belief assignments [9, 12, 14] and the purpose of this chapter is not to browse in details all fusion rules but only to analyze and compare the main ruels used with DST and DSmT approaches (Dempster's, PCR5, SAC rules) and the TCN fusion rule. Since these rules have already been presented in details in chapters 1 and 12, they will not be repeated in this chapter. Our main goal is to show their performance on a very simple Target Type Tracking example.

[5]The index of the source has been omitted for simplicity.

13.3 The Target Type Tracking Problem

13.3.1 Formulation of the problem

The Target Type Tracking Problem can be simply stated as follows:

- Let $k = 1, 2, ..., k_{max}$ be the time index and consider M possible target types $T_i \in \Theta = \{\theta_1, \dots, \theta_M\}$ in the environment; for example $\Theta = \{Fighter, Cargo\}$ and $T_1 \triangleq Fighter$, $T_2 \triangleq Cargo$; or $\Theta = \{Friend, Foe, Neutral\}$, etc.

- at each instant k, a target of true type $T(k) \in \Theta$ (not necessarily the same target) is observed by an attribute-sensor (we assume a perfect target detection probability here).

- the attribute measurement of the sensor (say noisy Radar Cross Section for example) is then processed through a classifier which provides a decision $T_d(k)$ on the type of the observed target at each instant k.

- The sensor is in general not totally reliable and is characterized by a $M \times M$ confusion matrix
$$\mathbf{C} = [c_{ij} = P(T_d = T_j | \text{True Target Type} = T_i)]$$

Question: How to estimate $T(k)$ from the sequence of declarations obtained from the unreliable classifier up to time k, i.e. how to build an estimator $\hat{T}(k) = f(T_d(1), \dots, T_d(k))$ of $T(k)$?

13.3.2 Proposed issues

We propose in this work four methods for solving the Target Type Tracking Problem. All methods assume the same Shafer's model for the frame of Target Types Θ and also use the same information (vacuous belief assignment as prior belief and same sequence of *measurements*, i.e. same set of classifier declarations to get a fair comparative analysis). Three of proposed issues are based on the ordinary combination of belief functions and the fourth - on a new class of fusion rules, based on particular fuzzy operations.

The principle of our estimators is based on the sequential combination of the current basic belief assignment (drawn from classifier decision, i.e. our *measurements*) with the prior bba estimated up to current time from all past classifier declarations. In the first approach, the Demspter's rule is used for estimating the current Target type, while in the next three approaches we use PCR5, TCN and SAC rules.

Here is how our Target Type Tracker (TTT) works:

a) Initialization step (i.e. $k = 0$). Select the target type frame $\Theta = \{\theta_1, \dots, \theta_M\}$ and set the prior bba $m^-(.)$ as vacuous belief assignment, i.e $m^-(\theta_1 \cup \dots \cup \theta_M) = 1$ since one has no information about the first target type that will be observed.

b) Generation of the current bba $m_{obs}(.)$ from the current classifier declaration $T_d(k)$ based on attribute measurement. At this step, one takes $m_{obs}(T_d(k)) = c_{T_d(k)T_d(k)}$ and all the unassigned mass $1 - m_{obs}(T_d(k))$ is then committed to total ignorance $\theta_1 \cup \dots \cup \theta_M$.

c) Combination of current bba $m_{obs}(.)$ with prior bba $m^{-}(.)$ to get the estimation of the current bba $m(.)$. Symbolically we will write the generic fusion operator as $\oplus$, so that $m(.) = [m_{obs} \oplus m^{-}](.) = [m^{-} \oplus m_{obs}](.)$. The combination $\oplus$ is done according either Demspter's rule (i.e. $m(.) = m_D(.)$) or PCR5, SAC and TCN rules (i.e. $m(.) = m_{PCR5}(.)$, $m(.) = m_{SACR}(.)$ and $\tilde{m}(.) = m_{TCN}(.)$).

d) Estimation of True Target Type is obtained from $m(.)$ by taking the singleton of Θ, i.e. a Target Type, having the maximum of belief (or eventually the maximum Pignistic Probability[6] [11]).

e) set $m^{-}(.) = m(.)$; do $k = k + 1$ and go back to step b).

13.4 Simulation results

In order to evaluate the performances of all considered estimators and to have a fair comparative analysis of all fusion rules (Dempster's, PCR5, TCN and SAC), we did a set of Monte-Carlo simulations on a very simple scenario for a 2D Target Type frame, i.e. $\Theta = \{(F)ighter, (C)argo\}$ for two classifiers, a good one C_1 and a poor one C_2 corresponding to the following confusion matrices:

$$\mathbf{C}_1 = \begin{bmatrix} 0.995 & 0.005 \\ 0.005 & 0.995 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_2 = \begin{bmatrix} 0.65 & 0.35 \\ 0.35 & 0.65 \end{bmatrix}$$

In our scenario we consider that there are two closely-spaced targets: one Cargo (C) and one Fighter(F). Due to circumstances, attribute measurements received are predominately from one or another, and both target generates actually one single (unresolved kinematics) track. In the real world, the tracking system should in this case maintain two separate tracks: one for cargo and one for fighter, and based on the classification, allocate the measurement to the proper track. But in difficult scenario like this one, there is no way in advance to know the true number of targets because they are unresolved and that's why only a single track is maintained. Of course, the single track can further be split into two separate tracks as soon as two different targets are declared based on the attribute tracking. This is not the purpose of our work however since we only want to examine how work PCR5, TCN, SAC and Dempster's rules for Target Type Tracking. To simulate such scenario, a true Target Type sequence (the groundtruth) over 100 scans was generated according figures 13.1, 13.2, 13.3 and 13.4 below. The sequence starts with the observation of a Cargo Type (i.e. we call it Type 2) and then the observation of the Target Type switches onto Fighter Type (we call it Type 1) with different time step $T[scans]$ as follows: (Fig.13.1 - T has a variable number of scans, Fig.13.2 - $T = 10$ scans, Fig.13.3 - $T = 5$ scans and Fig.13.4 - $T = 3$ scans). Our goal is to investigate what is the behavior of different fusion rules in case of variable switches' time step and also in cases of equal switches' time step, when target type changes appear to be more frequent, or in other words, to test until which point the proposed fusion rules are able to detect and to adapt to the occurring type's changes. As a simple analogy, tracking the target type changes committed to the same (hidden unresolved) track can be interpreted as tracking color changes of a chameleon moving in a tree on its leaves and on its trunk.

[6] We don't provide here the results based on Pignistic Probabilities since in our simulations the conclusions are unchanged when working with max. of belief or max. of Pign. Proba.

Our simulation consists of 1000 Monte-Carlo runs and we compute and show in the sequel the averaged performances of the four fusion rules. At each time step k the decision $T_d(k)$ is randomly generated according to the corresponding row of the confusion matrix of the classifier given the true Target Type (known in simulations). Then the algorithm presented in the previous section is applied. The complete Matlab source code of our simulation is freely available upon request to authors.

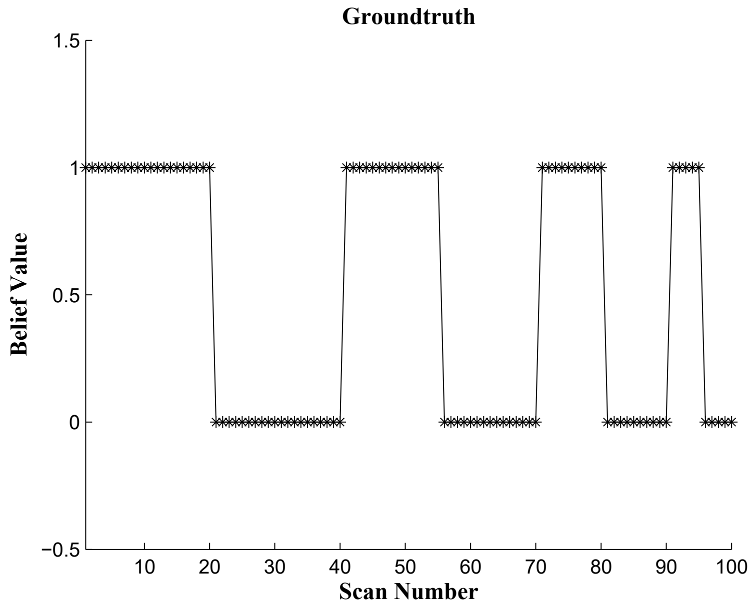

Figure 13.1: Sequence of True Target Type, T-variable number of scans

13.4.1 Results for classifier 1

Figures 13.5 - 13.8 show the belief masses, committed to Cargo type, obtained by our Target Type Trackers based on Demspter's rule (red curves -x-), PCR5 rule (blue curves -pentagram-), TCN rule (green curves -diamond-), SAC rule (magenta curves -o-). Figures 13.9 - 13.12 show the belief masses, committed to Fighter type. The investigations are for periods of target type switches respectively: figures 13.5 and 13.9 for T-variable time step; figures 13.6 and 13.10 for $T = 10$ scans; figures 13.7 and 13.11 for $T = 5$ scans; figures 13.8 and 13.12 for $T = 3$ scans. The target type classifier is $C1$.

It can be seen that the TTT based on Dempster's rule and for a very good classifier is unable to track properly the quick changes of target type. This phenomenon is due to the too long integration time necessary to the Demspter's rule for recovering the true belief estimation.

Demspter's rule presents a very long latency delay (about 8 scans in case of $T = 10$ scans) as we can see during the first type switch when almost all the basic belief mass is committed onto only one element of the frame. This rule does not provide a symmetric target type estimation - it is evident that graphics representing the estimated probability masses before and after the switching points are not settled in interval around the expected average value of mass

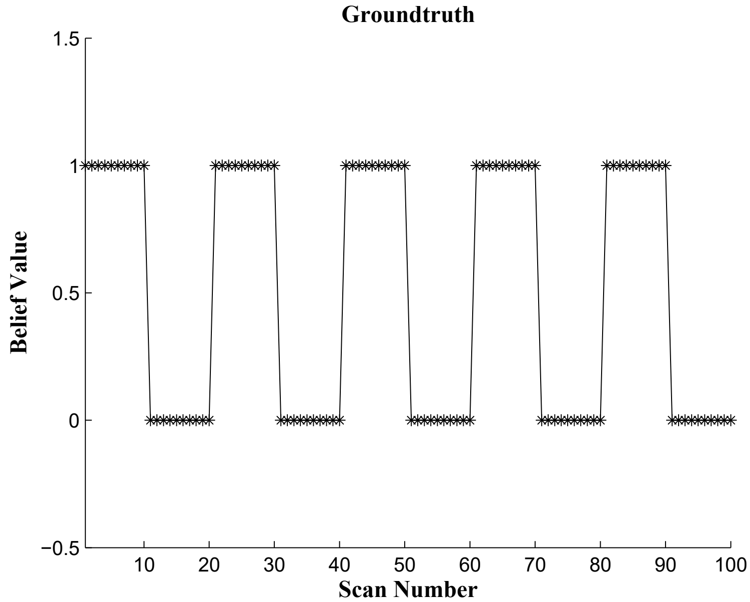

Figure 13.2: Sequence of True Target Type, $T = 10$ scans

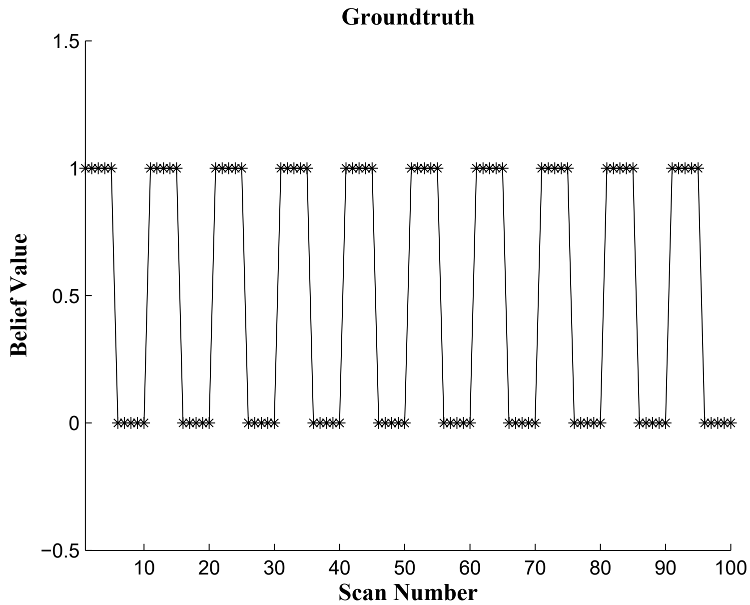

Figure 13.3: Sequence of True Target Type, $T = 5$ scans

$m(C) = 0.5$. In this case of very good target type classifier SAC rule, followed by PCR5 and TCN rules can quickly detect the type changes. They properly re-estimate the belief masses, providing a symmetric type estimation contrariwise to Dempster's rule. So in this configuration the TTT based on Demspter's rule works almost blindly since it is unable to detect the fighter in most of scans where the true target type is a Fighter.

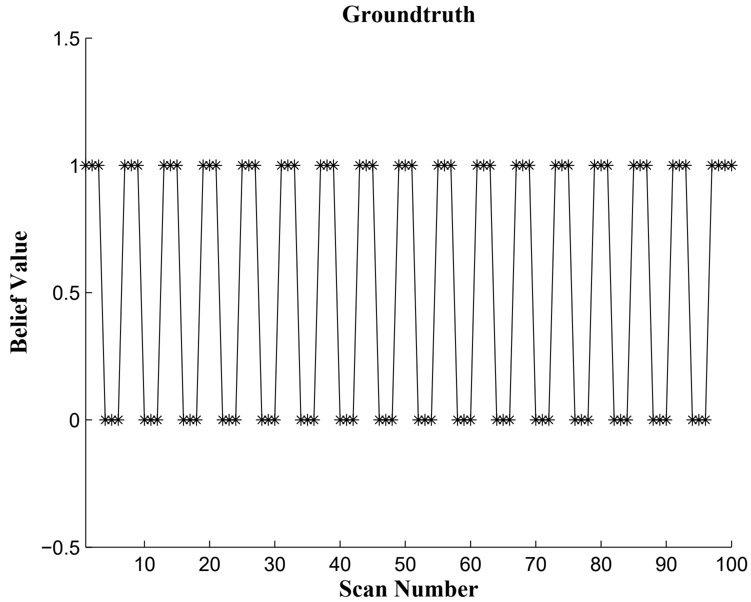

Figure 13.4: Sequence of True Target Type, $T = 3$ scans

Figures 13.5-13.12 show clearly the efficiency of PCR5, SAC and TCN rules with respect to Demspter's rule. Comparing the results obtained for T with variable time step, $T = 10 scans$, $T = 5 scans$ and $T = 3 scans$, one can make the conclusion, that the processes of reacting and adapting to the type changes for PCR5, TCN and SAC rules do not depend on the duration of switching interval. Their behavior is quite stable and effective.

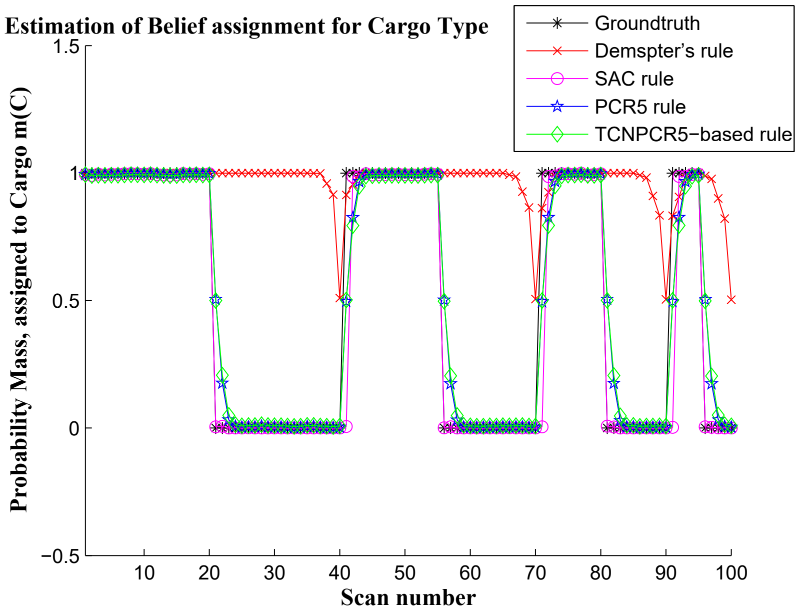

Figure 13.5: Belief mass for Cargo Type,T-variable step, case 1

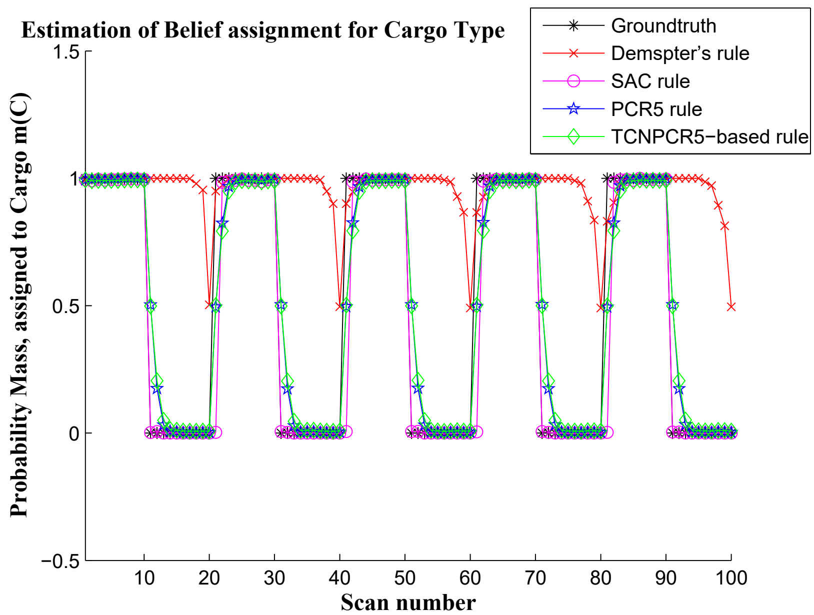

Figure 13.6: Belief mass for Cargo Type, $T = 10$ scans, case 1

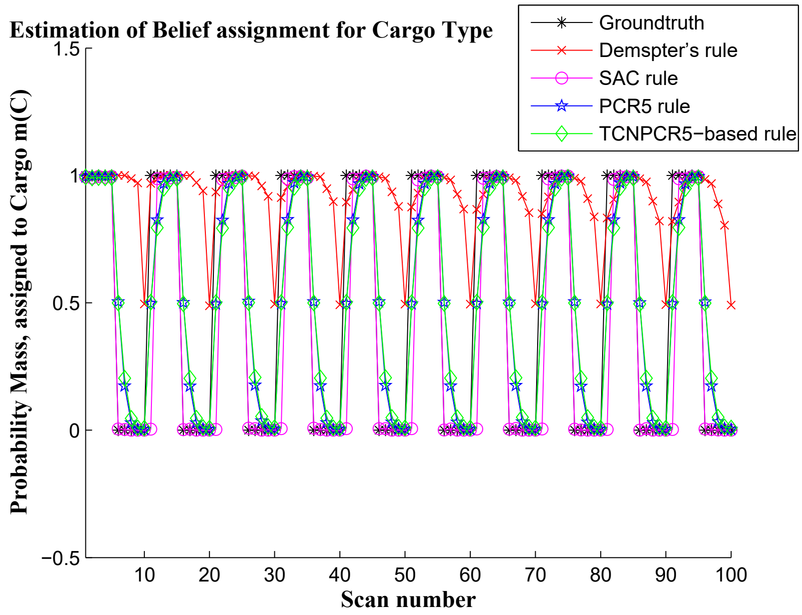

Figure 13.7: Belief mass for Cargo Type, $T = 5$ scans, case 1

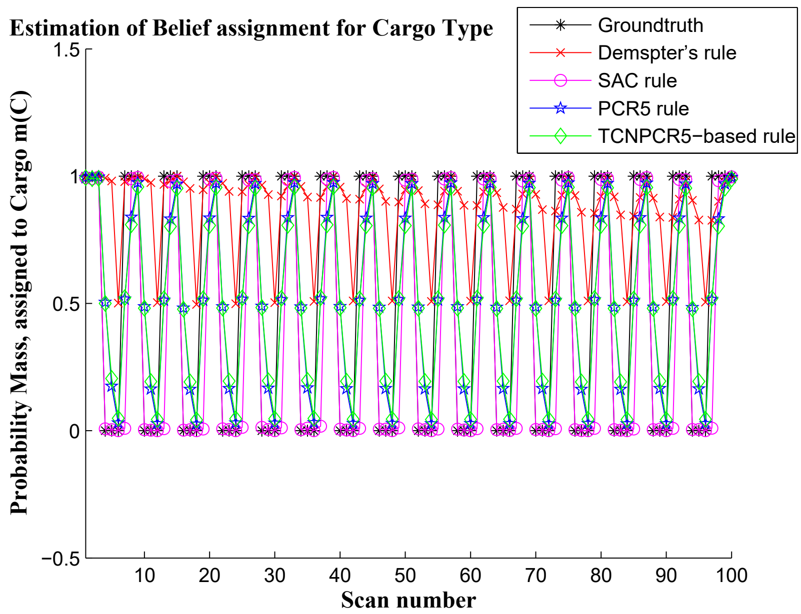

Figure 13.8: Belief mass for Cargo Type, $T = 3$ scans, case 1

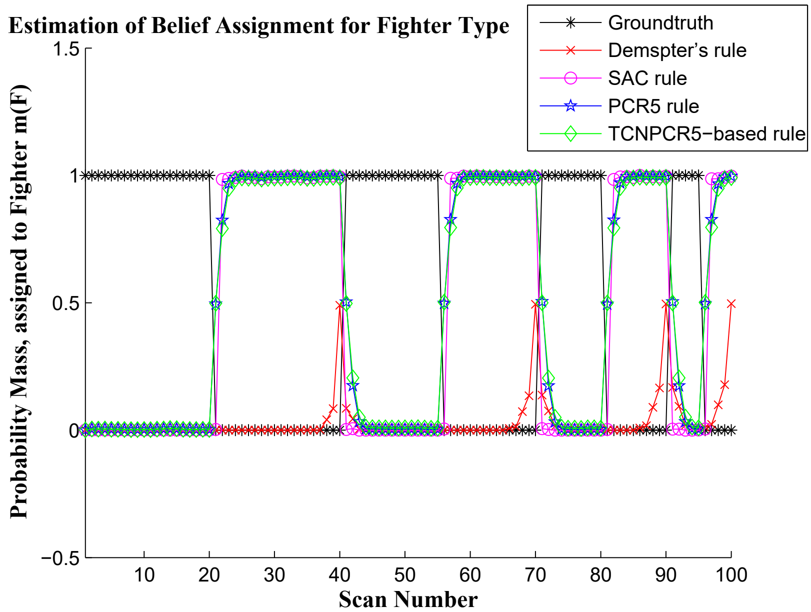

Figure 13.9: Belief mass for Fighter Type,T-variable step, case 1

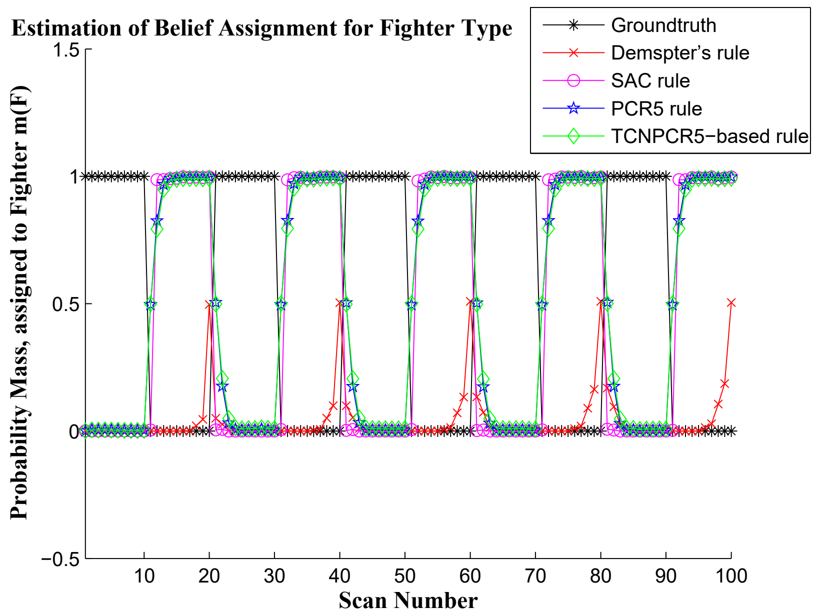

Figure 13.10: Belief mass for Fighter Type, $T = 10$ scans, case 1

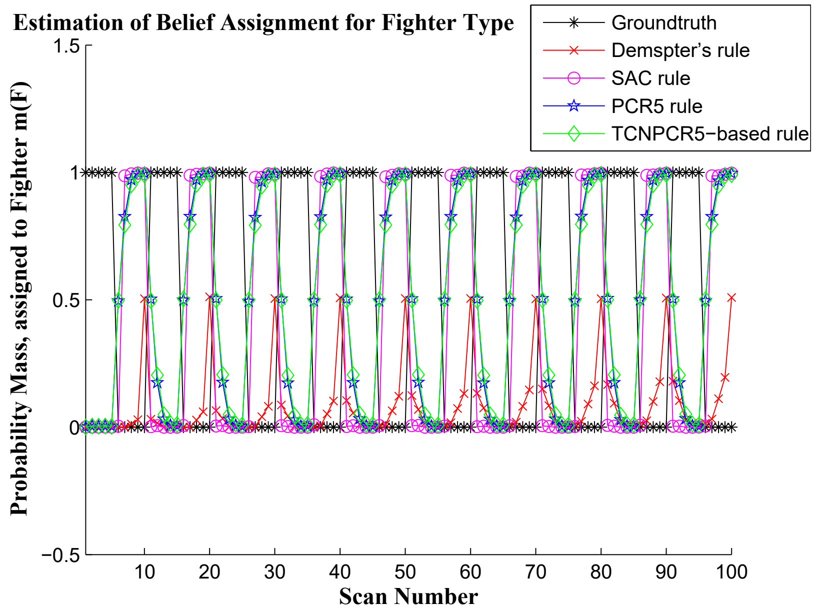

Figure 13.11: Belief mass for Fighter Type, $T = 5$ scans, case 1

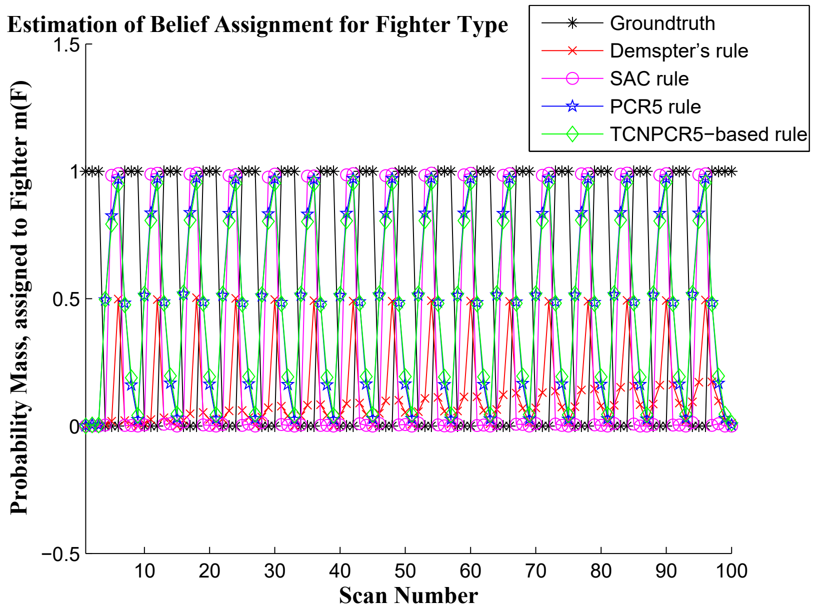

Figure 13.12: Belief mass for Fighter Type, $T = 3$ scans, case 1

13.4.2 Results for classifier 2

Figures 13.13 - 13.16 show the belief masses, committed to Cargo type, obtained by our Target Type Trackers based on Demspter's rule (red curves -x-), PCR5 rule (blue curves -pentagram-), TCN rule (green curves -diamond-), SAC rule (magenta curves -o-). Figures 13.17 - 13.20 show the belief masses, committed to Fighter type. The investigations are for periods of target type switches respectively: figures 13.13 and 13.17 - for T with variable time step, figures 13.14 and 13.18 - for $T = 10$ scans, figures 13.15 and 13.19 - for $T = 5$ scans, figures 13.16 and 13.20 - for $T = 3$ scans. The target type classifier is $C2$.

Paradoxically, we can observe that Demspter's rule seems to work better with a poor classifier than with a good one, because we can see from the red curves that Dempster's rule in that case produces small change detection peaks (with always an important latency delay although). This phenomenon is actually not so surprising and comes from the fact that the belief mass of the true type has not well been estimated by Dempster's rule (since the mass is not so close to its extreme value) and thus the bad estimation of Target Type facilitates the ability of Dempster's rule to react to new incoming information and detect changes. An asymetric Target type estimation is detected as in the case of a very good classifier. When from Demspter's rule, one obtains an over-confidence onto only one focal element of the power-set, it then becomes very difficult for the Dempster's rule to readapt automatically, efficiently and quickly to any changes of the state of the nature which varies with the time and this behavior is very easy to check either analytically or through simple simulations. The major reason for this unsatisfactory behavior of Dempster's rule can be explained with its main weakness: counterintuitive averaging of strongly biased evidence, which in the case of poor classifier is not valid.

What is important according to the performances of PCR5, TCN and SAC rule is that in this case of the poor classifier PCR5 provides the best adaptation to the type changes and quick re-estimation of probability mass, assigned to corresponding target type. It is followed by TCN rule. Both of the rules (PCR5 and TCN) provide a symmetric type estimation in term of probability mass. In the same time SAC rule reacts more slowly than PCR5 and TCN and demonstrates the bad behavior of Dempster's rule, providing an asymetric target type estimation. The process of reacting and adapting to the type changes for PCR5, TCN and SAC rules do not depend on the duration of switching interval even in the case of considered poor classifier.

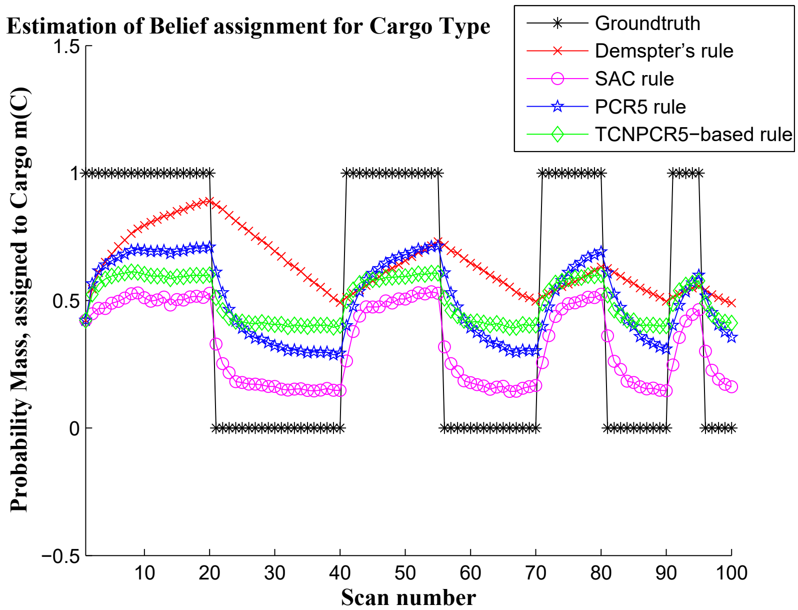

Figure 13.13: Belief mass for Cargo Type, T-variable step, case 2

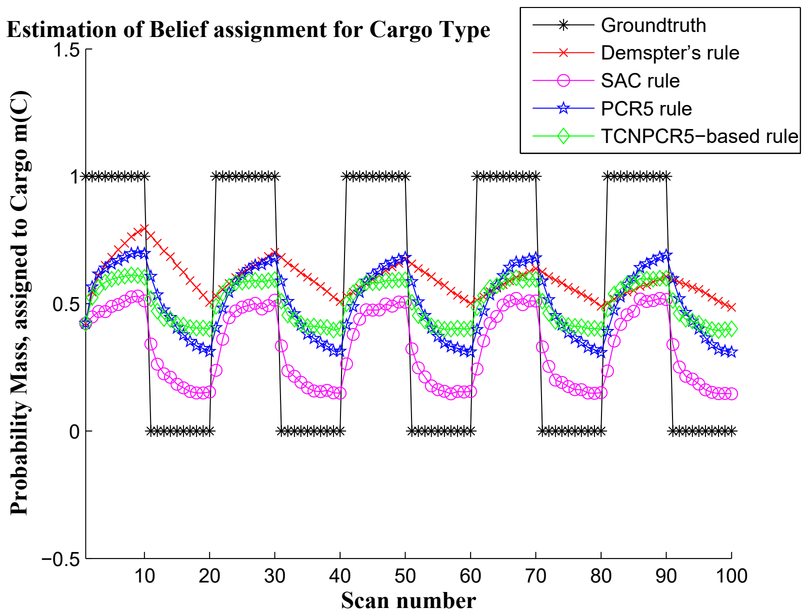

Figure 13.14: Belief mass for Cargo Type, $T = 10$ scans, case 2

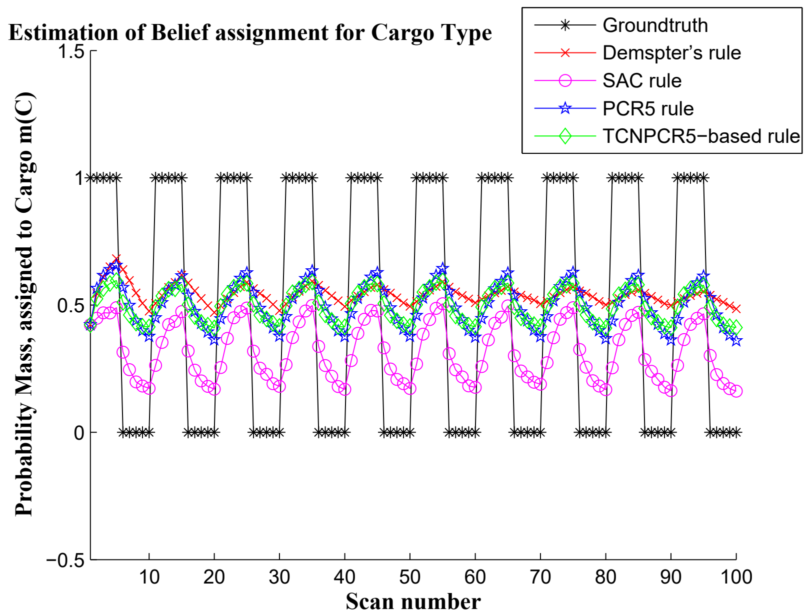

Figure 13.15: Belief mass for Cargo Type, $T = 5$ scans, case 2

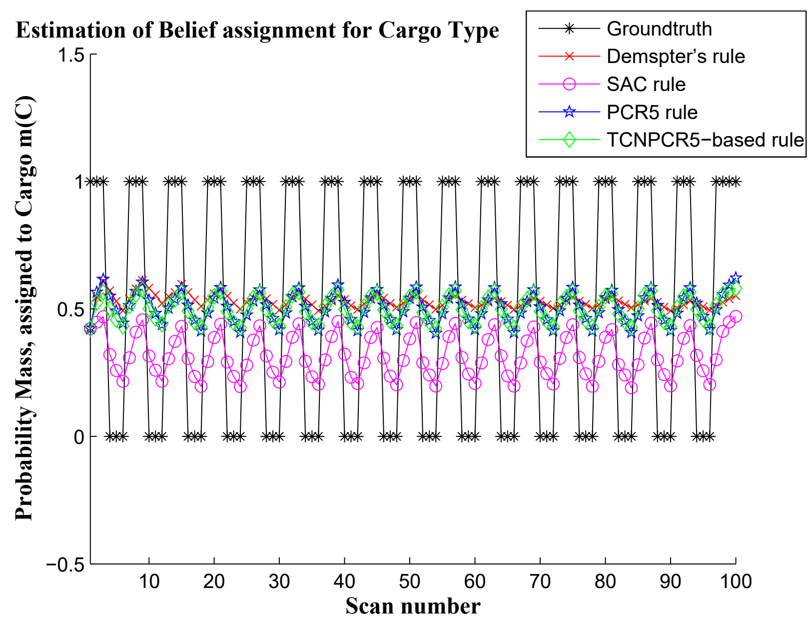

Figure 13.16: Belief mass for Cargo Type, $T = 3$ scans, case 2

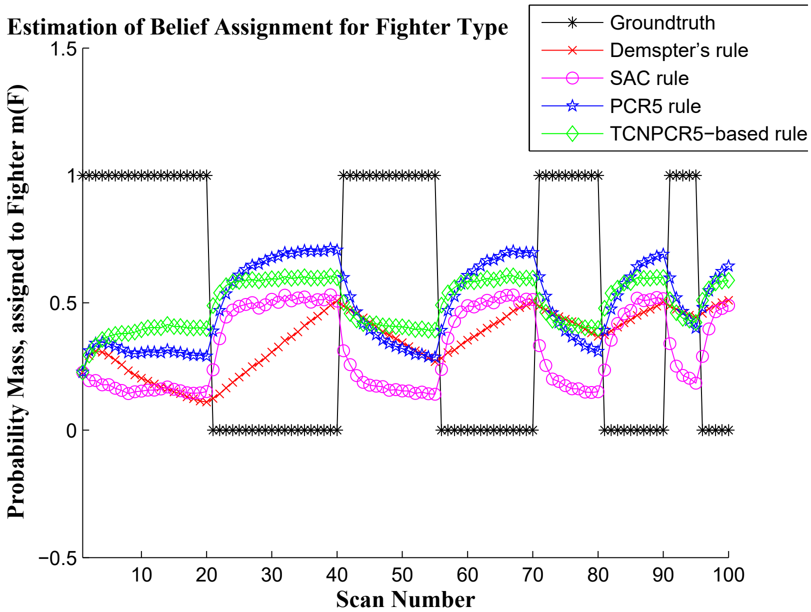

Figure 13.17: Belief mass for Fighter Type, variable step, case 2

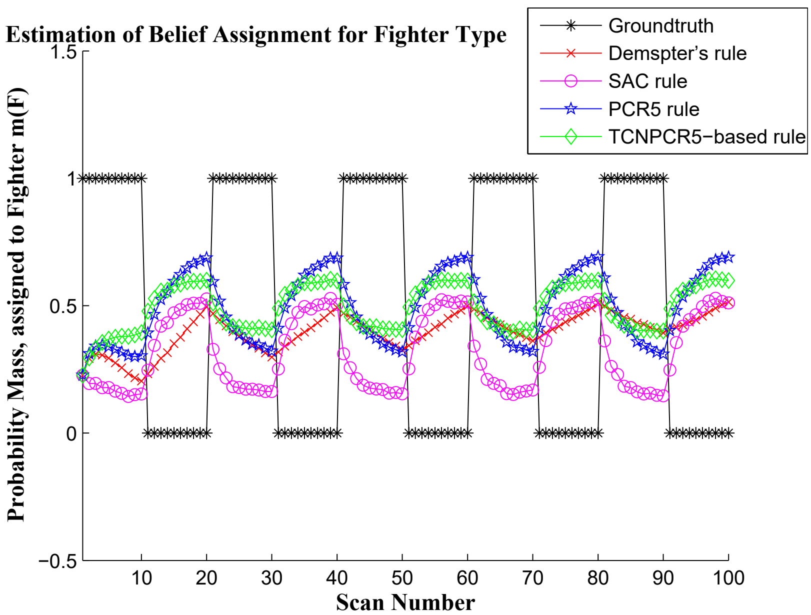

Figure 13.18: Belief mass for Fighter Type, $T = 10$scans, case 2

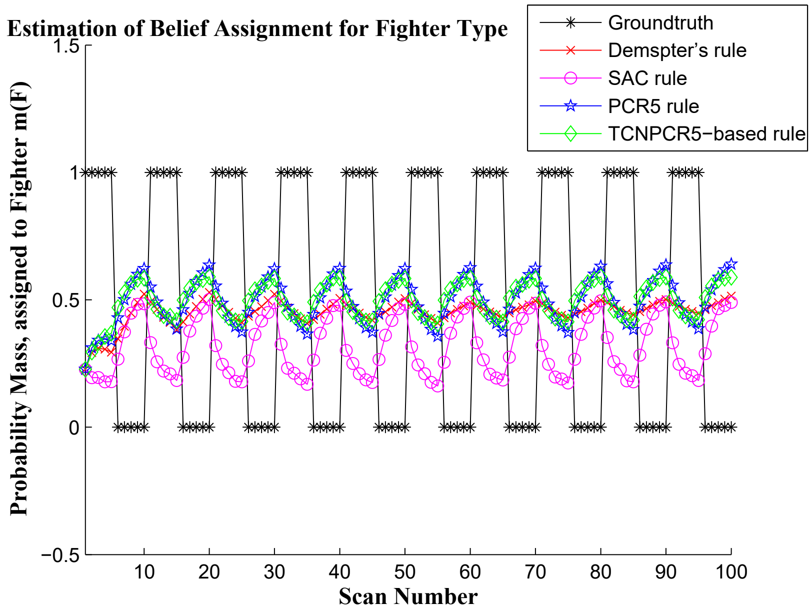

Figure 13.19: Belief mass for Fighter Type, $T = 5$scans, case 2

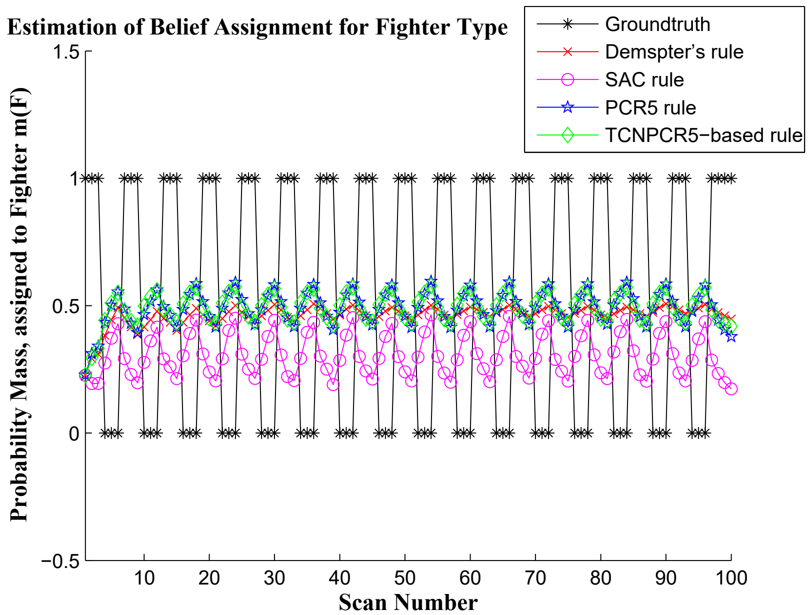

Figure 13.20: Belief mass for Fighter Type, $T = 3$ scans, case 2

13.5 Conclusions

Four Target Type Trackers (TTT) have been proposed and compared in this chapter. Our trackers are based on four combinational rules for temporal attribute data fusion for target type estimation: 1) Dempster's rule drawn from Dempster-Shafer Theory (DST); 2) Proportional Conflict Redistribution rule no. 5, PCR5 rule drawn from Dezert-Smarandache Theory (DSmT); 3) new class fusion rule, based on fuzzy T-Conorm and T-Norm operators (TCN); 4) new Symmetric Adaptive Combination (SAC) rule, drawn as a particular mixture of disjunctive and conjunctive rules. Our comparative analysis shows through a very simple scenario and Monte-Carlo simulation that PCR5, TCN and SAC rules allow a very efficient Target Type Tracking, reducing drastically the latency delay for correct Target Type decision, while Dempster's rule demonstrates risky behavior, keeping indifference to the detected target type changes. The temporal fusion process utilizes the new knowledge in an incremental manner and hides the possibility for arising bigger conflicts between the new incoming and the previous updated evidence. Dempster's rule cannot detect quickly and efficiently target type changes, and thus to track them correctly. It hides the risk to produce counter-intuitive and non adequate results. Dempster's rule and the SAC rule do not provide a symmetric target type estimation. Our PCR5/TCN/SAC-based Target Type Trackers are totally new, efficient and promising to be incorporated in real-time Generalized Data Association - Multi Target Tracking systems (GDA-MTT). The process of reacting and adapting to the type changes for PCR5, TCN and SAC rules do not depend on the duration of switching interval in both cases - of well defined and of poor classifier. It provides an important result on the behavior of these three rules with respect to Dempster's rule.

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Chapter 14

A DSmT-based Fusion Machine for Robot's Map Reconstruction

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Abstract: *Characteristics of uncertainty and imprecision, even imperfection is presented from knowledge acquisition in map reconstruction of autonomous mobile robots. Especially in the course of building grid map using sonar, this characteristic of uncertainty is especially servere. Jean Dezert and Florentin Smarandache have recently proposed a new information fusion theory (DSmT), whose greatest merit is to deal with uncertainty and conflict of information, and also proposed a series of proportional conflict redistribution rules (PRC1∼PCR5), therein, presently PCR5 is the most precise rule to deal with conflict factor according to its authors, though the complexity of computation might be increased correspondingly. In this chapter, according to the fusion machine based on the theory of DSmT coupled with PCR5, we not only can fuse information of the same reliable degree from homogeneous or heterogeneous sensors, but also the different reliable degree of evidential sources with the discounting theory. Then we established the belief model for sonar grid map, and constructed the generalized basic belief assignment function (gbbaf). Pioneer II virtual mobile robot with 16 sonar range finders on itself served as the experiment platform, which evolves in a virtual environment with some obstacles (discernable objects) and 3D Map was rebuilt online with our self-developing software platform. At the same time, we also compare it from other methods (i.e. Probability theory, Fuzzy theory and Dempster-Shafer Theory (DST)). The results of the comparison shows the new tool to have a better performance in map reconstruction of mobile robot. It also supplied with a foundation to study the Self-Localization And Mapping (SLAM) problem with the new tool further.*

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14.1 Introduction

The study on exploration of entirely unknown environment for intelligent mobile robots has been a popular and difficult subject for experts in the robotic field for a long time. Robots do not know the environment around themselves, that is, they have no experienced knowledge about the environment such as size, shape, layout of the environment, and also no signs such as beacons, landmarks, allowing them to determine their location about robot within the environment. Thus, the relation between self-localization and map building for mobile robot is like the chicken and egg problem [3, 16]. This is because if the mobile robot builds the map of the environment, then it must know the real position of itself within the environment; at the same time, if the robot wants to know its own position, then it must have a referenced map of the environment. Though it is hard to answer this question, some intelligent sensors such as odometer, electronic compass, sonar detector, laser range finder and vision sensor are installed on the mobile robot as if a person has perceptive organs.

How to manage and utilize this perceptive information acquired by organs, it's a new subject in information fusion, which will play an important role herein. As far as we know, experts have not yet given a unified expression. Just aiming to the practical field or system, proposed architecture of control such as hierarchical, concentrative, distributive and composite, and then according to the different integrated hierarchy, we compared the validity of all kinds of classical (Probability) and intelligent (Fuzzy, Neural-Networks (NN), Rough Set theory, Dempster-Shafer theory (DST), etc.) arithmetic. As far as the mobile robot is concerned, the popular arithmetic of self-localization in an unknown environment relying on interoceptive sensors (odometer, electronic compass) and exteroceptive sensors (sonar detector, laser range finder and visual sensor) is Markov location [10] or Monte Carlo location [28]. The map of the environment is built by applying some arithmetic such as Probability theory, Fuzzy Set theory and DST. The information of environment can be expressed as grid map, geometrical feature or topological map, etc., where the grid map is the most popular arithmetic expression [8, 9]. In this chapter, a new tool of the Fusion Machine based on DSmT [5, 6, 22] coupling with PCR5 is introduced to apply to the map reconstruction of mobile robots. DSmT mentioned here that has been proposed by Jean Dezert and Florentin Smarandache based on Bayesian theory and Dempster-Shafer theory [21] recently is a general, flexible and valid arithmetic of fusion. Its largest advantage is that it can deal with uncertain and imprecise information effectively, which supplies with a powerful tool to deal with uncertain information acquired by sonar detector in the course of building the grid map. Moreover, through the rule of PCR5, which is also proposed by Jean Dezert and Florentin Smarandache [23–25], we can refine and redistribute the conflict mass to improve the precision and correctness of fusion. The comparison of the new tool from other methods is done to testify it to have a better performance to solve the puzzle.

14.2 The fusion machine

14.2.1 General principle

At first, here the fusion machine is referred to a theory tool to combine and integrate the imperfect information without preprocessing it (i.e. filter the information) according to the different combination rules (i.e. DST, DSmT, etc.). It even redistributes the conflict masses to other basic belief masses according to the constraints of system using the different redistribution rules (i.e. PCR1$\sim$PCR5, minC [2], WAO [11], etc.). Of course, how to adopt the fusion rule must be considered according to the different application. Here we consider the application, and give a special fusion machine (shown in Fig. 14.1). In Fig. 14.1, k sources of evidences (i.e. the inputs) provide basic belief assignments over a propositional space generated by elements of a frame of discernment and set operators endowed with eventually a given set of integrity constraints, which depend on the nature of elements of the frame. The set of belief assignments need then to be combined with a fusion operator. Since in general the combination of uncertain information yields a degree of conflict, say K, between sources, this conflict must be managed by the fusion operator/machine. The way the conflict is managed is the key of the fusion step and makes the difference between the fusion machines. The fusion can be performed globally/optimally (when combining the sources in one derivation step all together) or sequentially (one source after another as in Fig. 14.1). The sequential fusion processing (well adapted for temporal fusion) is natural and more simple than the global fusion but in general remains only suboptimal if the fusion rule chosen is not associative, which is the case for most of fusion rules, but Dempster's rule. In this chapter, the sequential fusion based on the PCR5 rule is chosen because PCR5 has shown good performances in works and because the sequential fusion is much more simple to implement and to test. The *optimal* (global) PCR5 fusion rule formula for k sources is possible and has also been proposed [23] but is much more difficult to implement and has not been tested yet. A more efficient PCR rule (denoted PCR6) proposed very recently by Martin and Osswald in [15], which outperforms PCR5, could be advantageously used in the fusion machine instead PCR5. Such idea is currently under investigation and new results will be reported in a forthcoming publication. We present in more details in next section the DSmT-based fusion machine.

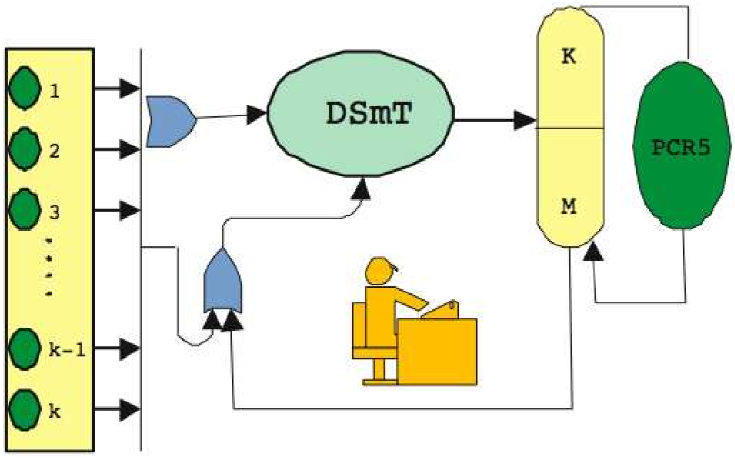

Figure 14.1: A kind of sequential fusion machine

14.2.2 Basis of DSmT

DSmT (Dezert-Smarandache Theory) is a new, general and flexible arithmetic of fusion, which can solve the fusion problem of different tiers including data-tier, feature-tier and decision-tier, and even, not only can solve the static problem of fusion, but also can solve the dynamic one. Especially, it has a prominent merit that it can deal with uncertain and highly conflicting information [5, 6, 22].

14.2.2.1 Simple review of DSmT

1) Let $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, be the frame of discernment, which includes n finite focal elements $\theta_i (i = 1, \cdots, n)$. Because the focal elements are not precisely defined and separated, so that no refinement of Θ in a new larger set Ω_{ref} of disjoint elementary hypotheses is possible.
2) The hyper-power set D^Θ is defined as the set of all compositions built from elements of Θ with $\cup$ and $\cap$ (Θ generates D^Θ under operators $\cup$ and $\cap$) operators such that

a) $\emptyset, \theta_1, \theta_2, \cdots, \theta_n \in D^\Theta$.

b) If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.

c) No other elements belong to D^Θ, except those obtained by using rules a) or b).

3) General belief and plausibility functions
Let $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$ be the general frame of discernment. For every evidential source S, let us define a set of map of $m(\cdot) : D^\Theta \to [0, 1]$ associated to it (abandoning Shafer's model) by assuming here that the fuzzy/vague/relative nature of elements $\theta_i (i = 1, \cdots, n)$ can be non-exclusive, as well as no refinement of Θ into a new finer exclusive frame of discernment Θ_{ref} is possible. The mapping $m(\cdot)$ is called a generalized basic belief assignment function if it satisfies

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1,$$

then $m(A)$ is called A's generalized basic belief assignment function (gbbaf). The general belief function and plausibility function are defined respectively in almost the same manner as within the DST, i.e.

$$\text{Bel}(A) = \sum_{B \in D^\Theta, B \subseteq A} m(B) \tag{14.1}$$

$$\text{Pl}(A) = \sum_{B \in D^\Theta, B \cap A \neq \emptyset} m(B) \tag{14.2}$$

4) Classical (free) DSm rule of combination
Let $\mathcal{M}^f(\Theta)$ be a free DSm model. The classical (free) DSm rule of combination (denoted (DSmC) for short) for $k \geq 2$ sources is given $\forall A \neq \emptyset, and A \in D^\Theta$as follows:

$$m_{\mathcal{M}^f(\Theta)}(A) \cong [m_1 \oplus \cdots \oplus m_k](A) = \sum_{\substack{X_1, \cdots, X_k \in D^\Theta \\ X_1 \cap \cdots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i) \tag{14.3}$$

14.2.2.2 Fusion of unreliable sources

1) On the necessity of discounting sources
In fact, sources of information are unreliable in real systems due to the sources with different knowledge and experience. For example, from the point of view of the mobile robots' sensors, the metrical precision and resolution with laser range finder are both higher than that with sonar sensor. Even if they are the same sonar sensors, then they have also different precision due to the manufacturing and other factors. Under this condition, if we treat data of unreliable information sources as data of reliable sources to be fused, then the result is very unreliable

and even reverses. Thus, unreliable resources must be considered, and then DSmT based on the discounting method [7, 12, 21, 26] does well in dealing with unreliable sensors.

2) Principle of discounting method

Let's consider k evidential sources of information $(S_1, S_2, \ldots, S_k)$, here we work out a uniform way in dealing with the homogeneous and heterogeneous information sources. So we get the discernment frame $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$, $m(\cdot)$ is the basic belief assignment, let $m_i(\cdot) : D^\Theta \to [0,1]$ be a set of maps, and let p_i represent reliable degree supported by S_i $(i = 1, 2, \ldots, k)$, considering $\sum_{A \in D^\Theta} m_i(A) = 1$, let $I_t = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_n$ express the total ignorance, and then let $m_i^g(I_t) = 1 - p_i + p_i m_i(I_t)$ represent the belief assignment of the total ignorance for global system (*after discounting*), and then this is because of existing occurrence of malfunction, that is, $\sum_{A \in D^\Theta} m_i(A) = p_i$, we assign the quantity $1 - p_i$ to the total ignorance again. Thus, the rule of combination for DSmT based on discounting method with $k \geq 2$ evidential sources is given as in the formula (14.3), i.e. the conjunctive consensus on the hyper-power set by $m^g_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ and $\forall A \neq \emptyset \in D^\Theta$,

$$m^g_{\mathcal{M}^f(\Theta)}(A) \cong [m_1^g \oplus \ldots \oplus m_k^g](A) = \sum_{\substack{X_1, \cdots, X_k \in D^\Theta \\ X_1 \cap \cdots \cap X_k = A}} \prod_{i=1}^{k} p_i m_i(X_i) \tag{14.4}$$

14.2.3 The PCR5 fusion rule

When integrity constraints are introduced in the model, one has to deal with the conflicting masses, i.e. all the masses that would become assigned to the empty set through the DSmC rule. Many fusion rules (mostly based on Shafer's model) have been proposed [20] for managing the conflict. Among these rules, Dempster's rule [21] redistributes the total conflicting mass over all propositions of 2^Θ through a simple normalization step. This rule has been the source of debates and criticisms because of its unexpected/counter-intuitive behavior in some cases. Many alternatives have then been proposed [20, 22] for overcoming this drawback. In DSmT, we have first extended the Dubois & Prade's rule [7, 22] for taking into account any integrity constraints in the model and also the possible dynamicity of the model and the frame. This first general fusion rule, called DSmH (DSm Hybrid) rule, consists just in transferring the partial conflicts onto the partial ignorances[1]. The DSmH rule has been recently and advantageously replaced by the more sophisticated Proportional Conflict Redistribution rule no.5 (PCR5). According to Smarandache and Dezert, PCR5 does a better redistribution of the conflicting mass than Dempster's rule since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the partial conflicting masses only to the sets involved in the conflict and proportionally to their masses put in the conflict, considering the conjunctive normal form of the partial conflict. PCR5 is quasi-associative and preserves the neutral impact of the vacuous belief assignment. Since PCR5 is presented in details in [23], we just remind PCR5 rule for only two sources[2]: $m_{PCR5}(\emptyset) = 0$, and for all $X \in G \setminus \{\emptyset\}$,

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G \setminus \{X\} \\ X \cap Y = \emptyset}} \left[\frac{m_1^2(X) m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2^2(X) m_1(Y)}{m_2(X) + m_1(Y)} \right], \tag{14.5}$$

[1] Partial ignorance being the disjunction of elements involved in the partial conflicts.

[2] A general expression of PCR5 for an arbitrary number ($s > 2$) of sources can be found in [23].

where all sets are in canonical form and $m_{12}(X) = \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2}} m_1(X_1) \cdot m_2(X_2)$ corresponds to the conjunctive consensus on X between the two sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded.

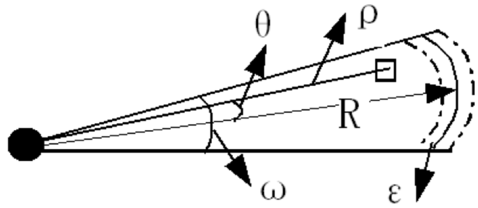

Figure 14.2: Sketch of the principle of sonar

14.3 Modeling of Sonar Grid Map Building Based on DSmT

Here we mainly discuss a sonar sensor, whose working principle (shown as Fig. 14.2) is: producing sheaves of cone-shaped wave and detecting the objects by receiving the reflected wave. Due to the restriction of sonar physical characteristic, metrical data has uncertainty as follows:
a) Beside its own error of making, the influence of external environment is also very great, for example, temperature, humidity, atmospheric pressure and so on.
b) Because the sound wave spreads outwards through a form of loudspeaker, and there exists a cone-shaped angle, we cannot know the true position of object detected among the fan-shaped area, with the enlargement of distance between sonar and it.
c) The use of many sonar sensors will result in interference with each other. For example, when the i-th sonar gives out detecting wave towards an object of irregular shape, if the angle of incidence is too large, the sonar wave might be reflected out of the receiving range of the i-th sonar sensor or also might be received by other sonar sensors.
d) Because sonar sensors utilize the reflection principle of sound wave, if the object absorbs most of heavy sound wave, the sonar sensor might be invalid.

Pointing to the characteristics of sonar's measurement, we construct a model of uncertain information acquired from grid map using sonar based on DSmT. Here we suppose there are two focal elements in system, that is, $\Theta = \{\theta_1, \theta_2\}$. Where θ_1 means grid is empty, θ_2 means occupied, and then we can get its hyper-power set $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. Every grid in environment is scanned $k \geq 5$ times, each of which is viewed as source of evidence. Then we may define a set of map aiming to every source of evidence and construct the general basic belief assignment functions (gbbaf) as follows: $m(\theta_1)$ is defined as the gbbaf for grid-unoccupied (empty); $m(\theta_2)$ is defined as the gbbaf for grid-occupied; $m(\theta_1 \cap \theta_2)$ is defined as the gbbaf for holding grid-unoccupied and occupied simultaneous (conflict). $m(\theta_1 \cup \theta_2)$ is defined as the gbbaf for grid-ignorance due to the restriction of knowledge and present experience (here referring to the gbbaf for these grids still not scanned presently), it reflects the degree of ignorance of grid-unoccupied or occupied.

The gbbaf of a set of map $m(\cdot) : D^\Theta \to [0, 1]$ is constructed by authors such as the formulae (14.6)$\sim$(14.9) according to sonar physical characteristics.

$$m(\theta_1) = E(\rho)E(\theta) = \begin{cases} (1-(\rho/R)^2)\lambda \quad \text{if} \quad \begin{cases} R_{min} \leq \rho \leq R \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{14.6}$$

$$m(\theta_2) = O(\rho)O(\theta) = \begin{cases} e^{-3\rho_v(\rho-R)^2}\lambda \quad \text{if} \quad \begin{cases} R_{min} \leq \rho \leq R+\epsilon \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{14.7}$$

$$m(\theta_1 \cap \theta_2) = \begin{cases} [1-[\frac{2(\rho-R+2\epsilon)}{R}]^2]\lambda \quad \text{if} \quad \begin{cases} R_{min} \leq \rho \leq R \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{14.8}$$

$$m(\theta_1 \cup \theta_2) = \begin{cases} \tanh(2(\rho-R))\lambda \quad \text{if} \quad \begin{cases} R \leq \rho \leq R_{max} \\ 0 \leq \theta \leq \omega/2 \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{14.9}$$

where $\lambda = E(\theta) = O(\theta)$ is given by (see [8] for justification)

$$\lambda = \begin{cases} 1-(2\theta/\omega)^2 \quad \text{if} \quad 0 \leq |\theta| \leq \omega/2 \\ 0 \quad \text{otherwise} \end{cases} \tag{14.10}$$

where ρ_v in formula (14.7) is defined as an environment adjusting variable, that is, the less the object is in environment, the greater the variable ρ_v is, and makes the function of $m(\theta_2)$ more sensitive. Here let ρ_v be one. $E(\cdot)$ and $O(\cdot)$ are expressed as the *Effect Function* of ρ, θ to grid's empty or occupancy. In order to insure the sum of all masses to be one, we must renormalize it. The analysis on the characteristics of gbbaf are shown as Fig. 14.3∼Fig. 14.7, when $R = 1.5m$.

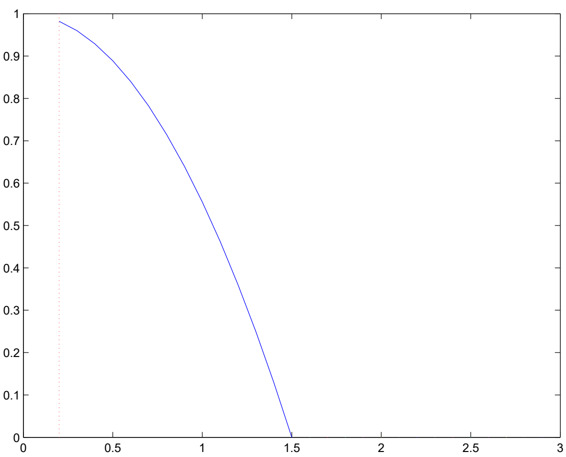

Figure 14.3: $m(\theta_1)$ as function of ρ given by (14.6)

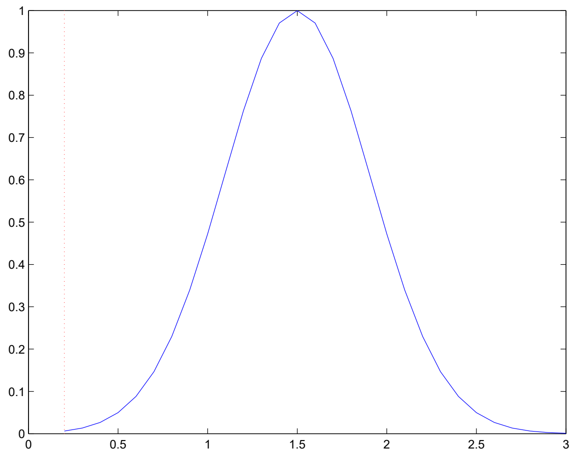

Figure 14.4: $m(\theta_2)$ as function of ρ given by (14.7)

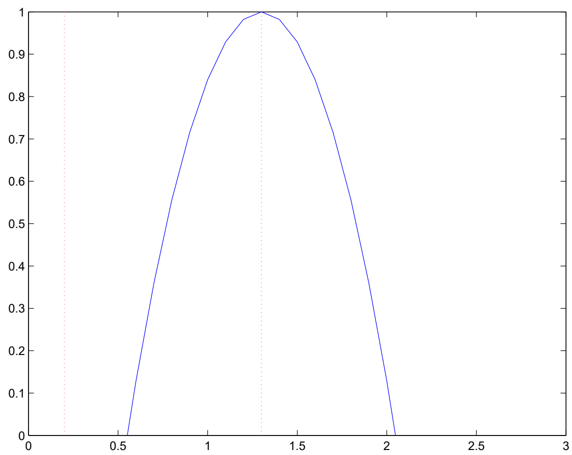

Figure 14.5: $m(\theta_1 \cap \theta_2)$ as function of ρ given by (14.8)

Seen from Fig. 14.3, $m(\theta_1)$ has a falling tendency with the increasing of distance between grid and sonar, and has the maximum at R_{min} and zero at R. From the point of view of the working principle of sonar, the more the distance between them approaches the measured value, the more that grid might be occupied. Thus the probability that grid indicated is empty is very low, of course the gbbaf of grid-unoccupied is given a low value.

From Fig. 14.4, $m(\theta_2)$ takes on the distribution of Gaussian function with respect to the addition of distance between them, has the maximum at R, which answers for the characteristic of sonar acquiring information.

From Fig. 14.5, $m(\theta_1 \cap \theta_2)$ takes on the distribution of a parabola function with respect to the addition of distance between them. In fact, when $m(\theta_1)$ equals $m(\theta_2)$, $m(\theta_1 \cap \theta_2)$ has the maximum there. But it is very difficult and unnecessary to find the point of intersection of the two functions. Generally, we let the position of $R - 2\varepsilon$ replace the point of intersection. Experience indicates that its approximate value is more rational.

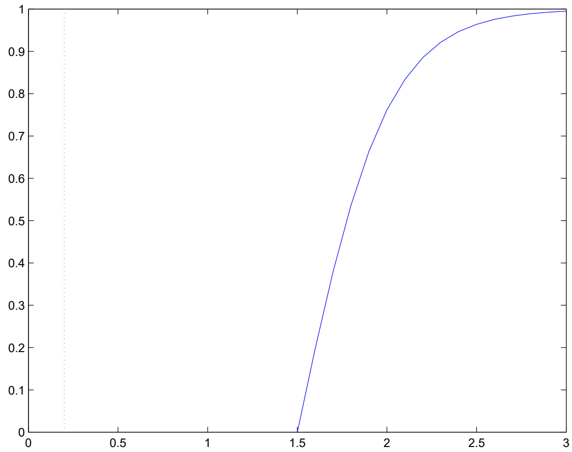

Figure 14.6: $m(\theta_1 \cup \theta_2)$ as function of ρ given by (14.9)

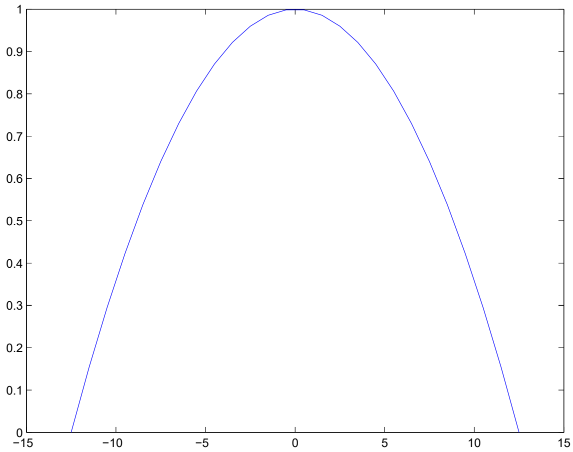

Figure 14.7: λ as function of θ given by (14.10)

From Fig. 14.6, $m(\theta_1 \cup \theta_2)$ takes on the distribution of the hyperbola function with respect to the addition of distance between them, and zero at R. This function reflects well the ignorance of grid information at $R \leq \rho \leq R_{max}$. The relation between θ and λ is reflected in Fig. 14.7, where the more the position of grid approaches the central axis, the greater λ becomes, that is, the greater the contribution to belief assignment is. Otherwise, the lower it is. In short, the general basic belief assignment functions (gbbaf) entirely fit with the characteristic of sonar acquiring information. This supplies a theoretic foundation for dealing with uncertain information in grid map building.

14.4 Sonar Grid Map Building Based on Other Methods

To apply the probability theory and fuzzy set theory to map building, at first, two functions of uncertainty are introduced. Here the working environment U of robot is separated into $m \times n$

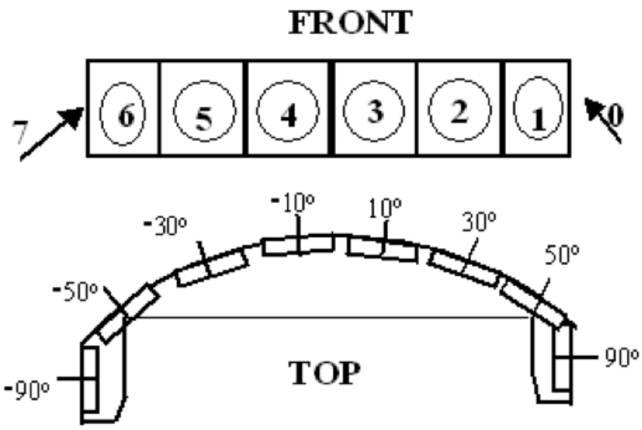

Figure 14.8: Sketch of the layout of sonars

rectangle grids of same size. Every grid is represented by G_{ij}, $U = \{G_{ij} | i \in [1, m], j \in [1, n]\}$, according to the reference [29]. Two functions are applied to represent the uncertainty of sonar as follows:

$$\Gamma(\theta) = \begin{cases} 1 - 21\left(\frac{\theta\pi}{180}\right)^2, & \text{if } 0 \leq |\theta| \leq 12.5^o, \\ 0, & \text{if } |\theta| > 12.5^o. \end{cases} \tag{14.11}$$

$$\Gamma(\rho) = 1 - (1 + \tanh(2(\rho - \rho_v)))/2, \tag{14.12}$$

where θ represents the angle between the center-axis and the spot (i, j) measured in Fig. 14.2. ρ_v is the pre-defined value, which reflects the smooth transferring point from the certainty to uncertainty. $\Gamma(\theta)$ shows that the nearer by center- axis is the spot (i, j), the larger is the density of the wave. $\Gamma(\rho)$ shows that the farther away from the sonar is it, the lower is the reliability, while the nearer by the sonar it is, the higher is the reliability of correct measurement.

1) Probability Theory

Elfes and Moravec [8, 9] firstly represented the probability of the grid occupied by obstacles with probability theory. Then Thrun, Fox and Burgard [27], Olson [17], Romero and Morales [19] also proposed the different methods of map reconstruction by themselves based on probability theory. According to the above methods, we give the general description of map building based on probability theory. To avoid an amount of computation, we suppose that all grids are independent. For every grid G_{ij}, let $s(G_{ij}) = E$ represent the grid empty, while $s(G_{ij}) = O$ represent the grid occupied and $P[s(G_{ij}) = E]$ and $P[s(G_{ij}) = O]$ the probabilities of these events with the constraint $P[s(G_{ij}) = E] + P[s(G_{ij}) = O] = 1$. According to the physical characteristics, the probability model to map the sonar perception datum is given by

$$P\left[s(G_{ij})=O|R\right]=P\left[s(\rho,\theta)=O|R\right]=\begin{cases}(1-\lambda')/2, & \text{if } 0\le\rho<R-2\varepsilon,\\ [1-\lambda'\left(1-(2+a)^2\right)]/2, & \text{if } R-2\varepsilon\le\rho<R-\varepsilon,\\ [1+\lambda'\left(1-a^2\right)]/2, & \text{if } R-\varepsilon\le\rho<R+\varepsilon,\\ 1/2, & \text{if } \rho\ge R+\varepsilon\end{cases} \tag{14.13}$$

where $\lambda'=\Gamma(\theta)\cdot\Gamma(\rho)$ and $a=(\rho-R)/\varepsilon$. Seen from Eq. (14.13), the mapping between the sonar data and probability answers for the physical characteristics of sonar. For the data outside the measurement range, the probability value is 0.5, that is, the uncertainty is the largest. When the distance between the grid in the range and the sonar is less than the measurement, the nearer by the sonar it is, the less is the possibility of grid occupied, while the nearer by the location of the measurement is the grid, and the larger is the possibility of grid occupied.

The fusion algorithm for multi-sensors is given according to the Bayesian estimate as follows:

$$P\left[s(G_{ij})=O|R_1,\cdots,R_{k+1}\right]=\frac{P\left[s(G_{ij})=O|R_{k+1}\right]\cdot P\left[s(G_{ij})=O|R_1,\cdots,R_k\right]}{\sum\limits_{X\in\{E,O\}}P\left[s(G_{ij})=X|R_{k+1}\right]\cdot P\left[s(G_{ij})=X|R_1,\cdots,R_k\right]} \tag{14.14}$$

Remark: In order to make the equation (14.14) hold, we must suppose at the beginning of map building

$$\forall G_{ij}\in U,\qquad P\left[s(G_{ij})=E\right]=P\left[s(G_{ij})=O\right]=0.5$$

2) Fuzzy Set Theory

Map building based on fuzzy logic is firstly proposed by Giuseppe Oriolo et al. [18]; they define two fuzzy sets Ψ (represents grid empty) and Ω (represents grid occupied), which of size all are equal to U, correspondingly, their membership functions are μ_Ψ and μ_Ω. Similarly, we can get fuzzy model to map the sonar perception datum.

$$\mu^{S(R)}\Psi(G_{ij})=\lambda'\cdot f_\Psi(\rho,R) \tag{14.15}$$

$$\mu^{S(R)}\Omega(G_{ij})=\lambda'\cdot f_\Omega(\rho,R), \tag{14.16}$$

where

$$f_\Psi(\rho,R)=\begin{cases}k_E, & \text{if } 0\le\rho<R-\varepsilon,\\ k_E((R-\rho)/\varepsilon)^2, & \text{if } R-\varepsilon\le\rho<R,\\ 0, & \text{if } \rho\ge R.\end{cases}$$

$$f_\Omega(\rho,R)=\begin{cases}k_O, & \text{if } 0\le\rho<R-\varepsilon,\\ k_O((R-\rho)/\varepsilon)^2, & \text{if } R-\varepsilon\le\rho<R+\varepsilon,\\ 0, & \text{if } \rho\ge R.\end{cases}$$

Here, $f_\Psi(\rho,R)$ represents the influence of ρ,R on the membership μ_Ψ of grid G_{ij}. $f_\Omega(\rho,R)$ represents the influence of ρ,R on the membership μ_Ω of grid G_{ij}. k_E and k_O are constants with $0<k_E\le1$ and $0<k_O\le1$. λ' is same as the definition in (14.13). Seen from the Eq.

(14.15) and (14.16), if the grid has the great possibility of occupation, then the membership is μ_Ψ small, while the membership μ_Ω is great. If the grid is outside the measuring range, then $\mu_\Psi = 0$ and $\mu_\Omega = 0$. The fusion algorithm for multi-sensors is given according to the union operator in the fuzzy theory as follow: $\forall X \in \{\Psi, \Omega\}$,

$$\mu_X^{S(R_1,\cdots,R_{k+1})}(G_{ij}) = \mu_X^{S(R_1,\cdots,R_k)}(G_{ij}) + \mu_X^{S(R_{k+1})(G_{ij})} - \mu_X^{S(R_1,\cdots,R_k)}(G_{ij}) \cdot \mu_X^{S(R_{k+1})}(G_{ij}). \quad (14.17)$$

Remark: Initially, we suppose $\mu_\Psi(G_{ij}) = \mu_\Omega(G_{ij}) = 0$ $(\forall G_{ij} \in U)$. According to membership of every grid, we can get the final map representation as follows:

$$M = \overline{\Psi^2 \cap \overline{\Omega} \cap \overline{A} \cap \overline{I}} \quad (14.18)$$

Here, $A = \Psi \cap \Omega$, $I = \overline{\Psi} \cap \overline{\Omega}$. At the same time, the rule of fuzzy intersection is

$$\mu_{i \cap j}(G_{ij}) = \mu_i(G_{ij}) \cdot \mu_j(G_{ij}),\ \forall\ G_{ij} \in U,$$

and the rule of fuzzy complementation is

$$\mu_{\overline{i}}(G_{ij}) = 1 - \mu_i(G_{ij}),\ \forall\ G_{ij} \in U.$$

The larger is the membership of G_{ij} belonging to the fuzzy set M, the greater is the possibility of the grid occupied.

3) Dempster-Shafer Theory (DST)

The idea of using belief functions for representing someone's subjective feeling of uncertainty was first proposed by Shafer [21], following the seminal work of Dempster [4] about upper and lower probabilities induced by a multi-valued mappings. The use of belief functions as an alternative to subjective probabilities for representing uncertainty was later justified axiomatically by Smets [26], who introduced the Transferable Belief Model (TBM), providing a clear and coherent interpretation of the various concepts underlying the theory.

Let θ_i $(i = 1, 2, \ldots, n)$ be some exhaustive and exclusive elements (hypotheses) of interest taking on values in a finite discrete set Θ, called the frame of discernment. Let us assume that an agent entertains beliefs concerning the value of θ_i, given a certain evidential corpus. We postulate that these beliefs may be represented by a belief structure (or belief assignment), i.e. a function from 2^Θ to $[0, 1]$ verifying $\sum_{A \subseteq \Theta} m(A) = 1$ and $m(\emptyset) = 0$ for all $A \subseteq \Theta$, the quantity $m(A)$ represents the mass of belief allocated to proposition "$\theta_i \subseteq A$", and that cannot be allocated to any strict sub-proposition because of lack of evidence. The subsets A of Θ such that $m(A) > 0$ are called the focal elements of m. The information contained in the belief structure may be equivalently represented as a belief function *bel*, or as a plausibility function *pl*, defined respectively as $bel(A) = \sum_{B \subseteq A} m(B)$ and $pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$. The quantity $bel(A)$, called the belief in A, is interpreted as the total degree of belief in A (i.e. in the proposition "$\theta_i \subseteq A$"), whereas $pl(A)$ denotes plausibility of A, i.e. the amount of belief that could potentially be transferred to A, taking into account the evidence that does not contradict that hypothesis.

Now we assume the simplest situation that two distinct pieces of evidence induce two belief structures m_1 and m_2. The orthogonal sum of m_1 and m_2, denoted as $m = m_1 \oplus m_2$ is defined as:

$$m(A) = K^{-1} \sum_{B \cap C = A} m_1(B) m_2(C), \quad (14.19)$$

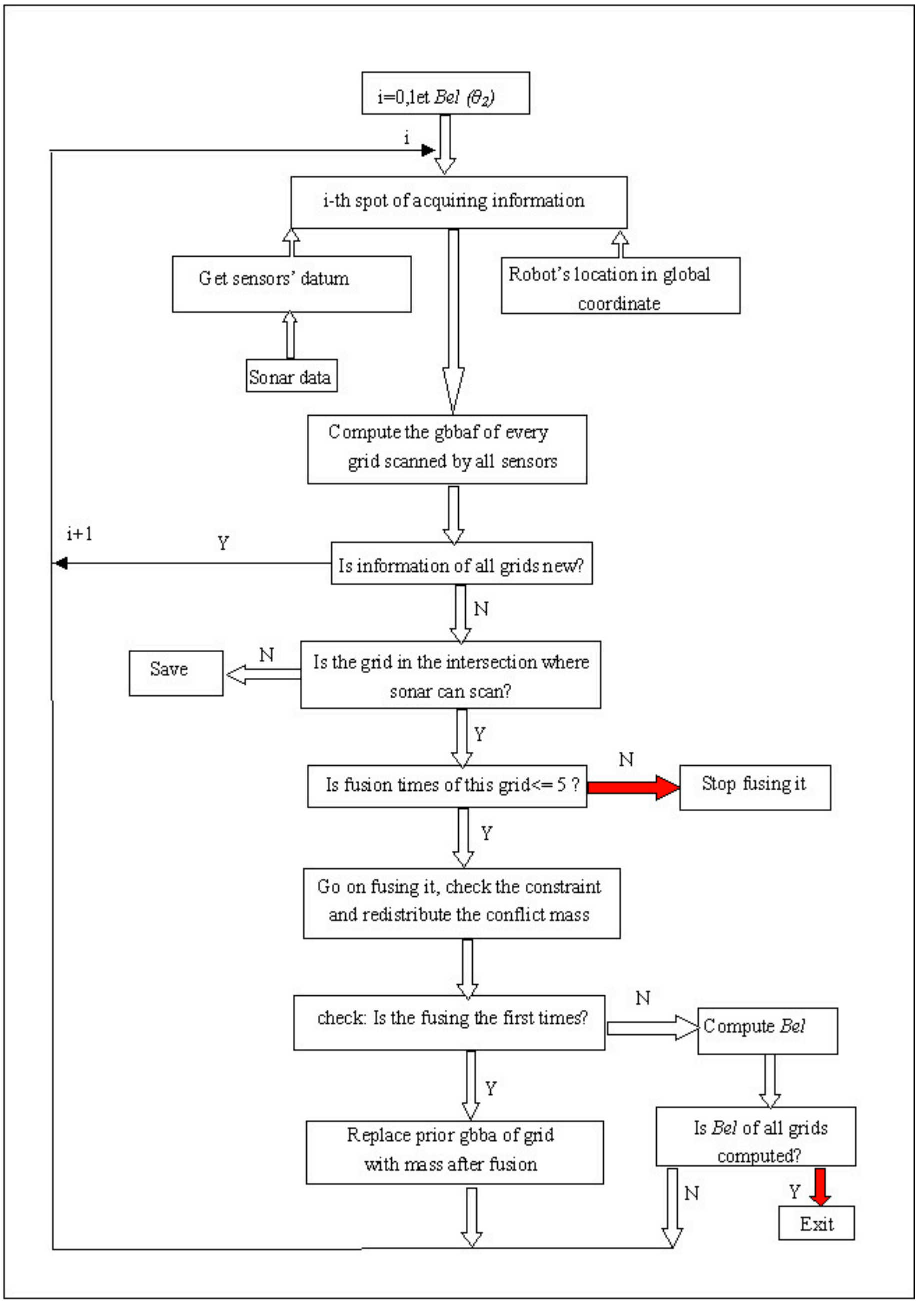

Fig. 9: Flowchart of procedure of sonar map building based on DSmT

Figure 14.9: Flowchat of procedure of sonar map building based on DSmT and PCR5

Here $K = 1 - \sum_{B \cap C = \emptyset} m_1(B)m_2(C)$ for $A \neq \emptyset$ and $m(\emptyset) = 0$. The orthogonal sum (also called Dempster's rule of combination) is commutative and associative. It plays a fundamental operation for combining different evidential sources in evidence theory. DST as an information fusion method has been applied to the environment exploration and map reconstruction [1, 13]. This method can assure to have a precise result in fusing the same or different multi-sources information from the sensors on the robot. According to the requirement of sonar grid map building, we need only to consider a simple 2D frame $\Theta = \{\theta_1, \theta_2\}$, where θ_1 means "the grid is empty", and θ_2 means "the grid is occupied", and then we work with basic belief assignments defined on its power set $2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. According to DST, let $m_{DST}(\emptyset) = 0$, here $m_{DST}(\theta_1)$ is defined as the basic belief assignment function (bbaf) for grid-empty, $m_{DST}(\theta_2)$ is defined as the basic belief assignment function (bbaf) for grid-occupied, $m_{DST}(\theta_1 \cup \theta_2)$ is defined as the basic belief assignment function for grid-ignorance. We may also construct basic belief assignment function such as $m_{DST}(\theta_1) = m(\theta_1)$, $m_{DST}(\theta_2) = m(\theta_2)$, $m_{DST}(\theta_1 \cup \theta_2) = m(\theta_1 \cup \theta_2)$. bbaf reflects still the characteristics of uncertainty for sonar grip map building in Fig. 14.2. Though here we define the same bbaf as DSmT, considering the definition of DST must be satisfied, we must renormalize them while acquiring sonar grid information [13, 14]. The new basic belief assignment after fusing two evidence sources from the sonar range finders can be obtained by the combination rule in Eq. (14.19).

14.5 Simulation Experiment

The experiment consists in simulating the autonomous navigation of a virtual Pioneer II Robot carrying 16 simulated sonar detectors in a 5000mm×5000mm square array with an unknown obstacle/object. The map building with sonar sensors on the mobile robot is done from the simulator of SRIsim (shown in Fig.14.10) of ActivMedia company and our self-developing experimental or simulation platform together. (shown in Fig. 14.11). Here the platform developed with the tool software of visual c++ 6.0 and OpenGL servers as a client end, which can connect the server end (also developed by ourselves, which connects the SRIsim and the client). When the virtual robot runs in the virtual environment, the server end can collect many information (i.e. the location of robot, sensors reading, velocity .etc) from the SRIsim. Through the protocol of TCP/IP, the client end can get any information from the server end and fuse them. The Pioneer II Robot may begin to run at arbitrary location; here we choose the location (1500mm, 2700mm) with an 88 degrees angle the robot faces to. We let the robot move at speeds of transverse velocity 100mm/s and turning-velocity 50degree/s around the object in the *world* map plotted by the Mapper (a simple plotting software), which is opened in the SRIsim shown in Fig. 14.10.

We adopt grid method to build map. Here we assume that all the sonar sensors have the same reliability. The global environment is divided into 50×50 lattices (which of size are same). The object in Fig. 14.10 is taken as a regular rectangular box. When the virtual robot runs around the object, through its sonar sensors, we can clearly recognize the object and know its appearance, and even its location in the environment. To describe the experiment clearly, the flowchart of procedure of sonar map is given in Fig. 14.9. The main steps of procedure based on the new tool are given as follows:

1. Initialize the parameters of robot (i.e. initial location, moving velocity, etc.).

2. Acquire 16 sonar readings, and robot's location, when the robot is running (Here we set the first timer, of which interval is 100 ms.).

3. Compute gbbaf of the fan-form area detected by each sonar sensor.

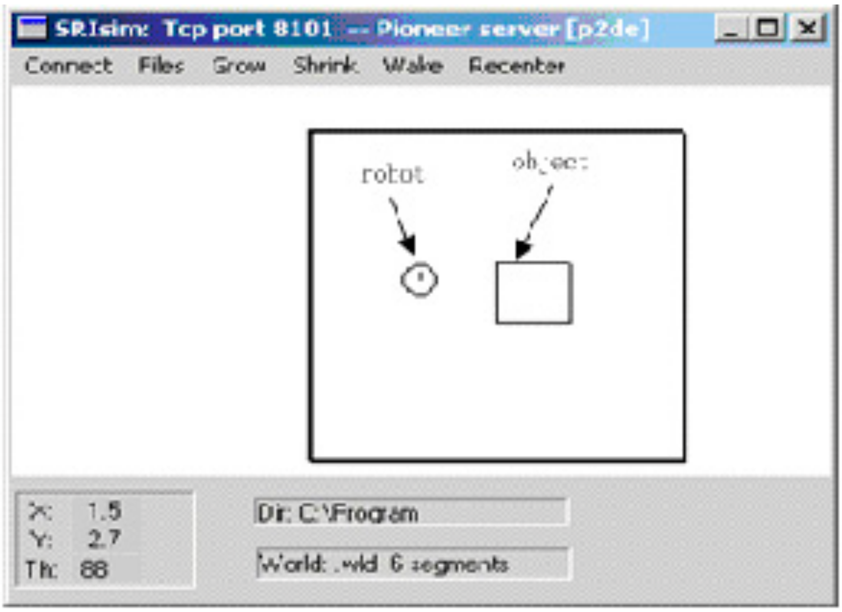

Figure 14.10: A world map opened in the SRIsim

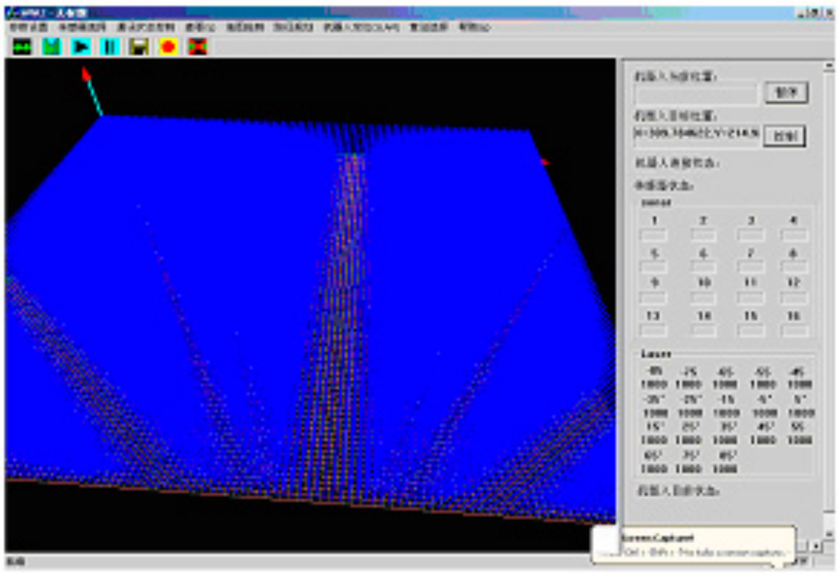

Figure 14.11: The platform for simulation or real experiment

4. Check whether some grids are scanned more than 5 times by sonar sensors (Same sonar in different location, or different sonar sensors. Of course, here we suppose each sonar sensor has the same characteristics.)? If "yes", then go to next step, otherwise, go to step 2.

5. According to the combination rule and the PCR5 in (14.3) and (14.5) respectively, we can get the new basic belief masses, and redistribute the conflicting mass to the new basic belief masses in the order of the sequential fusion, until all 5 times are over.

6. Compute the credibility/total belief of occupancy $bel(\theta_2)$ of some grids, which have been fused according to (14.1).

7. Update the map of the environment. (Here we set the second timer, of which interval is 100 ms) Check whether all the grids have been fused? If "yes", then stop robot and exit. Otherwise, go to step 2.

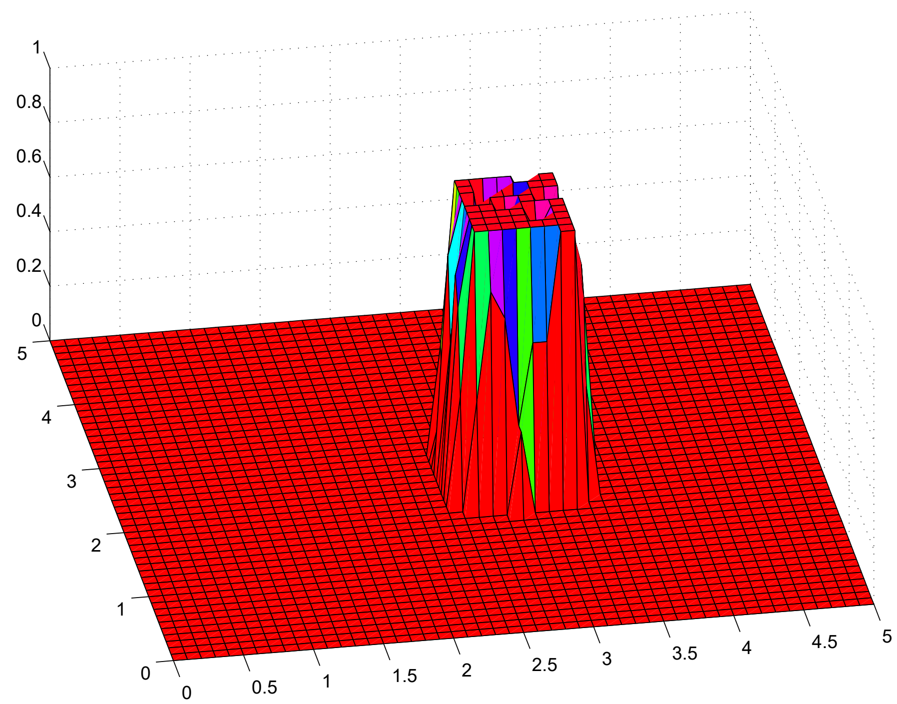

Figure 14.12: Map reconstruction with DSmT coupling with PCR5 (3D)

Finally, we rebuild the map shown in the Fig. 14.12 (3D) and Fig. 14.13 (2D) with the new tool. We also rebuild the map by other methods (i.e. probability theory, fuzzy theory, DST) in Fig. 14.14–Fig. 14.16. Because the other methods are not new, here we don't give the detailed steps. If the reader has some interest in them, please refer to their corresponding reference. We give the result of comparison in Table 14.5. Through the figure and table, it can be concluded that:

1) In Fig. 14.12, the Z axis shows the Belief of every grid occupied. The value 0 indicates that the grid is fully empty, and the value 1 indicates that this grid is fully occupied. This facilitates very much the development of human-computer interface of mobile robot exploring unknown, dangerous and sightless area.

2) Low coupling. Even if there are many objects in grid map, but there occurs no phenomenon of the apparently servered, but actually connected. Thus, it supplies with a powerful evidence for self-localization, path planning and navigation of mobile robot.

3) High validity of calculation. The fusion machine considering the restrained spreading arithmetic is adopted [29], and overcomes the shortcoming that the global grids in map must be reckoned once for sonar scanning every time, and improves the validity of calculation.

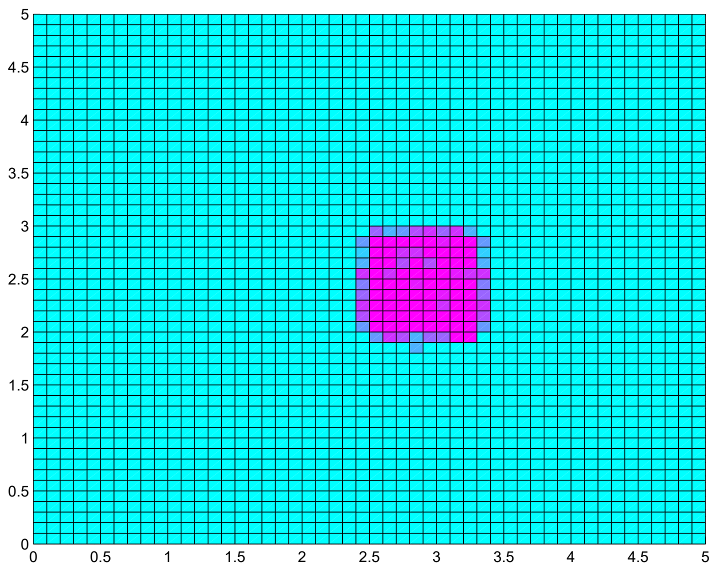

Figure 14.13: Map reconstruction with DSmT coupling with PCR5 (2D)

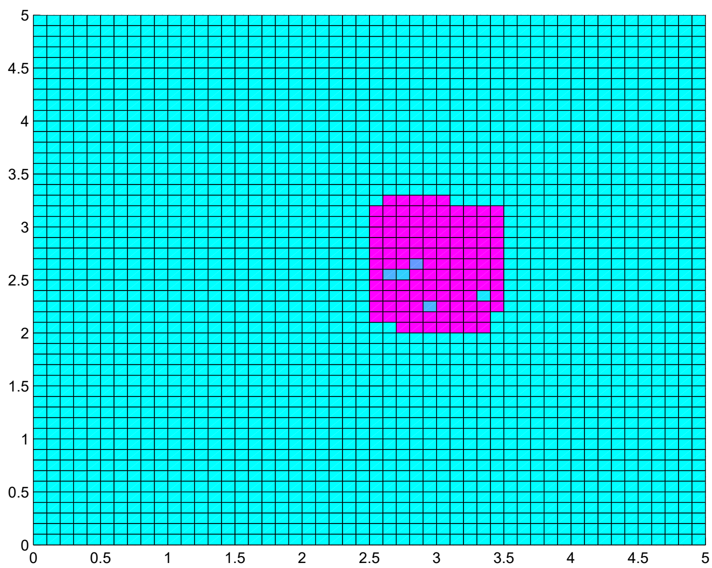

Figure 14.14: Map reconstruction with probability theory (2D)

4) Seen from the Fig. 14.12, the new tool has a better performance than just DSmT in building map, (see also [13, 14]), because of considering the conflict factor, and redistributing the conflict masses to other basic belief masses according to the PCR5.

5) Seen from the Table 14.5, Probability theory spends the least time, while Fuzzy theory spends the most time. But Map reconstruction with probability theory has very low precision and high mistaken judging rate shown in Fig. 14.14. Though the new tool spends a little more time than

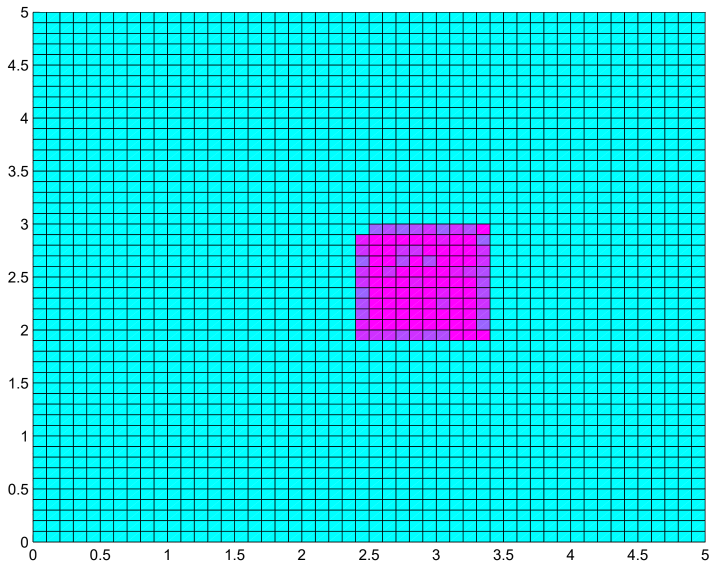

Figure 14.15: Map reconstruction with fuzzy theory (2D)

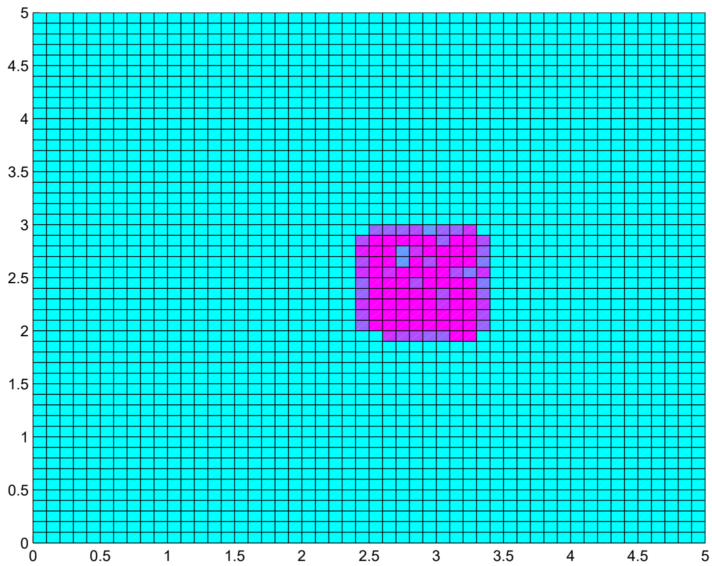

Figure 14.16: Map reconstruction with DS theory (2D)

probability theory. However, it has very high precision shown in Fig. 14.12 and Fig. 14.13 the same as the fuzzy theory and very low mistaken judging rate shown in Fig. 14.15. In fact, the comparison in map building between DST and DSmT have been made in details by us in [13]. Of course, here DST presents high precision and general mistaken rate shown in Fig. 14.16 without considering the PCR rules and other conflict redistribution rules. We don't compare the new tool from the fusion machine based on DST coupling with PCR5. Through the analysis

	Probability Theory	Fuzzy Set Theory	DST	DSmT
Total spent time (ms)	18146	35823	19673	19016
Map precision	Very low	Very high	High	Very high
Mistaken rate	High	Low	Medium	Very low

Table 14.1: Comparison of the performances of the different approaches

of comparison among the four tools in Table 14.5, we testify the new tool to play a better role in map building.

14.6 Conclusion

In this chapter, we have applied a fusion machine based on DSmT coupled with PCR5 for mobile robot's map building in a small environment. Then we have established the belief model for sonar grid map, and constructed the generalized basic belief assignment function. Through the simulation experiment, we also have compared the new tool with the other methods, and got much better performances for robot's map building. Since it is necessary to consider also robot's self-localization as soon as the size of environment becomes very large, complex, and irregular, we are currently doing some research works in Self-Localization And Mapping (SLAM) based on this new tool which improves the robustness and practicability of the fusion processing. In conclusion, our study has supplied a shortcut for human-computer interface for mobile robot exploring unknown environment and has established a firm foundation for the study of dynamic unknown environment and multi-robots' building map and SLAM together.

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Chapter 15

Reducing DSmT hybrid rule complexity through optimization of the calculation algorithm

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Abstract: *The work covered here had for objective to write a Matlab^{TM} program able to execute efficiently the DSmT hybrid rule of combination. As we know the DSmT hybrid rule of combination is highly complex to execute and requires high amounts of resources. We have introduced a novel way of understanding and treating the rule of combination and thus were able to develop a Matlab^{TM} program that would avoid the high level of complexity and resources needs.*

15.1 Introduction

The purpose of DSmT [3] was to introduce a theory that would allow to correctly fuse data, even in presence of conflicts between sources of evidence or in presence of constraints. However, as we know, the DSmT hybrid rule of combination is very complex to compute and to use in data fusion compared to other rules of combination [4]. We will show in the following sections, that there's a way to avoid the high level of complexity of DSmT hybrid rule of combination permitting to program it into Matlab$^{\text{TM}}$. An interesting fact to know is that the code developed and presented in this chapter is the first one known to the authors to be complete and functional. A partial code, useful for the calculation of the DSmT hybrid rule of combination, is presented in [3]. However, its function is to calculate complete hyper-power sets, and its execution took us over a day for $|\Theta| = 6$. This has made it impossible to have a basis for comparison of efficiency for our code, which is able to execute a complete DSmH combination in a very short period of time. We will begin by a brief review of the theory used in the subsequent sections, where there will be presented a few definitions followed by a review of Dempster-Shafer theory and its problem with mass redistribution. We will then look at Dezert-Smarandache theory and its complexity. It is followed by a section presenting the methodology used to avoid the complexity of DSmT hybrid rule of combination. We will conclude with a short performance analysis and with the developed Matlab$^{\text{TM}}$ code in appendix.

15.2 Theories

15.2.1 Definitions

A minimum of knowledge is required to understand DSmT, we'll thus begin with a short review of important concepts.

- *Frame of discernment* (Θ) : $\Theta = \{\theta_1, \theta_2, \ldots \theta_n\}$. It's the set including every possible object θ_i.

- *Power set* $\left(2^\Theta\right)$: represents the set of all possible sets using the objects of the frame of discernment Θ. It includes the empty set and excludes intersections. The power set is closed under union. With the frame of discernment defined above, we get the power set

$$2^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, \ldots \{\theta_n\}, \{\theta_1, \theta_2\}, \ldots \{\theta_1, \theta_2, \ldots \theta_n\}, \ldots, \Theta\}.$$

- *Hyper-power set* $\left(D^\Theta\right)$: represents the set of all possible sets using the objects of the frame of discernment Θ. The hyper-power sets are closed under union and intersection and includes the empty set. With the frame of discernment $\Theta = \{\theta_1, \theta_2\}$, we get the hyper-power set $D^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, \{\theta_1 \cap \theta_2\}, \{\theta_1 \cup \theta_2\}\}$.

- *Belief* $(\mathrm{Bel}(A))$: is an evaluation of the minimal level of certainty, or trust, that a set can have.

- *Plausibility* $(\mathrm{Pl}(A))$: is an evaluation of the maximal level of certainty, or trust, that a set can have.

- *Constraint* : a set considered impossible to obtain.

- *Basic belief assignment* (bba) : $m\ : 2^\Theta \to [0,1]$, so the mass given to a set $A \subseteq \Theta$ follows $m(A) \in [0,1]$.

- *Core of* Θ $(\mathcal{K})$: The set of all focal elements of Θ, where a focal element is a subset A of Θ such that $m(A) > 0$.

15.2.2 Dempster-Shafer Theory

The DST rule of combination is a conjunctive normalized rule working on the power set as described previously. It combines information with intersections, meaning that it works only with the bba's intersections. The theory also makes the hypothesis that the sources of evidence are mathematically independent. The i^{th} bba's source of evidence is denoted m_i. Equation (15.1) describes the DST rule of combination where K is the conflict. The conflict in DST is defined as in equation (15.2).

$$(m_1 \oplus m_2)(C) = \frac{1}{1-K} \sum_{A \cap B = C} m_1(A)\, m_2(B) \qquad \forall C \subseteq \Theta \tag{15.1}$$

$$K = \sum_{\substack{A, B \subseteq \Theta \\ A \cap B = \emptyset}} m_1(A)\, m_2(B) \tag{15.2}$$

15.2.2.1 DST combination example

Let's consider the case where we have an air traffic surveillance officer in charge of monitoring readings from two radars. The radars constitute our two sources of evidence. In this case, both radars display a target with the level of confidence (bba) of its probable identity. Radar 1 shows that it would be an F-16 aircraft (θ_1) with $m_1(\theta_1) = 0.50$, an F-18 aircraft (θ_2) with $m_1(\theta_2) = 0.10$, one of both with $m_1(\theta_1 \cup \theta_2) = 0.30$, or it might be another airplane (θ_3) with $m_1(\theta_3) = 0.10$. Collected data from radars 1 and 2 are shown in table 15.1. We can easily see from that table that the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ is sufficient to describe this case.

The evident contradiction between the sources causes a conflict to be resolved before interpreting the results. Considering the fact that the DST doesn't admit intersections, we'll have to discard some possible sets. Also, the air traffic surveillance officer got intelligence information recommending exclusion of the case $\{\theta_3\}$, creating a constraint on $\{\theta_3\}$. Table 15.2 represents the first step of the calculation before the redistribution of the conflicting mass.

(2^Θ)	$m_1(A)$	$m_2(A)$
$\{\theta_1\}$	0.5	0.1
$\{\theta_2\}$	0.1	0.6
$\{\theta_3\}$	0.1	0.2
$\{\theta_1, \theta_2\}$	0.3	0.1

Table 15.1: Events from two sources of evidence to combine

As we can see in table 15.2, the total mass of conflict is $\sum m(\emptyset) = 0.59$. So among all the possible sets, 0.59 of the mass is given to $\emptyset$. This would make it the most probable set. Using equation (15.1) the conflict is redistributed proportionally among focal elements. Results are given in tables 15.3 and 15.4. Finally, we can see that the most probable target identity is an F-18 aircraft.

The problem, which was predictable by analytical analysis of equation (15.1), occurs when conflict (K) get closer to 1. As K grows closer to 1, the DST rule of combination tends to give incoherent results.

15.2.3 Dezert-Smarandache Theory

Instead of the power set, used in DST, the DSmT uses the hyper-power set. DSmT is thus able to work with intersections. They also differ by their rules of combination. DSmT developed

	$m_1(\theta_1)$ 0.5	$m_1(\theta_2)$ 0.1	$m_1(\theta_3)$ 0.1	$m_1(\theta_1 \cup \theta_2)$ 0.3
$m_2(\theta_1)$ 0.1	θ_1 0.05	$\theta_1 \cap \theta_2 = \emptyset$ 0.01	$\theta_1 \cap \theta_3 = \emptyset$ 0.01	θ_1 0.03
$m_2(\theta_2)$ 0.6	$\theta_1 \cap \theta_2 = \emptyset$ 0.30	θ_2 0.06	$\theta_2 \cap \theta_3 = \emptyset$ 0.06	θ_2 0.18
$m_2(\theta_3)$ 0.2	$\theta_1 \cap \theta_3 = \emptyset$ 0.10	$\theta_2 \cap \theta_3 = \emptyset$ 0.02	$\theta_3 = \emptyset$ 0.02	$(\theta_1 \cup \theta_2) \cap \theta_3 = \emptyset$ 0.06
$m_2(\theta_1 \cup \theta_2)$ 0.1	θ_1 0.05	θ_2 0.01	$(\theta_1 \cup \theta_2) \cap \theta_3 = \emptyset$ 0.01	$\theta_1 \cup \theta_2$ 0.03

Table 15.2: Results from disjunctive combination of information from table 15.1 before mass redistribution

	$m_1(\theta_1)$ 0.5	$m_1(\theta_2)$ 0.1	$m_1(\theta_3)$ 0.1	$m_1(\theta_1 \cup \theta_2)$ 0.3
$m_2(\theta_1)$ 0.1	θ_1 $\frac{0.05}{1-0.59}$	$\emptyset$ 0	$\emptyset$ 0	θ_1 $\frac{0.03}{1-0.59}$
$m_2(\theta_2)$ 0.6	$\emptyset$ 0	θ_2 $\frac{0.06}{1-0.59}$	$\emptyset$ 0	θ_2 $\frac{0.18}{1-0.59}$
$m_2(\theta_3)$ 0.2	$\emptyset$ 0	$\emptyset$ 0	$\emptyset$ 0	$\emptyset$ 0
$m_2(\theta_1 \cup \theta_2)$ 0.1	θ_1 $\frac{0.05}{1-0.59}$	θ_2 $\frac{0.01}{1-0.59}$	$\emptyset$ 0	$\theta_1 \cup \theta_2$ $\frac{0.03}{1-0.59}$

Table 15.3: Results from disjunctive combination of information from table 15.1 with mass redistribution

in [3], possesses two[1] rules of combination which are able to work around the conflicted mass redistribution problem:

- Classic DSm rule of combination (DSmC), which is based on the free model $M^f(\Theta)$

$$m(C) = \sum_{A \cap B = C} m_1(A)\, m_2(B) \qquad A, B \in D^\Theta, \forall C \in D^\Theta \tag{15.3}$$

- Hybrid DSm rule of combination (DSmH), which is able to work with many types of constraints

[1]Actually more fusion rules based on Proportional Conflict Redistributions (PCR) have been presented in Part 1 of this book. The implementation of these new PCR rules will be presented and discussed in a forthcoming publication.

$m_{1\oplus 2}(\emptyset)$	0.000
$m_{1\oplus 2}(\theta_1)$	0.317
$m_{1\oplus 2}(\theta_2)$	0.610
$m_{1\oplus 2}(\theta_1 \cup \theta_2)$	0.073

Table 15.4: Final results for the example of DST combination

$$m_{M(\Theta)}(A) = \phi(A)\left[S_1(A) + S_2(A) + S_3(A)\right] \tag{15.4}$$

$$S_1(A) = \sum_{X_1 \cap X_2 = A} m_1(X_1)\, m_2(X_2) \qquad \forall X_1, X_2 \in D^\Theta \tag{15.5}$$

$$S_2(A) = \sum_{[(u(X_1)\cup u(X_2))=A]\vee[((u(X_1)\cup u(X_2))\in\emptyset)\wedge(A=I_t)]} m_1(X_1)\, m_2(X_2) \qquad \forall X_1, X_2 \in \emptyset \tag{15.6}$$

$$S_3(A) = \sum_{X_1 \cup X_2 = A} m_1(X_1)\, m_2(X_2) \qquad \forall X_1, X_2 \in D^\Theta \quad \text{and} \quad X_1 \cap X_2 \in \emptyset \tag{15.7}$$

Note that $\phi(A)$ in equation (15.4) is a binary function resulting in 0 for empty or impossible sets and in 1 for focal elements. In equation (15.6), $u(X)$ represents the union of all objects of set X. Careful analysis of equation (15.7) tells us that it's the union of all objects of sets X_1 and X_2, when it is not empty. Finally, also from equation (15.6), I_t represents the total ignorance, or the union of all objects part of the frame of discernment. Further information on how to understand and proceed in the calculation of DSmH is available in subsequent sections.

15.2.3.1 DSmC combination example

This example cannot be resolved by DST because of highly conflicting sources of evidence (K tends toward 1). Sources' information shown in table 15.5 gives us, with DSmC, the results displayed in table 15.6. As we can see, no mass is associated to an empty set since DSmC is based on free DSm model which does not allow integrity constraints (by definition). Final results for the present example, given by table 15.7, tell us that the most probable identity of the target to identify is an hybrid of objects of type θ_1 and θ_2.

15.3 How to avoid the complexity

15.3.1 Simpler way to view the DSmT hybrid rule of combination

First of all, one simple thing to do in order to keep the use of resources at low levels is to keep only the useful data. For example, table 15.6 shouldn't be entered as is in a program but reduced to the equivalent table 15.8. This way, the only allocated space to execute the calculation is the data space we actually need.

(D^{Θ})	$m_1(A)$	$m_2(A)$
$\{\theta_1\}$	0.8	0.0
$\{\theta_2\}$	0.0	0.9
$\{\theta_3\}$	0.2	0.1
$\{\theta_1, \theta_2\}$	0.0	0.0

Table 15.5: Events from two sources of evidence to combine

	$m_1(\theta_1)$ 0.8	$m_1(\theta_2)$ 0.0	$m_1(\theta_3)$ 0.2	$m_1(\theta_1 \cup \theta_2)$ 0.0
$m_2(\theta_1)$ 0.0	θ_1 0.00	$\theta_1 \cap \theta_2$ 0.00	$\theta_1 \cap \theta_3$ 0.00	θ_1 0.00
$m_2(\theta_2)$ 0.9	$\theta_1 \cap \theta_2$ 0.72	θ_2 0.00	$\theta_1 \cap \theta_3$ 0.18	θ_2 0.00
$m_2(\theta_3)$ 0.1	$\theta_1 \cap \theta_3$ 0.08	$\theta_2 \cap \theta_3$ 0.00	θ_3 0.02	$(\theta_1 \cup \theta_2) \cap \theta_3$ 0.00
$m_2(\theta_1 \cup \theta_2)$ 0.0	θ_1 0.00	θ_2 0.00	$(\theta_1 \cup \theta_2) \cap \theta_3$ 0.00	$\theta_1 \cup \theta_2$ 0.00

Table 15.6: Results from DSmC rule of combination with table 15.1 information

(θ_1)	0.00	$(\theta_1 \cap \theta_2)$	0.72	$(\theta_1 \cup \theta_2)$	0.00
(θ_2)	0.00	$(\theta_1 \cap \theta_3)$	0.08	$(\theta_1 \cup \theta_3)$	0.00
(θ_3)	0.02	$(\theta_2 \cap \theta_3)$	0.18	$(\theta_2 \cup \theta_3)$	0.00
$(\emptyset)$	0.00	$(\theta_1 \cap \theta_2) \cup \theta_3$	0.00	$(\theta_1 \cup \theta_2) \cap \theta_3$	0.00
$(\theta_1 \cup \theta_2 \cup \theta_3)$	0.00	$(\theta_1 \cap \theta_3) \cup \theta_2$	0.00	$(\theta_1 \cup \theta_3) \cap \theta_2$	0.00
$(\theta_1 \cap \theta_2 \cap \theta_3)$	0.00	$(\theta_2 \cap \theta_3) \cup \theta_1$	0.00	$(\theta_2 \cup \theta_3) \cap \theta_1$	0.00

Table 15.7: Final results for the example of DSmC combination ($m_{1\oplus 2}$)

This is especially important for full explicit calculation of equation (15.4). As the number of possible objects and/or the number of possible sources of evidence grows, we would avoid extraordinary increase in resources needs (since the increase follows Dedekind's sequence progression in the worst case [3]).

15.3.1.1 Simple procedure for effective DSmH

Instead of viewing DSmH as a mathematical equation, we propose to view it as a procedure. Table 15.9 displays that procedure. Obviously, it is still equivalent to the mathematical equation, but this way has the advantage of being very easily understood and implemented. The ease of its implementation is due to the high resemblance of the procedure to pseudo-code, a common step in software engineering.

	$m_1(\theta_1)$ 0.8	$m_1(\theta_3)$ 0.2
$m_2(\theta_2)$ 0.9	$\theta_1 \cap \theta_2$ 0.72	$\theta_1 \cap \theta_3$ 0.18
$m_2(\theta_3)$ 0.1	$\theta_1 \cap \theta_3$ 0.08	θ_3 0.02

Table 15.8: Reduced version of table 15.6

Step S1 $(\theta_1 \cap \theta_2)$	If $(\theta_1 \cap \theta_2)$ is a constraint, then continue at step S3, otherwise, the mass $m_1(X_1)\,m_2(X_2)$ is added to the mass $A = (\theta_1 \cap \theta_2)$.
Step S3 $(\theta_1 \cup \theta_2)$	If $(\theta_1 \cup \theta_2)$ is a constraint, then continue at step S2, otherwise, the mass $m_1(X_1)\,m_2(X_2)$ is added to the mass $A = (\theta_1 \cup \theta_2)$.
Step S2 $(u(X_1) \cup u(X_2))$	If $(u(X_1) \cup u(X_2))$ is a constraint, then add mass to I_t, otherwise, the mass $m_1(X_1)\,m_2(X_2)$ is added to the mass $A = (u(X_1) \cup u(X_2))$.

Table 15.9: Procedure to apply to each pair of sets (X_1, X_2) until its combined mass is given to a set

15.3.2 Notation system used

15.3.2.1 Sum of products

The system we conceived treats information in terms of *union of intersections* or *sum of products*. The sum (ADD) is being represented by union ($\cup$), and the product (MULT) by intersection ($\cap$). We have chosen this, instead of *product of sums*, to avoid having to treat parenthesis. We could also use the principles developed for logic circuits such as Karnaugh table, Boolean rules, etc. Here are few examples of this notation:

- $\theta_1 \cap \theta_2 \cap \theta_3 = \theta_1\theta_2\theta_3 = [1, MULT, 2, MULT, 3]$
- $\theta_1 \cup \theta_2 \cup \theta_3 = \theta_1 + \theta_2 + \theta_3 = [1, ADD, 2, ADD, 3]$
- $(\theta_1 \cap \theta_2) \cup \theta_3 = \theta_1\theta_2 + \theta_3 = [1, MULT, 2, ADD, 3] = [3, ADD, 1, MULT, 2]$
- $(\theta_1 \cup \theta_2) \cap \theta_3 = (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) = \theta_1\theta_3 + \theta_2\theta_3 = [1, MULT, 3, ADD, 2, MULT, 3]$
- $(\theta_1 \cap \theta_2 \cap \theta_3) \cup (\theta_4 \cap \theta_5) = \theta_1\theta_2\theta_3 + \theta_4\theta_5 = [1, MULT, 2, MULT, 3, ADD, 4, MULT, 5]$

15.3.2.2 Conversion between *sum of products* and *product of sums* notation

As we have seen above, we will use the *sum of products* as our main way of writing sets. However, as we will later see, we will need to use the *product of sums* or intersections of unions in some parts of our system to simplify the calculation process. More specifically, this dual system of notation, introduced in the last two columns of table 15.10, was done so we would be able to use the same algorithm to work with the matrix of unions and the matrix of intersections. Table 15.10 thus presents the notation used, accompanied with its equivalent mathematical notation. We can see in the *sum of products* notation in table 15.10, that a line represents a monomial of product type (e.g. $\theta_1\theta_3$) and that lines are then summed to get unions (e.g. $\theta_1\theta_3 + \theta_2$). In the *product of sums* notation, we have the reversed situation where lines represents a monomial of sum type (e.g. $\theta_1 + \theta_3$) and that lines are then multiplied to get intersections (e.g. $\theta_2 (\theta_1 + \theta_3)$).

Mathematical	Matlab input/output	Sum of products	Product of sums
$\{\theta_1\}$	$[1]$	$\begin{bmatrix} 1 \end{bmatrix}$	$\begin{bmatrix} 1 \end{bmatrix}$
$\{\theta_1 \cup \theta_2\}$	$[1, ADD, 2]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \end{bmatrix}$
$\{\theta_1 \cap \theta_2\}$	$[1, MULT, 2]$	$\begin{bmatrix} 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$\{(\theta_2) \cup (\theta_1 \cap \theta_3)\}$	$[2, ADD, 1, MULT, 3]$	$\begin{bmatrix} 2 & \\ 1 & 3 \end{bmatrix}$	–
$\{(\theta_1 \cup \theta_2) \cap (\theta_2 \cup \theta_3)\}$	–	–	$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$
$\{(\theta_1 \cap \theta_2) \cup (\theta_2 \cap \theta_3)\}$	$[1, MULT, 2, ADD, 2, MULT, 3]$	$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$	–
$\{(\theta_2) \cap (\theta_1 \cup \theta_3)\}$	–	–	$\begin{bmatrix} 2 & \\ 1 & 3 \end{bmatrix}$

Table 15.10: Equivalent notations for events

The difficult part is the conversion step from the *sum of products* to the *product of sums* notation. For the simple cases, such as the ones presented in the first three lines of table 15.10 consist only in changing matrices lines into columns and columns into lines. For simplification in the conversion process we also use the absorption rule as described in equation (15.8) which is derived from the fact that $(\theta_1\theta_2) \subseteq \theta_1$. Using that rule, we can see how came the two last rows of table 15.10 by looking at the process detailed in equations (15.9) and (15.10).

$$\theta_1 + \theta_1\theta_2 = \theta_1 \tag{15.8}$$

$$(\theta_1 \cup \theta_2) \cap (\theta_2 \cup \theta_3) = (\theta_1 + \theta_2)(\theta_2 + \theta_3) = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2 + \theta_2\theta_3 = \theta_1\theta_3 + \theta_2 \tag{15.9}$$

$$(\theta_1 \cap \theta_2) \cup (\theta_2 \cap \theta_3) = \theta_1\theta_2 + \theta_2\theta_3 = \theta_2 (\theta_1 + \theta_3) \tag{15.10}$$

However, in the programmed Matlab$^{\text{TM}}$ code, the following procedure is used and works for any case. It's based on the use of DeMorgan's laws as seen in equations (15.11) and (15.12). Going through DeMorgan twice let's us avoid the use of negative sets. Hence, we will still respect DSm theory even with the use of this mathematical law. The use of absorption rule, as described in equation (15.8) also helps us achieve better simplification.

$$\overline{A}\,\overline{B} = \overline{A + B} \tag{15.11}$$

$$\overline{A} + \overline{B} = \overline{A\,B} \tag{15.12}$$

Here's how we proceed for the case of conversion from a set in *sum of products* to a set in *product of sums* notation. It's quite simple actually, we begin with an inversion of operators (changing additions ($\cup$) for multiplications ($\cap$) and multiplications for additions), followed by distribution of products and a simplification step. We then end it with a second inversion of operators. Since we have used the inversion two times, we don't have to indicate the not operator, ($\overline{\overline{A}} = A$).

Let's now proceed with a short example. Suppose we want to convert the set shown in equation (15.13) to a set in product of sums notation. We proceed first as said, with an inversion of operators, which gives the set in equation (15.14). We then distribute the multiplication as we did to get the set in equation (15.15). This is then followed by a simplification giving us equation (15.16), and a final inversion of operators gives us the set in equation (15.17). The set in equation (15.17) represents the *product of sums* notation version of the set in equation (15.13), which is in *sum of products*. A simple distribution of products and simplification can get us back from (15.17) to (15.13).

$$\theta_1 + \theta_2\theta_3 + \theta_2\theta_4 \tag{15.13}$$

$$\overline{\overline{(\theta_1)}\ \left(\overline{\theta_2} + \overline{\theta_3}\right)\ \left(\overline{\theta_2} + \overline{\theta_4}\right)} \tag{15.14}$$

$$\overline{\overline{\theta_1}\,\overline{\theta_2} + \overline{\theta_1}\,\overline{\theta_2}\,\overline{\theta_4} + \overline{\theta_1}\,\overline{\theta_2}\,\overline{\theta_3} + \overline{\theta_1}\,\overline{\theta_3}\,\overline{\theta_4}} \tag{15.15}$$

$$\overline{\overline{\theta_1}\,\overline{\theta_2} + \overline{\theta_1}\,\overline{\theta_3}\,\overline{\theta_4}} \tag{15.16}$$

$$(\theta_1 + \theta_2)\,(\theta_1 + \theta_3 + \theta_4) \tag{15.17}$$

15.3.3 How simple can it be

We have completed conception of a Matlab$^{\mathrm{TM}}$ code for the dynamic case. We've tried to optimize the code but some work is still necessary. It's now operational for a restricted body of evidence and well behaved. Here's an example of the input required by the system with the events from table 15.11. We will also proceed with θ_2 as a constraint making the following constraints too:

- $\theta_1 \cap \theta_2 \cap \theta_3$
- $\theta_1 \cap \theta_2$
- $\theta_2 \cap \theta_3$
- $(\theta_1 \cup \theta_3) \cap \theta_2$

Note that having θ_2 as a constraint, has an impact on more cases than the enumerated ones above. In fact, if we obtain cases like $\theta_1 \cup \theta_2$ for instance, since θ_2 is a constraint, the resulting case would then be θ_1. We will have to consider this when evaluating final bba for the result.

As we can see, only focal elements are transmitted to and received from the system. Moreover, these focal elements are all in *sum of products*. The output also include Belief and Plausibility values of the result.

Notice also that we have dynamic constraints capability, meaning that we can put constraints on each step of the combination. They can also differ at each step of the calculation. Instead of considering constraints only at the final step of combination, this system is thus able to

(D^Θ)	$m_1(A)$	$m_2(A)$	$m_3(A)$
$\{\theta_1\}$	0.7	0.0	0.1
$\{\theta_2\}$	0.0	0.6	0.1
$\{\theta_3\}$	0.2	0.0	0.5
$\{\theta_1 \cup \theta_2\}$	0.0	0.0	0.3
$\{\theta_1 \cup \theta_3\}$	0.0	0.2	0.0
$\{\theta_2 \cup \theta_3\}$	0.0	0.2	0.0
$\{\theta_2 \cap \theta_3\}$	0.1	0.0	0.0

Table 15.11: Information from three sources

reproduce real data fusion conditions where constraints may vary. Three different cases are presented here, keeping the same input information but varying the constraints conditions.

Complete commented listing of the produced Matlab$^{\text{TM}}$ code is available in the appendix. For the present section, only the parameters required in input and the output are displayed.

```
% Example with dynamic constraints kept stable
% INPUT FOR THE MATLAB PROGRAM

number_sources    = 3;   kind    = ['dynamic'];
info(1).elements = {[1],[3], [2, MULT, 3]};  info(1).masses   = [0.7, 0.2, 0.1];
info(2).elements = {[2],[1, ADD, 3], [2, ADD, 3]};  info(2).masses   = [0.6, 0.2, 0.2];
info(3).elements = {[1], [2], [3], [1, ADD, 2]};  info(3).masses   = [0.1, 0.1, 0.5, 0.3];
constraint{1}    = {[2], [1, MULT, 2], [2, MULT, 3],...
                   [1, MULT, 2, ADD, 3, MULT, 2], [1, MULT, 2, MULT, 3]};
constraint{2}    = {[2], [1, MULT, 2], [2, MULT, 3],...
                   [1, MULT, 2, ADD, 2, MULT, 3], [1, MULT, 2, MULT, 3]};

% OUTPUT OF THE MATLAB PROGRAM

DSm hybrid                    Plausibility              Belief
1 : m=0.28800000               1 : m=1.00000000          1 : m=0.82000000
1 MULT 3 : m=0.53200000        1 MULT 3 : m=1.00000000   1 MULT 3 : m=0.53200000
3 : m=0.17800000               3 : m=1.00000000          3 : m=0.71000000
1 ADD 3 : m=0.00200000         1 ADD 3 : m=1.00000000    1 ADD 3 : m=1.00000000

% Example with dynamic constraints applied only once at the end
% CONSTRAINTS INPUT FOR THE MATLAB PROGRAM

constraint{1}    = {};
constraint{2}    = {[2], [1, MULT, 2], [2, MULT, 3],...
                   [1, MULT, 2, ADD, 2, MULT, 3], [1, MULT, 2, MULT, 3]};

% OUTPUT OF THE MATLAB PROGRAM

DSm hybrid                    Plausibility              Belief
1 : m=0.36800000               1 : m=1.00000000          1 : m=0.61000000
1 MULT 3 : m=0.24200000        1 MULT 3 : m=1.00000000   1 MULT 3 : m=0.24200000
3 : m=0.39000000               3 : m=1.00000000          3 : m=0.63200000

% Example with dynamic constraints varying between steps of calculation
% CONSTRAINTS INPUT FOR THE MATLAB PROGRAM
```

```
constraint{1}    = {[2, MULT, 3], [2, ADD, 3]};
constraint{2}    = {[2], [1, MULT, 2], [2, MULT, 3],...
                  [1, MULT, 2, ADD, 2, MULT, 3], [1, MULT, 2, MULT, 3]};

% OUTPUT OF THE MATLAB PROGRAM

DSm hybrid                    Plausibility               Belief
1 : m=0.31200000               1 : m=1.00000000           1 : m=0.55400000
1 ADD 3 : m=0.14800000         1 ADD 3 : m=1.00000000     1 ADD 3 : m=1.00000000
1 MULT 3 : m=0.24200000        1 MULT 3 : m=1.00000000    1 MULT 3 : m=0.24200000
3 : m=0.29800000               3 : m=1.00000000           3 : m=0.54000000
```

15.3.4 Optimization in the calculation algorithm

15.3.4.1 How does it work

Being treated by a vectorial interpreter, our Matlab$^{\mathrm{TM}}$ code had to be adapted in consequence. We have also been avoiding, as much as we could, the use of `for` and `while` loops.

Our Matlab$^{\mathrm{TM}}$ code was conceived with two main matrices, one containing intersections, the other one containing unions. The input information is placed into a matrix identified as the fusion matrix. When building this matrix, our program puts in a vector each unique objects that will be used, hence defining total ignorance (I_t) for the case in input. Each elements of this matrix is a structure having two fields: sets and masses. Note also that only the first row and column of the matrix is filled with the input information. The rest of the matrix will contain the result.

It is easier to proceed with the intersection between two sets A and B using *product of sums* and to proceed with the union $A \cup B$ using *sum of products*. Because of that, we have chosen to keep the intersection matrix in the *product of sums* notation and the union matrix in the *sum of products* while working on these matrices separately.

To build the union matrix, we use information from the fusion matrix with the *sum of products* notation. The intersection matrix uses the *product of sums* notation for its construction with the information from the fusion matrix. However, once the intersection matrix is built, a simple conversion to the *sum of products* notation is done as we have described earlier. This way, data from this table can be compatible with those from the fusion and the union matrices.

Once the basis of the union matrix is defined, a calculation of the content is done by evaluating the result of the union of focal elements combination $m_1(X_i)\, m_2(X_j)$. The equivalent is done with the intersection matrix, replacing the union with an intersection obviously. Once the calculation of the content of the intersection matrix completed, it is converted to the sum of product notation.

The next step consist to fill up the fusion matrix with the appropriate information depending on the presence of constraints and following the procedure described earlier for the calculation of the DSmH combination rule.

In the case we want to fuse information from more than two sources, we could choose to fuse the information dynamically or statically. The first case is being done by fusing two sources at a time. The latter case considers information from all sources at once. Note however that our code is only able to proceed with the calculation dynamically for the time being. We will now proceed step by step with a full example, interlaced with explanations on the procedure, using the information from table 15.11 and the constraints described in section 15.3.3.

Table 15.12 gives us the union result from each combination of focal elements from the first two sources of evidence. The notation used in the case for union matrices is the *sum of products*. In the case of table 15.13, the intersection matrix, it is first built in the *product of sums* notation so the same calculation algorithm can be used to evaluate the intersection result from each combination of focal elements from the first two sources of evidence as it was used in the union matrix. As we'll see, a conversion to the *sum of products* notation is done to be able to obtain table 15.14.

m_1 \ m_2	$\begin{bmatrix} 2 \end{bmatrix}$ 0.6	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.2
$\begin{bmatrix} 1 \end{bmatrix}$ 0.7	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.42	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.14
$\begin{bmatrix} 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.12	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.04	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.04
$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \end{bmatrix}$ 0.06	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.02

Table 15.12: Union matrix with bba's m_1, m_2 information from table 15.11 in *sum of products* notation

m_1 \ m_2	$\begin{bmatrix} 2 \end{bmatrix}$ 0.6	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.2
$\begin{bmatrix} 1 \end{bmatrix}$ 0.7	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.42	$\begin{bmatrix} 1 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & \\ 2 & 3 \end{bmatrix}$ 0.14
$\begin{bmatrix} 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.12	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04
$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.06	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.02

Table 15.13: Intersection matrix with bba's m_1, m_2 information from table 15.11 in *product of sums* notation

We obtained $\theta_2\theta_3$ as a result in the second result cell in the last row of table 15.13 because the intersection $(\theta_2 \cdot \theta_3) \cdot (\theta_1 + \theta_3)$ gives us $\theta_1\theta_2\theta_3 + \theta_2\theta_3$ which, following absorption rule, gives us $\theta_2\theta_3$. The same process occurs on the second result cell in the first row of the same table where $\theta_1 \cdot (\theta_1 + \theta_3) = \theta_1 + \theta_1\theta_3 = \theta_1$.

m_1 \ m_2	$\begin{bmatrix} 2 \end{bmatrix}$ 0.6	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.2
$\begin{bmatrix} 1 \end{bmatrix}$ 0.7	$\begin{bmatrix} 1 & 2 \end{bmatrix}$ 0.42	$\begin{bmatrix} 1 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ 0.14
$\begin{bmatrix} 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.12	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04
$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.06	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.02

Table 15.14: Intersection matrix with bba's m_1, m_2 information from table 15.11 in *sum of products* notation

From the tables 15.12 and 15.14 we proceed with the DSmH and choose, according to constraints, from which table the result will come. We might also have to evaluate $(u(X_1) \cup u(X_2))$, or give the mass to the total ignorance if the intersection and union matrices' sets are constrained. We've displayed the choice made in the fusion matrix in table 15.15 with these symbols $\cap$ (intersection), $\cup$ (union), **u** (union of the sum of objects of combined sets), **It** (total ignorance). As you will see, we have chosen a case where we have constraints applied at each step of combination, e.g. when $[m_1, m_2]$ and when $[m_1 \oplus m_2, m_3]$ are combined.

Table 15.16 is the simplified version of table 15.15 in which sets has been adapted to consider constraints. It's followed by table 15.17 which represents the results from the first combination.

As we can see in table 15.15, the first result cell from the first row was obtained from the union matrix because $\theta_1 \cap \theta_2$ is a constraint. Also, the first result cell from the last row was obtained from the union of the sum of objects of the combined sets because $\theta_2 \cap \theta_3$ is a constraint in the intersection table (table 15.14) at the same position, so is θ_2 in the union table (table 15.12).

m_1 \ m_2	$\begin{bmatrix} 2 \end{bmatrix}$ 0.6	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.2
$\begin{bmatrix} 1 \end{bmatrix}$ 0.7	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.42 $\cup$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.14 $\cap$	$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ 0.14 $\cap$
$\begin{bmatrix} 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.12 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04 $\cap$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04 $\cap$
$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.06 **u**	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02 $\cup$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.02 $\cup$

Table 15.15: Fusion matrix with bba's m_1, m_2 information from table 15.11 in *sum of products* notation

m_1 \ m_2	$\begin{bmatrix} 2 \end{bmatrix}$ 0.6	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.2
$\begin{bmatrix} 1 \end{bmatrix}$ 0.7	$\begin{bmatrix} 1 \end{bmatrix}$ 0.42 $\cup$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.14 $\cap$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.14 $\cap$
$\begin{bmatrix} 3 \end{bmatrix}$ 0.2	$\begin{bmatrix} 3 \end{bmatrix}$ 0.12 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04 $\cap$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.04 $\cap$
$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \end{bmatrix}$ 0.06 **u**	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.02 $\cup$

Table 15.16: Simplified fusion matrix with bba's m_1, m_2 information from table 15.11 in *sum of products* notation

On the first row of table 15.16, the first result giving us θ_1 is obtained because $\theta_1 \cup \theta_2 = \theta_1$ when θ_2 is a constraint. The same process gave us $\theta_1 \cap \theta_3$ in the last cell of the first row. In that case, we obtained that result having $\theta_1 \cap \theta_2$ as a constraint where $\theta_1\theta_2 + \theta_1\theta_3 = \theta_1\theta_3$. Since we have more than two sources and have chosen a dynamic methodology: once the first two sources combined, we will have to proceed with a second combination. This time, we combine the results from the first combination $m_1 \oplus m_2$ with the third event from source of evidence m_3.

[1]	[3]	$[1 \cup 3]$	$[1 \cap 3]$
0.56	0.28	0.02	0.14

Table 15.17: Result of the combination of m_1 and m_2 from table 15.11

Table 15.18 represents the union matrix from second combination.

$m_1 \oplus m_2$ \ m_3	$\begin{bmatrix} 1 \\ \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \\ \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \\ \end{bmatrix}$ 0.5	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.3
$\begin{bmatrix} 1 \\ \end{bmatrix}$ 0.56	$\begin{bmatrix} 1 \\ \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.168
$\begin{bmatrix} 3 \\ \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 3 \\ \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.084
$\begin{bmatrix} 1 & 3 \\ & \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 \\ \end{bmatrix}$ 0.014	$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix}$ 0.014	$\begin{bmatrix} 3 \\ \end{bmatrix}$ 0.07	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.042
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.002	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.002	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.01	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.006

Table 15.18: Union matrix with bba's $m_1 \oplus m_2, m_3$ information comes from tables 15.11 and 15.15 in *sum of products* notation

Table 15.19 and 15.20 are the intersection matrix with *product of sums* and *sum of products* notation respectively.

$m_1 \oplus m_2$ \ m_3	$\begin{bmatrix} 1 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \end{bmatrix}$ 0.5	$\begin{bmatrix} 1 & 2 \end{bmatrix}$ 0.3
$\begin{bmatrix} 1 \end{bmatrix}$ 0.56	$\begin{bmatrix} 1 \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 \end{bmatrix}$ 0.168
$\begin{bmatrix} 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix}$ 0.084
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.014	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.014	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.07	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.042
$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 1 \end{bmatrix}$ 0.002	$\begin{bmatrix} 2 & \\ 1 & 3 \end{bmatrix}$ 0.002	$\begin{bmatrix} 3 \end{bmatrix}$ 0.01	$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ 0.006

Table 15.19: Intersection matrix with bba's $m_1 \oplus m_2, m_3$ information from tables 15.11 and 15.15 in *product of sums* notation

$m_1 \oplus m_2$ \ m_3	$\begin{bmatrix} 1 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \end{bmatrix}$ 0.5	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.3
$\begin{bmatrix} 1 \end{bmatrix}$ 0.56	$\begin{bmatrix} 1 \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 & 2 \end{bmatrix}$ 0.056	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 \end{bmatrix}$ 0.168
$\begin{bmatrix} 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 2 & 3 \end{bmatrix}$ 0.028	$\begin{bmatrix} 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$ 0.084
$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.014	$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 0.014	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.07	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.042
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 1 \end{bmatrix}$ 0.002	$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ 0.002	$\begin{bmatrix} 3 \end{bmatrix}$ 0.01	$\begin{bmatrix} 1 & \\ 2 & 3 \end{bmatrix}$ 0.006

Table 15.20: Intersection matrix with bba's $m_1 \oplus m_2, m_3$ information from tables 15.11 and 15.15 in *sum of products* notation

Finally we get table 15.21 which consists of the final fusion matrix, table 15.22 which is a simplified version of 15.21, and table 15.23 which compiles equivalent results giving us the result of the DSmH for the information from table 15.11 with same constraints applied at each step of combination.

$m_1 \oplus m_2$ \ m_3	$\begin{bmatrix} 1 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \end{bmatrix}$ 0.5	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.3
$\begin{bmatrix} 1 \end{bmatrix}$ 0.56	$\begin{bmatrix} 1 \end{bmatrix}$ 0.056 $\cap$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.056 $\cup$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.28 $\cap$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.168 $\cap$
$\begin{bmatrix} 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.028 $\cap$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ 0.028 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.14 $\cap$	$\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$ 0.084 $\cap$
$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.014 $\cap$	$\begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix}$ 0.014 $\cup$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.07 $\cap$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.042 $\cap$
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 1 \end{bmatrix}$ 0.002 $\cap$	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 0.002 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.01 $\cap$	$\begin{bmatrix} 1 & \\ 2 & 3 \end{bmatrix}$ 0.006 $\cap$

Table 15.21: Fusion matrix with bba's $m_1 \oplus m_2, m_3$ information from table 15.11 and 15.15 in *sum of products* notation

15.3.5 Performances analysis

Since no other implementation of DSmT on D^Θ is known, we don't have the possibility of comparing it. However, we are able to track the evolution of the execution time with the growth in the number of objects or the number of sources. The same can be done with the memory requirement. Until another implementation of the DSmH is written, it is the only pertinent feasible performances analysis. The program usually gives us as output the DSmH calculation results with plausibility and belief values. However, the tests we have realized were done on the DSmH alone. The code, which can be found in the appendix, had also to be modified to output time and size of variables which can undoubtedly affect time of execution and probably size required by the program.

For the measurement of the time of execution, we have only used the `tic toc` Matlab^{TM} command between each tested cases. The `clear` command, which clears variables values, was also used to prevent Matlab^{TM} from altering execution time by using already existing variables.

$m_1 \oplus m_2$ \ m_3	$\begin{bmatrix} 1 \end{bmatrix}$ 0.1	$\begin{bmatrix} 2 \end{bmatrix}$ 0.1	$\begin{bmatrix} 3 \end{bmatrix}$ 0.5	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 0.3
$\begin{bmatrix} 1 \end{bmatrix}$ 0.56	$\begin{bmatrix} 1 \end{bmatrix}$ 0.056 $\cap$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.056 $\cup$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.28 $\cap$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.168 $\cap$
$\begin{bmatrix} 3 \end{bmatrix}$ 0.28	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.028 $\cap$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.028 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.14 $\cap$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.084 $\cap$
$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.14	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.014 $\cap$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.014 $\cup$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.07 $\cap$	$\begin{bmatrix} 1 & 3 \end{bmatrix}$ 0.042 $\cap$
$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.02	$\begin{bmatrix} 1 \end{bmatrix}$ 0.002 $\cap$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 0.002 $\cup$	$\begin{bmatrix} 3 \end{bmatrix}$ 0.01 $\cap$	$\begin{bmatrix} 1 \end{bmatrix}$ 0.006 $\cap$

Table 15.22: Simplified fusion matrix version of table 15.21 in *sum of products* notation

[1]	[3]	$[1 \cap 3]$	$[1 \cup 3]$
0.288	0.178	0.532	0.002

Table 15.23: Final result of DSmH for information from table 15.11

For the size of variable measurements, we have used the `whos` command at the end of the file `hybrid.m`. The program is divided into 22 files, however the main variables are contained in `hybrid.m`. Most of the functions of the programmed system calls very few other functions one into another. We also assume that once Matlab^{TM} leaves a function, it destroys all of its variables. We considered hence the memory size values obtained within `hybrid.m` a good lower estimate of the required memory size.

Note also that the tests were done on a Toshiba Satellite Pro 6100 station which has a Pentium M 4 running at 1.69 GHz, 2x512 MB of RAM PC2700, and an 80 GB hard drive running at 7200 rpm.

15.3.5.1 Execution time vs $|\Theta|$

Figure (15.1) shows us evolution of the execution time versus the cardinality of Θ for $|\Theta|$ going from 3 to 9. Since there are large number of possible testing parameters, we have chosen to perform the tests in a specific case. It consists of measuring the evolution of the execution time versus $|\Theta|$ while keeping the number of sources to 5 with the same information provided by each source for each point. Each source gives a bba with only six focal elements ($|\mathcal{K}| = 6$).

We have chosen also to put only six constraints on each point. Moreover, the constraints are dynamical and applied at each step of combination. As we can see on figure (15.1), time evolves exponentially with $|\Theta|$.

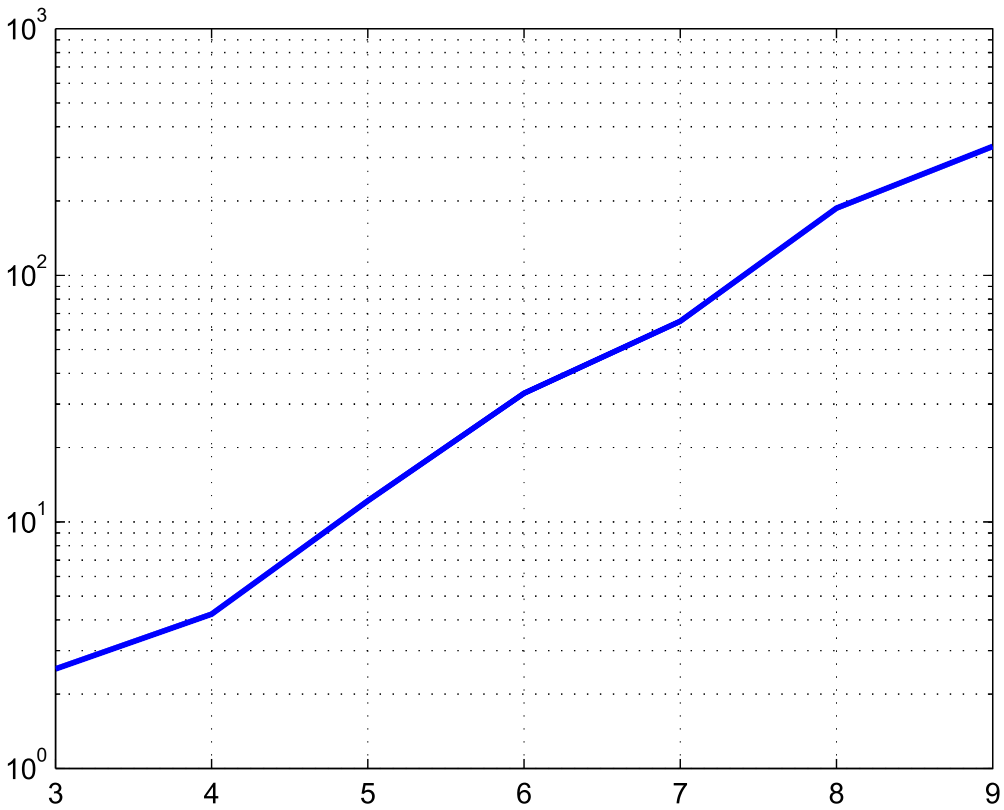

Figure 15.1: Evolution of execution time (sec) vs the cardinality of Θ

15.3.5.2 Execution time vs Number of sources

Figure (15.2) shows us the evolution of the execution time versus the number of sources going from 3 to 9. Since there are large number of possible testing parameters, we have chosen to perform the tests in a specific case. It consists of measuring the evolution of the execution time versus the number of sources while keeping $|\Theta|$ to 5 with information varying for each source for each point.

Each source gives a bba with only six focal elements ($|\mathcal{K}| = 6$). We have chosen also to put only six constraints on each point; moreover, the constraints are dynamical and applied at each step of combination. As we can see on figure (15.2), time also evolves exponentially with the number of sources.

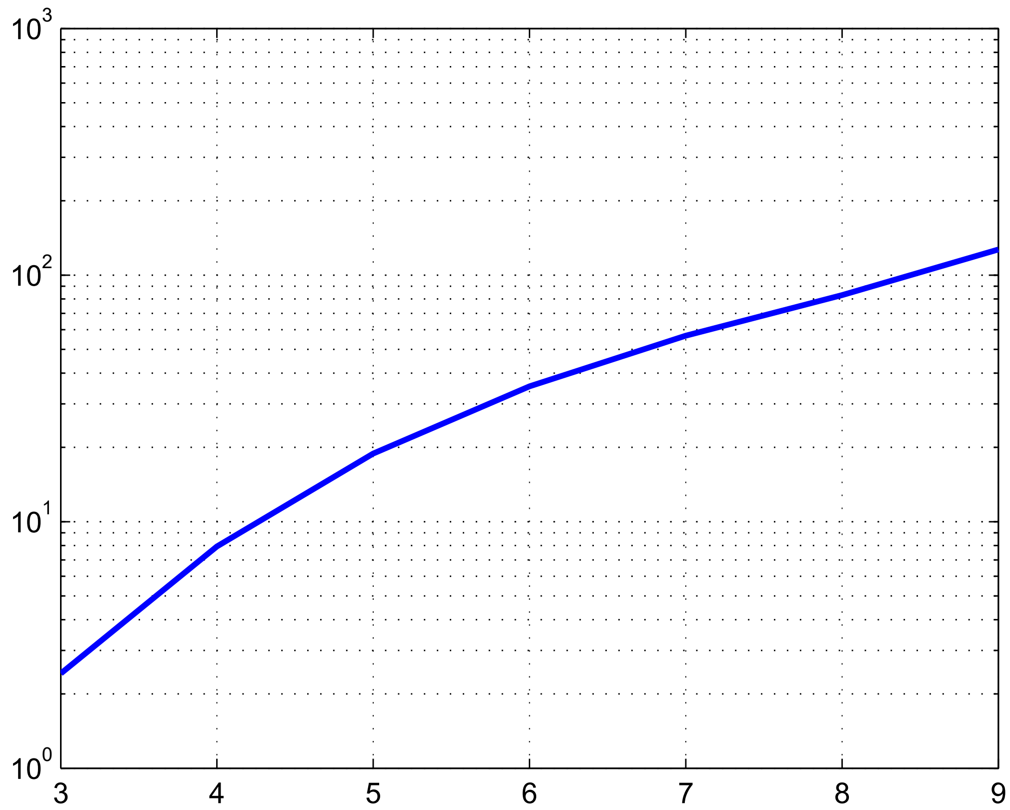

Figure 15.2: Evolution of execution time (sec) vs the number of sources

15.3.5.3 Execution time vs $|\mathcal{K}|$

Figure (15.2) shows us evolution of the execution time versus the core dimension or the number of non-zero masses going from 3 to 9. In this case, we have chosen to perform the tests while keeping $|\Theta|$ to 3 with a fixed number of sources of 5. We have chosen also to put only three constraints on each step of combination. As we can see on figure (15.3), time evolves almost linearly with the core dimension.

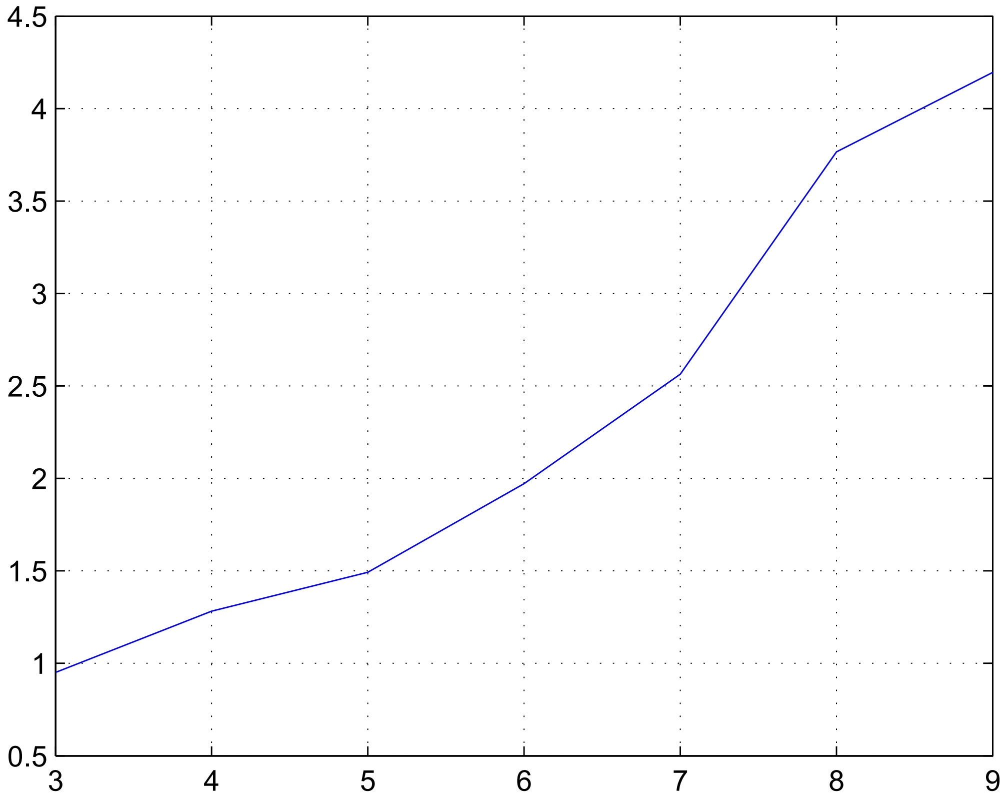

Figure 15.3: Evolution of execution time (sec) vs the cardinality of $\mathcal{K}$

15.3.5.4 Memory size vs the number of sources or $|\Theta|$

Figure (15.4) was realized under the same conditions as the input conditions for the execution time performance tests. We note that even with an increasing memory requirement, memory needs are still small. It is, of course, only the requirement for one of the many functions of our system. However, subdivisions of the code in many functions, the memory management system of Matlab^{TM} and the fact that we only keep the necessary information to fuse helps keeping it at low levels. Also, during the tests we have observed in the Windows XP Pro task manager the amount of system memory used by Matlab^{TM}. We've noted the memory use going from 79 MB before starting the test, to a peak usage of 86 MB.

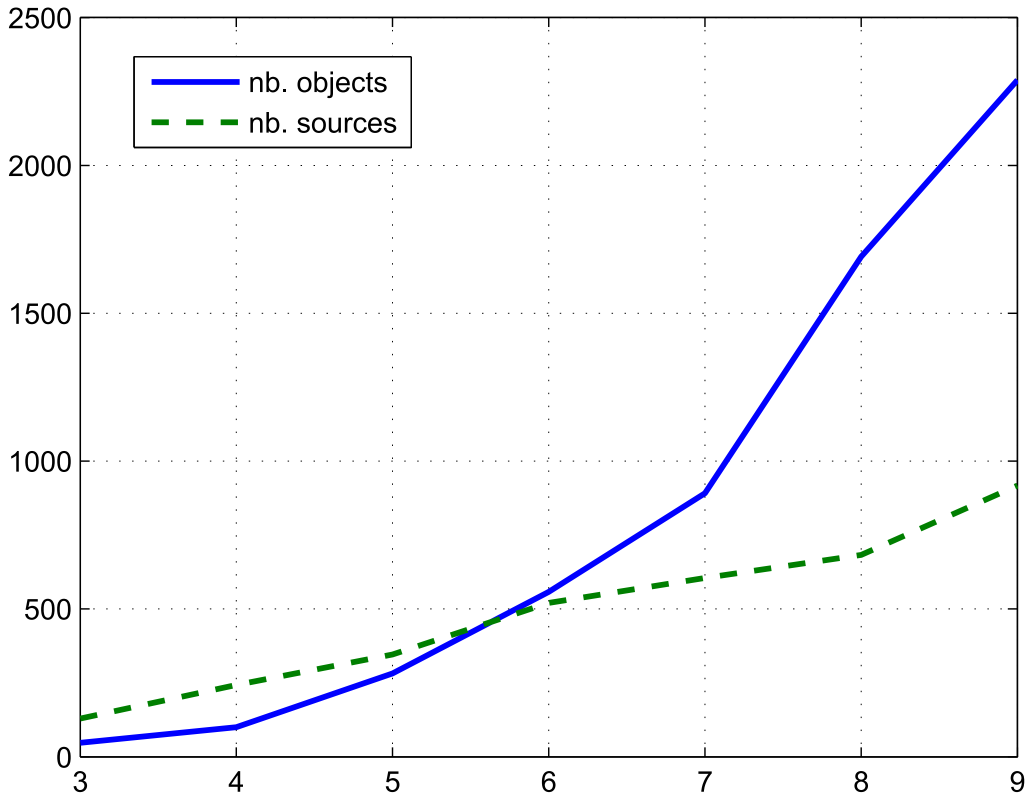

Figure 15.4: Evolution of memory size (KB) of hybrid.m workspace vs the number of sources or $|\Theta|$

We have also tried it once in static mode with a core dimension of 10 from five sources and ten constraints with three objects in the frame of refinement to see how much memory it would take. In that simple case, we went from 79 MB (before the test started) to 137 MB (a peak memory usage during the test). A huge hunger for resources was predictable for the static calculation mode with the enormous matrix it has to build with all the input information.

15.3.5.5 Further optimization to be done

Our code's algorithm is an optimization of the original DSmH calculation process. However, certain parts of our program remains to be optimized. First of all, the possibility of rejecting information and transferring it's mass to total ignorance in the case it's mass is too small or if we have too many information should be added. Second point, at many stages of our calculation, sorting is required. As we know, sorting is one of the most time consuming process in programs and it's also the case in our program. We've used two `for` loops for sorting within two other `for` loops to go through all the elements of the matrix within the file `ordre_grandeur.m`. So as the quantity of information grows, Matlab^{TM} might eventually have problems sorting the inputted information. The use of an optimized algorithm replacing this part is recommended. There's also the possibility of using the Matlab^{TM} command `sort` with some adaptations to be able to do the following sorting.

Our required sorting process in `ordre_grandeur.m` should be able to sort sets first according to the sets' size. Then, for equal sized sets, the sorting process should be able to sort in numerical order of objects. So the following set : $\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3 + \theta_1\theta_3$ should be ordered this way : $\theta_4 + \theta_1\theta_3 + \theta_2\theta_3 + \theta_1\theta_3\theta_4$. A sorting process is also in use within the file `tri.m` which is able to sort matrices or sets. However the sorting process should also be optimized there.

15.4 Conclusion

As we have seen, even with the apparent complexity of DSmH, it is still possible to engineer an efficient procedure of calculation. Such a procedure enables us to conceive an efficient Matlab^{TM} code. We have conceived such a code that can perform within a reasonable amount of time by limiting the number of `for` and `while` loops exploiting Matlab's^{TM} vectorial calculation capabilities. However, even if we have obtained an optimal process of evaluating DSmH, there's still work to be done to optimize some parts of our code involving sorting.

Two avenues can be taken in the future. The first one would be to increase optimization of the actual code, trying to reduce further the number of loops, particularly in sorting. The second avenue would now be to explore how to optimize and program new combination rules such as the adaptive combination rule (ACR) [1], and the proportional conflict redistribution (PCR) rule [1].

15.5 Acknowledgements

We acknowledge Defence R&D Canada for their financial and technical supports. Pascal Djiknavorian would also like to thank his parents and his little brother, Nejmeh, Mihran, and Edouard, for encouraging him, and for their support.

15.6 References

[1] Florea M. C., Dezert J., Valin P., Smarandache F., Jousselme A.-L. , *Adaptative combination rule and proportional conflict redistribution rule for information fusion*, in Proceedings of COGIS '06 Conference, Paris, France, March 2006.

[2] Gagnon M.-L. , Grenier D., *Rapport: Fusion de données, été 2005.*, Université Laval, Canada, 2005.

[3] Smarandache F., Dezert J. (Editors), *Advances and Applications of DSmT for Information Fusion.*, Collected Works, American Research Press, Rehoboth, June 2004.

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15.7 Appendix: Matlab™code listings

The code listed in this chapter is the property of the Government of Canada, DRDC Valcartier and has been developed by M.-L. Gagnon and P. Djiknavorian under the supervision of Dominic Grenier at Laval University, Quebec, Canada.

The code is available as is for educational purpose only. The authors can't be held responsible of any other usage. Users of the code use it at their own risks. For any other purpose, users of the code should obtain an autorization from the authors.

15.7.1 File : aff_ensemble.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function displaying elements and mass from a set
%
% info: elements and mass information to display
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function aff_ensemble(info)
%#inbounds
%#realonly
nI = length(info);
for k = 1 : nI
    %% displays only the non empty elements
    if ~isequal(info(k).elements,[])
    disp([num2str(info(k).elements) ' : m='  num2str(info(k).masses,'%12.8f')]);
    end
end
disp(' ')
```

15.7.2 File : aff_matrice.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function displaying elements and mass
%
% info: elements and mass information to display
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function aff_matrice(info)
%#inbounds
%#realonly
[m,n] = size(info);
%% go through all the objects
for k = 1 : m
    for g = 1 : n
        ensemble = info(k,g).elements
        for h = 1 : length(ensemble)
            disp([num2str(ensemble{h})]);
```

```
        end
        disp ([ 'm : ' num2str(info(k,g).masses,'%6.4f') ]);
    end
end
```

15.7.3 File : bon_ordre.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function ordering vectors in sets
%
% ensembleN, ensembleM : two sets in which we have to see if some values
% are identical, if so, they must be put at the same position
%
% ensembleNOut, ensembleMOut : output vector
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensembleMOut, ensembleNOut] = bon_ordre(ensembleM, ensembleN)
%#inbounds
%#realonly
ensembleMOut  = {};
ensembleNOut  = {};
ensemble1     = [];
ensemble2     = [];
ensemble_temp = [];
plus_grand    = 1;
%% go through all the objects
if length(ensembleN) >= length(ensembleM)
    ensemble1  = ensembleN;
    ensemble2  = ensembleM;
    plus_grand = 1;
else
    ensemble1  = ensembleM;
    ensemble2  = ensembleN;
    plus_grand = 2;
end
%% check if there is two identical sets, otherwise check vectors
for g = 1 : length(ensemble2)
    for h = 1 : length(ensemble1)
        if isequal(ensemble1{h},ensemble2{g})
            ensemble_temp = ensemble1{g};
            ensemble1{g}  = ensemble1{h};
            ensemble1{h}  = ensemble_temp;
        end
    end
end
if isequal(plus_grand,1)
    ensembleMOut = ensemble2;
    ensembleNOut = ensemble1;
elseif isequal(plus_grand,2)
    ensembleNOut = ensemble2;
    ensembleMOut = ensemble1;
end
```

15.7.4 File : calcul_DSm_hybrid_auto.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function to execute a DSm hybrid rule of combination
%               in dynamic or static mode
%
% Output: displayed in sum of product
%                                   sum for union
%                                   product for intersection
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte)
%#inbounds
%#realonly
global ADD
global MULT
ADD  = -2;
MULT = -1;
Iall       = [];
Ihyb       = [];
contraire  = [];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% compute the product of sum
[contraire_complet, contraire] = faire_contraire(info);
%% case with two sources
if isequal(nombre_source,2)
    Ihyb   = hybride(info, contrainte{1},contraire,2,nombre_source,contraire_complet);
    shafer = 0;
    Iall = depart(Ihyb,2);
    Ihyb = depart(Ihyb,1);
    disp('DSm hybride');
    aff_ensemble(Ihyb);
else
    %% case with more than two sources : check the type 'sorte' of DSmH
    %% case dynamic
    if isequal(sorte,'dynamique')
       Ihyb = hybride(info,contrainte{1},contraire,2,nombre_source,contraire_complet);
        for g = 3 : nombre_source
            Ihyb          = depart(Ihyb,2);
            ensemble_step = {};
            masses_step   = [];
            disp('DSm hybride');
            aff_ensemble(Ihyb)
            for h = 1 : length(Ihyb)
                ensemble_step{h} = Ihyb(h).elements;
                masses_step(h)   = Ihyb(h).masses;
            end
            info(1).elements    = {};               info(1).masses     = [];
            info(2).elements    = {};               info(2).masses     = [];
            info(1).elements    = ensemble_step;    info(1).masses     = masses_step;
            info(2)             = info(g);
            [contraire_complet, contraire] = faire_contraire(info);
            clear Ihyb;
     Ihyb = hybride(info,contrainte{g-1},contraire,2,nombre_source,contraire_complet);
```

```
        end
        %% replace numerical value of ADD and MULT by the text 'ADD','MULT'
        Iall = depart(Ihyb,2);
        Ihyb = depart(Ihyb,1);
        disp('DSm hybride');
        aff_ensemble(Ihyb);
    %% case static
    else
        Ihyb = hybride(info,contrainte{nombre_source -1},contraire,1, ...
                                           nombre_source,contraire_complet);
        %% replace numerical value of ADD and MULT by the text 'ADD','MULT'
        Iall = depart(Ihyb,2);
        Ihyb = depart(Ihyb,1);
        disp('DSm hybride');
        aff_ensemble(Ihyb);
    end
end
%% compute belief and plausibility
Isel = Iall;
fboe = {'pl' 'bel'};
for k=1:length(fboe)
    switch fboe{k}
        case 'pl'
            Pds = plausibilite(Isel,contrainte);
            disp('Plausibilite');
            Pds = depart(Pds,1);
            aff_ensemble(Pds);
        case 'bel'
            Bds = croyance(Isel);
            disp('Croyance');
            Bds = depart(Bds,1);
            aff_ensemble(Bds);
    end
end
```

15.7.5 File : calcul_DSm_hybride.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  main file to execute a DSm hybrid rule of combination
%               in dynamic or static mode
%
% Output: displayed in sum of product
%                                    sum for union
%                                    product for intersection
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
clc;

%#inbounds
%#realonly

global ADD
global MULT
```

```
ADD  = -2;
MULT = -1;

Iall       = [];
Ihyb       = [];
info       = [];
contrainte = [];
contraire  = [];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% WRITE EVENTS AND CONSTRAINTS IN SUM OF PRODUCT NOTATION  %

% nombre_source    = 2;
% info(1).elements = {[1], [1, ADD, 2], [1, ADD, 3], [2], [2, ADD, 3], [3]};
% info(1).masses   = [0.2, 0.17, 0.33, 0.03, 0.17, 0.1];
% info(2).elements = {[1], [2], [3]};
% info(2).masses   = [0.2, 0.4, 0.4];
%%contrainte{1}    = {};
% %contrainte{1}    = {[1, MULT, 2], [1, MULT, 3], [2, MULT, 3]};
% contrainte{1}    = {[1, MULT, 2], [1, MULT, 3], [2, MULT, 3],...
%                    [1, MULT, 2,ADD, 1, MULT, 3]...
%                    [1, MULT, 2,ADD, 2, MULT, 3]...
%                    [1, MULT, 3,ADD, 2, MULT, 3]...
%                    [1, MULT, 2,ADD, 1, MULT, 3, ADD, 2, MULT, 3]};

% nombre_source    = 3;   sorte    = ['dynamique'];
% info(1).elements = {[1],[3], [2, MULT, 3]};
% info(1).masses   = [0.7, 0.2, 0.1];
% info(2).elements = {[2],[1, ADD, 3], [2, ADD, 3]};
% info(2).masses   = [0.6, 0.2, 0.2];
% info(3).elements = {[1], [2], [3], [1, ADD, 2]};
% info(3).masses   = [0.1, 0.1, 0.5, 0.3];
% contrainte{1}    = {[2, MULT, 3], [2, ADD, 3]};
% contrainte{2}    = {[2], [1, MULT, 2], [2, MULT, 3],...
%                   [1, MULT, 2, ADD, 2, MULT, 3], [1, MULT, 2, MULT, 3]};

nombre_source    = 2;
info(1).elements = {[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3], [1]};
info(1).masses  =   [0.6, 0.4];
info(2).elements = {[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3], [1]};
info(2).masses  =   [0.4, 0.6];
contrainte{1}    = {};

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% compute the product of sum
[contraire_complet, contraire] = faire_contraire(info);
%% case with two sources
if isequal(nombre_source,2)
    Ihyb   = hybride(info, contrainte{1},contraire,2,nombre_source,contraire_complet);
    shafer = 0;
    Iall = depart(Ihyb,2);
    Ihyb = depart(Ihyb,1);
    disp('DSm hybride');
```

```
    aff_ensemble(Ihyb);
else
    %% case with more than two sources : check the type 'sorte' of DSmH
    %% case dynamic
    if isequal(sorte,'dynamique')
      Ihyb = hybride(info,contrainte{1},contraire,2,nombre_source,contraire_complet);
        for g = 3 : nombre_source
            Ihyb          = depart(Ihyb,2);
            ensemble_step = {};
            masses_step   = [];
            disp('DSm hybride');
            aff_ensemble(Ihyb)
            for h = 1 : length(Ihyb)
                ensemble_step{h} = Ihyb(h).elements;
                masses_step(h)   = Ihyb(h).masses;
            end
            info(1).elements    = {};              info(1).masses  = [];
            info(2).elements    = {};              info(2).masses  = [];
            info(1).elements    = ensemble_step;   info(1).masses  = masses_step;
            info(2)             = info(g);
            [contraire_complet, contraire] = faire_contraire(info);
            clear Ihyb;
            Ihyb  = hybride(info,contrainte{g-1},contraire,2,nombre_source, ...
                                                           contraire_complet);
        end
        %% replace numerical value of ADD and MULT by the text 'ADD','MULT'
        Iall = depart(Ihyb,2);
        Ihyb = depart(Ihyb,1);
        disp('DSm hybride');
        aff_ensemble(Ihyb);
    %% case static
    else
        Ihyb = hybride(info,contrainte{nombre_source -1},contraire,1,...
                                              nombre_source,contraire_complet);
        %% replace numerical value of ADD and MULT by the text 'ADD','MULT'
        Iall = depart(Ihyb,2);
        Ihyb = depart(Ihyb,1);
        disp('DSm hybride');
        aff_ensemble(Ihyb);
    end
end
%% compute belief and plausibility
Isel = Iall;
fboe = {'pl' 'bel'};
for k=1:length(fboe)
    switch fboe{k}
        case 'pl'
            Pds = plausibilite(Isel,contrainte);
            disp('Plausibilite');
            Pds = depart(Pds,1);
            aff_ensemble(Pds);
        case 'bel'
            Bds = croyance(Isel);
```

```
            disp('Croyance');
            Bds = depart(Bds,1);
            aff_ensemble(Bds);
    end
end
```

15.7.6 File : croyance.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function that computes belief
%
% I : final information for which we want to compute belief
%
% croyance_complet: output giving belief values and objects
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function croyance_complet = croyance(I)
%#inbounds
%#realonly
global ADD
global MULT
ADD  = -2;
MULT = -1;
info          = [];
matrice_monome = [];
ignorance     = [];
nombreElement  = 0;
ensemble      = {};
vecteur1      = [];
vecteur2      = [];
f             = 1;
j             = 1;
%% separates objects, remove words ADD and MULT
for g = 1 : length(I)
    if ~isempty(I(g).elements)
        ensemble{f} = I(g).elements;
        vecteur1(f) = I(g).masses;
        vecteur2(f) = 1;
        f           = f + 1;
    end
end
info(1).elements = ensemble;
info(1).masses   = vecteur1;
info(2).elements = ensemble;
info(2).masses   = vecteur2;
[matrice_monome,ignorance,nombreElement] = separation(info,1);
matrice_monome                           = ordre_grandeur(matrice_monome,2);
%% produces the union matrix
matrice_intersection_contraire = intersection_matrice(matrice_monome,1);
matrice_intersection_contraire = ordre_grandeur(matrice_intersection_contraire,2);
matrice_intersection_contraire = dedouble(matrice_intersection_contraire,2);
%% Those for which union equals the monome (by lines), we add their masses.
[m,n] = size(matrice_intersection_contraire);
for g = 2 : m
```

```
    for h = 2 : n
        if isequal(matrice_intersection_contraire(g,h).elements,...
                                                 matrice_monome(g,1).elements)
            resultat(j).elements = matrice_monome(g,1).elements;
            resultat(j).masses   = matrice_intersection_contraire(g,h).masses;
            j                    = j + 1;
        end
    end
end
croyance_complet = dedouble(resultat,1);
```

15.7.7 File : dedouble.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function that removes identical values and simplifies object
%
% matrice:  matrix to simplify, can be a set
% sorte:    indicates if input is a matrix or a set
%
% retour:   output once simplified
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [retour] = dedouble(matrice,sorte)
%#inbounds
%#realonly
global REPETE
REPETE = 0;
%% case set
if isequal(sorte,1)
    ensembleOut = [];
    j           = 1;
    %% go through elements of the set
    for g = 1 : length(matrice)
        for h = g + 1 : length(matrice)
            if isequal(matrice(h).elements,matrice(g).elements)
                matrice(h).elements = REPETE;
                matrice(g).masses   = matrice(g).masses + matrice(h).masses;
            end
        end
        if ~isequal(matrice(g).elements,REPETE) & ~isequal(matrice(g).masses,0)
            ensembleOut(j).elements = matrice(g).elements;
            ensembleOut(j).masses   = matrice(g).masses;
            j                       = j + 1;
        end
    end
    retour = ensembleOut;
%% case matrix
else
    [m,n]    = size(matrice);
    vecteur1 = [];
    vecteur2 = [];
    if m > 1
        u = 2;
        y = 2;
```

```
else
    u = 1;
    y = 1;
end
%% go through elements of the matrix
for h = u : m
    for g = y : n
        ensemble        = {};
        ensemble        = matrice(h,g).elements;
        j               = 1;
        nouvel_ensemble = {};
        %% go through all vectors of the matrix
        for k = 1 : length(ensemble)
            vecteur1 = ensemble{k};
            if ~isempty(vecteur1)
                for f = k + 1 : length(ensemble)
                    vecteur2 = ensemble{f};
                    %% check if there is two identical vectors
                    if ~isempty(vecteur2)
                        if isequal(vecteur1, vecteur2)
                            vecteur1 = REPETE;
                        else
                            %% check if a vector is included in another
                            %% 2 intersection 2union3  : remove 2union3
                            compris = 0;
                            for v = 1 : length(vecteur1)
                                for c = 1 : length(vecteur2)
                                    if isequal(vecteur1(v),vecteur2(c))
                                        compris = compris + 1;
                                    end
                                end
                            end
                            if length(vecteur1) < length(vecteur2)
                                if isequal(compris, length(vecteur1))
                                    vecteur2 = REPETE;
                                end
                            else
                                if isequal(compris, length(vecteur2))
                                    vecteur1 = REPETE;
                                end
                            end
                        end
                    ensemble{f} = vecteur2;
                    end
                end
                ensemble{k} = vecteur1;
            end
            if ~isequal(ensemble{k},REPETE)
                nouvel_ensemble{j} = ensemble{k};
                j                  = j + 1;
            end
        end
        matriceOut(h,g).elements = nouvel_ensemble;
```

```
            matriceOut(h,g).masses   = matrice(h,g).masses;
        end
    end
    matriceOut = tri(matriceOut,1);
    retour = matriceOut;
end
```

15.7.8 File : depart.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function putting ADD and MULT
%
% ensemble_complet: set for which we want to add ADD and MULT
%       each element is a cell including vectors
%       each vector is a product and a change of vector is a sum
% sorte: to know if it has to be in numerical value or not
%
% ensemble_final: output with ADD and MULT
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensemble_final] = depart(ensemble_complet,sorte)
%#inbounds
%#realonly
global A
global M
global ADDS
global MULTS
ADDS  = ' ADD ';
MULTS = ' MULT ';
A     = -2;
M     = -1;
ensemble       = [];
ensemble_final = [];
%% go through vectors of the set
for g = 1 : length(ensemble_complet)
        ensemble = ensemble_complet(g).elements;
        for k = 1 : length(ensemble)
            %% first time
            if isequal(k,1)
                if isequal(length(ensemble{k}),1)
                    if isequal(sorte,1)
                        ensemble_final(g).elements = [num2str(ensemble{1})];
                    else
                        ensemble_final(g).elements = [ensemble{1}];
                    end
                else
                    vecteur = ensemble{k};
                    for f = 1 : length(vecteur)
                        if isequal(f,1)
                            if isequal(sorte,1)
                                ensemble_final(g).elements = [num2str(vecteur(f))];
                            else
                                ensemble_final(g).elements = [vecteur(f)];
                            end
```

```
                else
            if isequal(sorte,1)
                ensemble_final(g).elements = [...
                                        ensemble_final(g).elements, ...
                                        MULTS, num2str(vecteur(f))];
            else
                ensemble_final(g).elements = [...
                                        ensemble_final(g).elements, ...
                                        M, vecteur(f)];
            end
                end
            end
        end
    %% puts ' ADD ' since change of vector
    else
        if isequal(sorte,1)
            ensemble_final(g).elements = ...
             [ensemble_final(g).elements, ADDS];
        else
            ensemble_final(g).elements = ...
             [ensemble_final(g).elements, A];
        end

        if isequal(length(ensemble{k}),1)
            if isequal(sorte,1)
                ensemble_final(g).elements = ...
                 [ensemble_final(g).elements, ...
                 num2str(ensemble{k})];
            else
                ensemble_final(g).elements = ...
                 [ensemble_final(g).elements, ...
                 ensemble{k}];
            end
        %% puts ' MULT '
        else
            premier = 1;
            vecteur = ensemble{k};
            for f = 1 : length(vecteur)
                    if premier == 1
                        if isequal(sorte,1)
                            ensemble_final(g).elements = ...
                             [ensemble_final(g).elements,...
                              num2str(vecteur(f))];
                        else
                            ensemble_final(g).elements = ...
                             [ensemble_final(g).elements, ...
                             vecteur(f)];
                        end
                        premier                    = 0;
                    else
                        if isequal(sorte,1)
                            ensemble_final(g).elements = ...
                             [ensemble_final(g).elements, ...
```

```
                                     MULTS, num2str(vecteur(f))];
                                else
                                    ensemble_final(g).elements = ...
                                     [ensemble_final(g).elements, ...
                                     M, vecteur(f)];
                                end
                            end
                    end
                end
            end
        end
    ensemble_final(g).masses = ensemble_complet(g).masses;
end
```

15.7.9 File : DSmH_auto.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% description: file from which we can call the function version of the DSmH
%
%%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
% The examples used in this file were available in :
%''Advances and Applications of DSmT for Information Fusion''
% written by par Jean Dezert and Florentin Smarandache, 2004
%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;  clc;
info       = [];
contrainte = [];
global ADD
global MULT
ADD  = -2;
MULT = -1;

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 1, Page 21');

 nombre_source    = 2;
 sorte            = ['dynamique'];
 info(1).elements = {[1],[2],[3],[1, ADD, 2]};
 info(1).masses   = [0.1, 0.4, 0.2, 0.3];
 info(2).elements = {[1],[2],[3],[1, ADD, 2]};
 info(2).masses   = [0.5, 0.1, 0.3, 0.1];
 contrainte{1}    = {[1, MULT, 2, MULT, 3],[1, MULT, 2],[2, MULT, 3],...
  [1, MULT, 3],[3],[1, MULT, 3, ADD, 2, MULT, 3],...
   [1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 2, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);
```

```
%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 5, Page 86');

 nombre_source    = 2;
 info(1).elements = {[1, MULT, 3],[3],[1, MULT, 2],[2],[1],...
                     [1, ADD, 3],[1, ADD, 2]};
 info(1).masses   = [0.1, 0.3, 0.1, 0.2, 0.1, 0.1, 0.1];
 info(2).elements = {[2, MULT, 3],[3],[1, MULT, 2],[2],[1],[1, ADD, 3]};
 info(2).masses   = [0.2, 0.1, 0.2, 0.1, 0.2, 0.2];
 contrainte{1}    = {[1, MULT, 3], [1, MULT, 2, MULT, 3],[1],...
                     [1, MULT, 2],[1, MULT, 2, ADD, 1, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 2, Page 90');

 nombre_source    = 2;
 info(1).elements = {[1, MULT, 2, MULT, 3],[2, MULT, 3],[1, MULT, 3],...
 [1, MULT, 3, ADD, 2, MULT, 3],[3],[1, MULT, 2],[1, MULT, 2, ADD, 2, MULT, 3],...
 [1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3],...
 [3, ADD, 1, MULT, 2],[2],[2, ADD, 1, MULT, 3],[2, ADD, 3],[1],...
 [1, ADD, 2, MULT, 3],[1, ADD, 3],[1, ADD, 2],[1, ADD, 2, ADD, 3]};
 info(1).masses   = [0.01,0.04,0.03,0.01,0.03,0.02,0.02,0.03,0.04,...
 0.04,0.02,0.01,0.2,0.01,0.02,0.04,0.03,0.4];
 info(2).elements = {[1, MULT, 2, MULT, 3],[2, MULT, 3],[1, MULT, 3],...
 [1, MULT, 3, ADD, 2, MULT, 3],[3],[1, MULT, 2],[1, MULT, 2, ADD, 2, MULT, 3],...
 [1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3],...
 [3, ADD, 1, MULT, 2],[2],[2, ADD, 1, MULT, 3],[2, ADD, 3],[1],...
 [1, ADD, 2, MULT, 3],[1, ADD, 3],[1, ADD, 2],[1, ADD, 2, ADD, 3]};
 info(2).masses   = [0.4,0.03,0.04,0.02,0.04,0.20,0.01,0.04,0.03,0.03,...
 0.01,0.02,0.02,0.02,0.01,0.03,0.04,0.01];
 contrainte{1}    = {[1, MULT, 2], [1, MULT, 2, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 7, Page 90');
```

```
 nombre_source    = 2;
 info(1).elements = {[1, MULT, 2, MULT, 3],[2, MULT, 3],[1, MULT, 3],...
 [1, MULT, 3, ADD, 2, MULT, 3],[3],[1, MULT, 2],[1, MULT, 2, ADD, 2, MULT, 3],...
 [1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3],...
 [3, ADD, 1, MULT, 2],[2],[2, ADD, 1, MULT, 3],[2, ADD, 3],[1],...
 [1, ADD, 2, MULT, 3],[1, ADD, 3],[1, ADD, 2],[1, ADD, 2, ADD, 3]};
 info(1).masses   = [0.01,0.04,0.03,0.01,0.03,0.02,0.02,0.03,0.04,...
 0.04,0.02,0.01,0.2,0.01,0.02,0.04,0.03,0.4];
 info(2).elements = {[1, MULT, 2, MULT, 3],[2, MULT, 3],[1, MULT, 3],...
[1, MULT, 3, ADD, 2, MULT, 3],[3],[1, MULT, 2],[1, MULT, 2, ADD, 2, MULT, 3],...
[1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3],...
[3, ADD, 1, MULT, 2],[2],[2, ADD, 1, MULT, 3],[2, ADD, 3],[1], ...
[1, ADD, 2, MULT, 3],[1, ADD, 3],[1, ADD, 2],[1, ADD, 2, ADD, 3]};
 info(2).masses   = [0.4,0.03,0.04,0.02,0.04,0.20,0.01,0.04,0.03,0.03,...
 0.01,0.02,0.02,0.02,0.01,0.03,0.04,0.01];
 contrainte{1}    = {[1, MULT, 2, MULT, 3], [2, MULT, 3], [1, MULT, 3], ...
 [1, MULT, 3, ADD, 2, MULT, 3], [3], [1, MULT, 2], ...
 [1, MULT, 2, ADD, 2, MULT, 3],  [1, MULT, 2, ADD, 1, MULT, 3], ...
 [1, MULT, 2, ADD, 1, MULT, 3, ADD, 2, MULT, 3], [3, ADD, 1, MULT, 2]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 3.2, Page 97');

 nombre_source    = 3;
 %sorte            = ['dynamique'];
 sorte            = ['statique'];
 info(1).elements = {[1],[2],[1, ADD, 2],[1, MULT, 2]};
 info(1).masses   = [0.1, 0.2, 0.3, 0.4];
 info(2).elements = {[1],[2],[1, ADD, 2],[1, MULT, 2]};
 info(2).masses   = [0.5, 0.3, 0.1, 0.1];
 info(3).elements = {[3],[1, MULT, 3],[2, ADD, 3]};
 info(3).masses   = [0.4, 0.3, 0.3];
 contrainte{1}    = {};
 contrainte{2}    = {[3],[1, MULT, 2, MULT, 3],[1, MULT, 3],...
 [2, MULT, 3],[1, MULT, 3, ADD, 2, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 3.5, Pages 99-100');
```

```
 nombre_source    = 3;
 sorte            = ['dynamique'];
 %sorte            = ['statique'];
 info(1).elements = {[1],[2]};
 info(1).masses   = [0.6, 0.4];
 info(2).elements = {[1],[2]};
 info(2).masses   = [0.7, 0.3];
 info(3).elements = {[1], [2], [3]};
 info(3).masses   = [0.5, 0.2, 0.3];
 contrainte{1}    = {};
 contrainte{2}    = {[3], [1, MULT, 3], [2, MULT, 3], ...
 [1, MULT, 2, MULT, 3], [1, MULT, 3, ADD, 2, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 3.6, Page 100');

 nombre_source    = 2;
 info(1).elements = {[1], [2], [1, MULT, 2]};
 info(1).masses   = [0.5, 0.4 0.1];
 info(2).elements = {[1], [2], [1, MULT, 3], [4]};
 info(2).masses   = [0.3, 0.2, 0.1, 0.4];
 contrainte{1}    = {[1, MULT, 3], [1, MULT, 2], [1, MULT, 3, MULT, 4],...
  [1, MULT, 2, MULT, 3], [1, MULT, 2, MULT, 4],[1, MULT, 2, ADD, 1, MULT, 3]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 5.2.1.3, Page 107');

 nombre_source    = 3;
 sorte            = ['dynamique'];
 %rep du livre = statique
 sorte            = ['statique'];
 info(1).elements = {[1],[3]};
 info(1).masses   = [0.6, 0.4];
 info(2).elements = {[2],[4]};
 info(2).masses   = [0.2, 0.8];
 info(3).elements = {[2], [4]};
 info(3).masses   = [0.3, 0.7];
 contrainte{1}    = {};
 contrainte{2}   = {[1, MULT, 3],[1, MULT, 2],[1, MULT, 4],[2, MULT, 3],...
```

```
 [2, MULT, 4],[3, MULT, 4],[1, MULT, 2, MULT, 3],[1, MULT, 2, MULT, 4],...
 [1, MULT, 3, MULT, 4],[2, MULT, 3, MULT, 4],[1, MULT, 2, MULT, 3, MULT, 4],...
 [1, MULT, 3, ADD, 1, MULT, 4],[1, MULT, 2, ADD, 1, MULT, 3],...
 [1, MULT, 2, ADD, 1, MULT, 4],[2, MULT, 3, ADD, 2, MULT, 4],...
 [1, MULT, 2, ADD, 2, MULT, 4],[1, MULT, 2, ADD, 2, MULT, 3],...
 [1, MULT, 3, ADD, 2, MULT, 3],[1, MULT, 3, ADD, 3, MULT, 4],...
 [2, MULT, 3, ADD, 3, MULT, 4],[1, MULT, 4, ADD, 2, MULT, 4],...
 [1, MULT, 4, ADD, 3, MULT, 4],[2, MULT, 4, ADD, 3, MULT, 4],...
 [1, MULT, 2, MULT, 3, ADD, 1, MULT, 2, MULT, 4],...
 [1, MULT, 3, ADD, 1, MULT, 4, ADD, 2, MULT, 3, ADD, 2, MULT, 4]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 5.2.2.2, Page 109');

 nombre_source    = 3;
 sorte           = ['dynamique'];
 %sorte            = ['statique'];
 info(3).elements = {[1],[2],[1, ADD, 2]};
 info(3).masses   = [0.4, 0.5, 0.1];
 info(2).elements = {[3],[4],[3, ADD, 4]};
 info(2).masses   = [0.3, 0.6, 0.1];
 info(1).elements = {[1], [1, ADD, 2]};
 info(1).masses   = [0.8, 0.2];
 contrainte{1}    = {};
 contrainte{2}    = {[1, MULT, 3],[1, MULT, 2],[1, MULT, 4],...
 [2, MULT, 3],[2, MULT, 4],[3, MULT, 4],[1, MULT, 2, MULT, 3],...
 [1, MULT, 2, MULT, 4],[1, MULT, 3, MULT, 4],[2, MULT, 3, MULT, 4],...
 [1, MULT, 2, MULT, 3, MULT, 4],[1, MULT, 3, ADD, 1, MULT, 4],...
 [1, MULT, 2, ADD, 1, MULT, 3],[1, MULT, 2, ADD, 1, MULT, 4],...
 [2, MULT, 3, ADD, 2, MULT, 4],[1, MULT, 2, ADD, 2, MULT, 4],...
 [1, MULT, 2, ADD, 2, MULT, 3],[1, MULT, 3, ADD, 2, MULT, 3],...
 [1, MULT, 3, ADD, 3, MULT, 4],[2, MULT, 3, ADD, 3, MULT, 4],...
 [1, MULT, 4, ADD, 2, MULT, 4],[1, MULT, 4, ADD, 3, MULT, 4],...
 [2, MULT, 4, ADD, 3, MULT, 4],[1, MULT, 2, MULT, 3, ADD, 1, MULT, 2, MULT, 4],...
 [1, MULT, 3, ADD, 1, MULT, 4, ADD, 2, MULT, 3, ADD, 2, MULT, 4]};


    calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

    disp(' ');
    info       = [];
    contrainte = [];
    disp('Example 5.4.2, Page 116');
```

```
 nombre_source    = 3;
 %sorte            = ['dynamique'];
 sorte            = ['statique'];
 info(1).elements = {[1],[4, ADD, 5]};
 info(1).masses   = [0.99, 0.01];
 info(3).elements = {[2],[3],[4, ADD, 5]};
 info(3).masses   = [0.98, 0.01, 0.01];
 info(2).elements = {[1], [2], [3], [4, ADD, 5]};
 info(2).masses   = [0.01, 0.01, 0.97, 0.01];
 contrainte{1}    = {};
 contrainte{2}    = {};
 contrainte{2}    = {[1, MULT, 3],[1, MULT, 2],[1, MULT, 4],[2, MULT, 3],...
 [2, MULT, 4],[3, MULT, 4],[1, MULT, 2, MULT, 3],[1, MULT, 2, MULT, 4],...
 [1, MULT, 3, MULT, 4],[2, MULT, 3, MULT, 4],[1, MULT, 2, MULT, 3, MULT, 4],...
 [1, MULT, 3, ADD, 1, MULT, 4],[1, MULT, 2, ADD, 1, MULT, 3],...
 [1, MULT, 2, ADD, 1, MULT, 4],[2, MULT, 3, ADD, 2, MULT, 4],...
 [1, MULT, 2, ADD, 2, MULT, 4],[1, MULT, 2, ADD, 2, MULT, 3],...
 [1, MULT, 3, ADD, 2, MULT, 3],[1, MULT, 3, ADD, 3, MULT, 4],...
 [2, MULT, 3, ADD, 3, MULT, 4],[1, MULT, 4, ADD, 2, MULT, 4],...
 [1, MULT, 4, ADD, 3, MULT, 4],[2, MULT, 4, ADD, 3, MULT, 4],...
 [1, MULT, 2, MULT, 3, ADD, 1, MULT, 2, MULT, 4],...
 [1, MULT, 3, ADD, 1, MULT, 4, ADD, 2, MULT, 3, ADD, 2, MULT, 4],...
  [1, MULT, 5],[2, MULT, 5],[3, MULT, 5],[4, MULT, 5],[1, MULT, 2, MULT, 5],...
  [1, MULT, 3, MULT, 5],[1, MULT, 4, MULT, 5],[2, MULT, 3, MULT, 5],...
  [2, MULT, 4, MULT, 5],[3, MULT, 4, MULT, 5],[1, MULT, 2, MULT, 3, MULT, 5],...
  [1, MULT, 2, MULT, 4, MULT, 5],[1, MULT, 3, MULT, 4, MULT, 5],...
  [2, MULT, 3, MULT, 4, MULT, 5],[1, MULT, 2, MULT, 3, MULT, 4, MULT, 5],...
  [1, MULT, 4, ADD, 1, MULT, 5],[2, MULT, 4, ADD, 2, MULT, 5],...
  [3, MULT, 4, ADD, 3, MULT, 5],[1, MULT, 2, MULT, 4, ADD, 1, MULT, 2, MULT, 5],...
  [1, MULT, 3, MULT, 4, ADD, 1, MULT, 3, MULT, 5],...
  [2, MULT, 3, MULT, 4, ADD, 2, MULT, 3, MULT, 5],...
  [1, MULT, 2, MULT, 3, MULT, 4, ADD, 1, MULT, 2, MULT, 3, MULT, 5]};


   calcul_DSm_hybrid_auto(nombre_source, sorte, info, contrainte);

%>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

   disp(' ');
```

15.7.10 File : enlever_contrainte.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% description: function removing constraints in sets
%
% ensemble_complet:  sets composed of S1, S2, S3
% contrainte_separe: constraints' sets : divided in cells with vectors :
%                    each vector is a product, and a change of vector = sum
%
% ensemble_complet: final set
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [ensemble_complet] = ...
```

```
                enlever_contrainte(ensemble_complet, contrainte_separe);

%#inbounds
%#realonly
global ENLEVE
ENLEVE = {};
ensemble_contrainte = {};
ensemble_elements   = [];
ensemble_produit    = [];

%go through contraints
for g = 1 : length(contrainte_separe)
    ensemble_contrainte = contrainte_separe{g};
    for h = 1 : length(ensemble_complet)
        %si la contrainte est en entier dans l'ensemble complet, l'enlever
        if isequal(ensemble_contrainte, ensemble_complet(h).elements)
            ensemble_complet(h).elements = ENLEVE;
            ensemble_complet(h).masses   = 0;
        end
    end
end

%go through contraints
for g = 1 : length(contrainte_separe)
    ensemble_contrainte = contrainte_separe{g};
    %si elle est un singleton
    if isequal(length(ensemble_contrainte),1) & ...
                                isequal(length(ensemble_contrainte{1}),1)

        for h = 1 : length(ensemble_complet)
            if ~isequal(ensemble_complet(h).elements, ENLEVE)
                ensemble_elements = ensemble_complet(h).elements;
                entre             = 0;
                for k = 1 : length(ensemble_elements)
                    %si une union, enleve
                    if isequal(ensemble_elements{k},ensemble_contrainte{1})
                        vecteur1             = ensemble_elements{k};
                        vecteur2             = ensemble_contrainte{1};
                        ensemble_elements{k} = setdiff(vecteur1, vecteur2);
                        entre                = 1;
                    end
                end

                if isequal(entre, 1)
                    j                     = 1;
                    ensemble_elements_new = [];
                    for k = 1 : length(ensemble_elements)
                        if ~isequal(ensemble_elements{k},[]);
                            ensemble_elements_new{j} = ensemble_elements{k};
                            j                        = j + 1;
                        end
                    end
                    ensemble_elements = [];
```

```
                    ensemble_elements = ensemble_elements_new;
                end
                ensemble_complet(h).elements = ensemble_elements;
            end
        end

    %otherwise, its an intersection
    elseif length(ensemble_contrainte) == 1

        ensemble_produit = ensemble_complet;
        for t = 1 : length(ensemble_produit)
            ensemble        = ensemble_produit(t).elements;
            j               = 1;
            entre           = 1;
            nouvel_ensemble = {};
            for h = 1 : length(ensemble)
                for y = 1 : length(ensemble_contrainte)
                    if isequal(ensemble{h}, ensemble_contrainte{y})
                        ensemble{h} = [];
                        entre       = 0;
                    else
                        nouvel_ensemble{j} = ensemble{h};
                        j                  = j + 1;
                    end
                end
            end
            ensemble_produit(t).elements = nouvel_ensemble;
            ensemble_complet(t).elements = ensemble_produit(t).elements;
        end
    end
end

%remove empty
for r = 1 : length(ensemble_complet)
    ensemble1       = ensemble_complet(r).elements;
    j               = 1;
    nouvel_ensemble = [];
    for s = 1 : length(ensemble1)
        if ~isequal(ensemble1{s},[])
            nouvel_ensemble{j} = ensemble1{s};
            j                  = j + 1;
        end
    end
    ensemble_complet(r).elements = nouvel_ensemble;
end

%combines identical elements
ensemble_complet = dedouble(ensemble_complet,2);
ensemble_complet = dedouble(ensemble_complet,1);
```

15.7.11 File : ensemble.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Description:  function regrouping equal structure from matrix
%
% matrice: the matrix to regroup
%
% ensembleOut: outputs the structure with sets of regrouped matrix
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensembleOut] = ensemble(matrice)
%#inbounds
%#realonly
ensembleOut = [];
[m,n]       = size(matrice);
j           = 1;

if ~(m < 2)
    if isequal(matrice(2,2).elements, [])
        u = 1;
        y = 1;
    else
        u = 2;
        y = 2;
    end
else
        u = 1;
        y = 1;
end
%% go through all sets of the matrix, put the equal ones togheter and sum
%% their mass
for g = u : m
    for h = y : n
        if isequal(g,u) & isequal(h,y) & ~isequal(matrice(g,h).elements,[])
            ensembleOut(j).elements = matrice(g,h).elements;
            ensembleOut(j).masses   = matrice(g,h).masses;
            j                       = j + 1;
        elseif ~isequal(matrice(g,h).elements,[])
            compris = 0;
            for f = 1 : length(ensembleOut)
                if isequal(matrice(g,h).elements, ensembleOut(f).elements)
                    ensembleOut(f).masses = ...
                          ensembleOut(f).masses + matrice(g,h).masses;
                    compris               = 1;
                end
            end
            if isequal(compris,0)
                 ensembleOut(j).elements = matrice(g,h).elements;
                 ensembleOut(j).masses   = matrice(g,h).masses;
                 j                       = j + 1;
            end
        end
    end
end
```

15.7.12 File : faire_contraire.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that changes the sum of products in product of
%               sums with ADD and MULT
%
% info:         set that we want to modify
%
% ensembleOut:  once in product of sums and in same format as the input
% contraire:    only the first two information
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensembleOut, contraire] = faire_contraire(info)
%#inbounds
%#realonly
ensembleOut = [];
j           = 1;
f           = 1;
temp        = [];
flag        = 3;
contraire   = [];
%% puts the sets in product of sums
[temp, ignorance, nombre] = separation(info,2);
temp                      = produit_somme_complet(temp);
temp                      = depart(temp,2);
%% puts back the sets in one set
for g = 1 : length(nombre)
    debut           = 1;
    d               = 1;
    ensembleElement = {};
    for h = 1 : nombre(g)
        if isequal(debut,1)
            ensembleElement{d}    = [temp(f).elements];
            ensembleOut(j).masses = temp(f).masses;
            debut                 = 0;
        else
            ensembleElement{d}    = [temp(f).elements];
            ensembleOut(j).masses = [ensembleOut(j).masses, temp(f).masses];
        end
        f   = f + 1;
        d   = d + 1;
    end
    %% ensembleOut: output, once in product of sums
    ensembleOut(j).elements = ensembleElement;
    %% contraire: only the first two elements of output
    if j < 3
        contraire(j).elements = ensembleOut(j).elements;
        contraire(j).masses   = ensembleOut(j).masses;
    end
    j = j + 1;
end
```

15.7.13 File : hybride.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:      function that executes the three steps of the DSmH
%
% info:             informations from the sources in product of sums
% contrainte:       contraints in sum of product
% contraire:        informations from sources in sum of products
%
% sorte:                indicates the type of fusion: dynamic ou static
% nombre_source:        number of source of evidence
% contraire_complet:    All the information in product of sum
%
% ensemble_complet:     final values (objects + masses)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensemble_complet] = ...
        hybride(info,contrainte,contraire,sorte,nombre_source,contraire_complet)
matrice_intersection = [];  matrice_union        = [];  matrice_monome      = [];
ensemble_step1       = [];  ensemble_step2       = [];  ensemble_step3      = [];
ensemble_complet     = [];  vecteur_singleton    = [];  contrainte_produit  = [];
ignorance            = [];  ensemble_complet_temp = [];
%% case static
if isequal(sorte,1)
    matrice_infos           = [];
    matrice_infos_contraire = [];
    for g = 1 : nombre_source
        [matrice_infos,ignorance,nombreElement]           = ...
                              separation_unique(info(g),matrice_infos);
        [matrice_infos_contraire,ignorance,nombreElement] = ...
               separation_unique(contraire_complet(g),matrice_infos_contraire);
    end
    %% compute the intersection matrix
    matrice_intersection = intersection_matrice(matrice_infos_contraire,2);
    matrice_intersection = somme_produit_complet(matrice_intersection);
    matrice_intersection = dedouble(matrice_intersection,2);
    matrice_intersection = ordre_grandeur(matrice_intersection,1);
    %% compute the union matrix
    matrice_intersection_contraire = intersection_matrice(matrice_infos,2);
    matrice_intersection_contraire = dedouble(matrice_intersection_contraire,2);
%% case dynamic
else
    %% Separates products of each objects, also computes total ignorance
    [matrice_monome,ignorance1,nombreElement]           = separation(info,1);
    [matrice_monome_contraire,ignorance2,nombreElement] = separation(contraire,1);
    ignorance                                           = [ignorance1];
    %% compute the union matrix
    matrice_intersection_contraire = intersection_matrice(matrice_monome,1);
    matrice_intersection_contraire = ...
                              ordre_grandeur(matrice_intersection_contraire,2);
    matrice_intersection_contraire = dedouble(matrice_intersection_contraire,2);
    %% compute the intersection matrix
    matrice_intersection = intersection_matrice(matrice_monome_contraire,1);
    matrice_intersection = somme_produit_complet(matrice_intersection);
```

```
    matrice_intersection = dedouble(matrice_intersection,2);
end
%% separates objects in contraints: will help compare the with intersection
if ~isempty(contrainte)
    [contrainte_separe, ignorance3, nombre] = separation(contrainte,3);
    contrainte_separe                       = tri(contrainte_separe,2);
end
%% compute S1, S2, S3
%% If there is no constraints, simply take S1
if isempty(contrainte)
    ensemble_complet = ensemble(matrice_intersection);
%% Otherwise, we have to go throught the three steps
else
    %% Go through intersection matrix, if objects = contraints, take union,
    %% if objects from union = contraints, take union of objects, if it's a
    %% contraints, take total ignorance.
    j     = 1;  source = 1;
    [m,n] = size(matrice_intersection);
    ss = 1:m; s = 1;
    gg = 1:n; g = 1;
    %% Go through each line (s) of the matrix process by accessing each
    %% objects, by column (g)
    while s ~= (length(ss)+1)
    while g ~= (length(gg)+1)
        %% take value from intersection matrix
        ensemble_step = matrice_intersection(s,g).elements;
        %% if the flag is not active, set it to '1'
        if ~(source > 10)
            source = 1;
        end
        %% Proceed if there is something at (s,g) matrix position
        if ~isequal(ensemble_step, [])
            intersection = 0;
            for h = 1 : length(contrainte_separe)
                %% If value from intersection matrix is equal to actual
                %% constraint and if it hasn't been equal to a previous
                %% constraint, OR, if the flag was active, then proceed to
                %% union matrix.
                if (isequal(contrainte_separe{h},ensemble_step) &...
                                isequal(intersection,0)) | isequal(source,22)
                    intersection  = 1; union = 0;
                    ensemble_step = [];
                    ensemble_step =  matrice_intersection_contraire(s,g).elements;
                    %% if the flag is not active for the union of objects
                    %% or to total ignorance, set it to '2'
                    if ~(source > 22)
                        source = 2;
                    end
                    for t = 1 : length(contrainte_separe)
                        %% If value from union matrix is equal to actual
                        %% constraint and if it hasn't been equal to a
                        %% previous constraint, OR, if the flag was active,
                        %% then proceed to union of objects calculation.
```

```
if (isequal(contrainte_separe{t},ensemble_step)&...
             isequal(union,0))  | isequal(source,33)
    union = 1; subunion = 0;
    nouveau_vecteur = [];
    ensemble_step   = {};
        ensemble1 = matrice_monome(s,1).elements;
        ensemble2 = matrice_monome(1,g).elements;
        b = 1;
        for f = 1 : length(ensemble1)
        vecteur = ensemble1{f};
            for d = 1 : length(vecteur)
                nouveau_vecteur{b} = [vecteur(d)];
                b                  = b + 1;
            end
        end
        for f = 1 : length(ensemble2)
            vecteur = ensemble2{f};
            for d = 1 : length(vecteur)
                nouveau_vecteur{b} = [vecteur(d)];
                b                  = b + 1;
            end
        end
        %% remove repetition
        for f = 1 : length(nouveau_vecteur)
            for r = f + 1 : length(nouveau_vecteur)
      if isequal(nouveau_vecteur{f},nouveau_vecteur{r})
                    nouveau_vecteur{r} = [];
                end
            end
        end
        y = 1;
        for r = 1 : length(nouveau_vecteur)
            if ~isequal(nouveau_vecteur{r},[])
                ensemble_step{y} = nouveau_vecteur{r};
                y                = y + 1;
            end
        end
         %% ordering
        matrice              = [];
        matrice(1,1).elements = ensemble_step;
        matrice(1,1).masses   = 0;
        matrice(2,2).elements = [];
        matrice     = ordre_grandeur(matrice,2);
        ensemble_step        = [];
        ensemble_step        = matrice(1,1).elements;
        %% if the flag is not active for ignorance
        if ~(source > 33)
            source = 3;
        end
    for r = 1 : length(contrainte_separe)
        %% If value from union of objects matrix is
        %% equal to actual constraint and if it
        %% hasn't been equal to previous constraint
```

```
                    %% OR, if the flag was active.
                    if (isequal(contrainte_separe{r}, ensemble_step)...
                        & isequal(subunion,0)) | isequal(source,44)
                        subunion = 1;
                        ensemble_step = {};
                        ensemble_step = ignorance;
                        source = 4;
                    end
                end
            end
        end
    end
end
ensemble_complet_temp = [];
ensemble_complet_temp(1).elements   = ensemble_step;
ensemble_complet_temp(1).masses = matrice_intersection(s,g).masses;
%% remove constraints of composed objects, if there is any
ensemble_step_temp = ...
   enlever_contrainte(ensemble_complet_temp,contrainte_separe);
%% once the contraints are all removed, check if the object are
%% empty. If not, increment output matrix position, if it is
%% empty, activate the flag following the position from where
%% the answer would have been taken and restart loop without
%% incrementing (s,g) intersection matrix position.
if ~isempty(ensemble_step_temp(1).elements)
    ensemble_step = [];
    ensemble_step = ensemble_step_temp(1).elements;
    ensemble_complet(j).elements = ensemble_step;
    ensemble_complet(j).masses  = ...
                          matrice_intersection(s,g).masses;
    ensemble_complet            = tri(ensemble_complet,1);
    j                           = j + 1;

else
    switch (source)
        %% CASE 4 is not used here. It's the case where there
        %% would be a constraint on total ignorance.
        case 1
           source = 22;
        case 2
           source = 33;
        case 3
           source = 44;
    end
    %% Will let the while loop repeat process for actual (s,g)
    g = g - 1;
end
    end %% 'end' for the "if ~isequal(ensemble_step, [])" line
    %% move forward in the intersection matrix
    g = g + 1;
end %g = 1 : n (columns of intersection matrix)

%% move forward in the intersection matrix
```

```
    s = s + 1;
    g = 1;
    end %s = 1 : m (lines of intersection matrix)
end
g = 1; s = 1;
%% Sort the content of the output matrix
ensemble_complet = tri(ensemble_complet,1);
%% Filter the ouput matrix to merge equal cells
ensemble_complet = dedouble(ensemble_complet,1);
```

15.7.14 File : intersection_matrice.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that computes the intersection matrix and masses
%
% sorte:             type of fusion [static | dynamic]
% matrice_monome:   initial information, once separated by objects with ADD
% and MULT removed. vector represents products, a change of vector the sum
% includes only the first line and column of the matrix
%
% matrice_intersection: return the result of intersections
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [matrice_intersection] = intersection_matrice(matrice_monome,sorte)
%% case dynamic
if isequal(sorte,1)
    matrice_intersection = [];
    [m,n]                = size(matrice_monome);
    ensembleN            = {};
    ensembleM            = {};

    %% go through the first line and column, fill the intersection matrix
    for g = 2 : m
        ensembleM = matrice_monome(g,1).elements;
        for h = 2 : n
            ensembleN                          = matrice_monome(1,h).elements;
            matrice_intersection(g,h).elements = [ensembleN,ensembleM];
            matrice_intersection(g,h).masses   = ...
                  matrice_monome(g,1).masses * matrice_monome(1,h).masses;
        end
    end
    matrice_intersection = dedouble(matrice_intersection,2);
    matrice_intersection = ordre_grandeur(matrice_intersection,2);
%% case static
else
    matrice_intersection  = [];
    matrice_intermediaire = [];
    [m,n]                 = size(matrice_monome);
    ensembleN             = {};
    ensembleM             = {};
    j                     = 1;
    s                     = 1;
    %% fill the intersection matrix by multipliying all at once
    for g = 1 : n
```

```
        ensembleM = matrice_monome(1,g).elements;
        if ~isequal(ensembleM,[])
            for h = 1 : n
                    ensembleN = matrice_monome(2,h).elements;
                    if ~isequal(ensembleN,[])
                        matrice_intermediaire(j,s).elements = [ensembleN,ensembleM];
                        matrice_intermediaire(j,s).masses   = ...
                           matrice_monome(2,h).masses * matrice_monome(1,g).masses;
                        s                                    = s + 1;
                    end
            end
        end
    end
    [r,t] = size(matrice_intermediaire);
    s     = 1;
    for g = 3 : m
        for h = 1 : t
            ensembleM = matrice_intermediaire(1,h).elements;
            for u = 1 : n
                ensembleN = matrice_monome(g,u).elements;
                if ~isequal(ensembleN,[])
                    matrice_intersection(1,s).elements = [ensembleN,ensembleM];
                    matrice_intersection(1,s).masses   = ...
                  matrice_intermediaire(1,h).masses * matrice_monome(g,u).masses;
                    s                                  = s + 1;
                end
            end
        end
        matrice_intermediaire =  matrice_intersection;
        matrice_intersection  = [];
        [r,t] = size(matrice_intermediaire);
        s = 1;
    end
    matrice_intersection = matrice_intermediaire;
    matrice_intersection = dedouble(matrice_intersection,2);
end
```

15.7.15 File : ordre_grandeur.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that orders vectors
%
% matrice:      matrix in which we order the vectors in the sets
%
% matriceOut:   output ordered matrix
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [matriceOut] = ordre_grandeur(matrice,sorte)
[m,n]        = size(matrice);
ensemble     = {};
ensembleTemp = [];
%% case static
if isequal(sorte,1)
    u = 1;
```

```
    y = 1;
%% case dynamic
else
    essai = matrice(2,2).elements;
    if isempty(essai)
        u = 1;
        y = 1;
    else
        u = 2;
        y = 2;
    end
end
%% Order by size vector of sets of matrix
for g = u : m
    for h = y : n
        ensemble = matrice(g,h).elements;
        for f = 1 : length(ensemble)
            for k = f + 1 : length(ensemble)
                if length(ensemble{k}) < length(ensemble{f})
                    ensembleTemp = ensemble{f};
                    ensemble{f}  = ensemble{k};
                    ensemble{k}  = ensembleTemp;
                elseif isequal(length(ensemble{k}), length(ensemble{f}))
                    vecteur1 = ensemble{k};
                    vecteur2 = ensemble{f};
                    changer  = 0;
                    for t = 1 : length(vecteur1)
                        if (vecteur1(t) < vecteur2(t)) & isequal(changer,0)
                           ensembleTemp = ensemble{f};
                            ensemble{f} = ensemble{k};
                            ensemble{k} = ensembleTemp;
                            changer     = 1;
                            break;
                        end
                    end
                end
            end
        end
        matriceOut(g,h).elements = ensemble;
        matriceOut(g,h).masses   = matrice(g,h).masses;
    end
end
```

15.7.16 File : plausibilite.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that calculates plausibility
%
% I:            final information for which we want plausibility
% contrainte:   initial constraints
%
% plausibilite_complet: returns plausibility and masses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
function plausibilite_complet = plausibilite(I,contrainte)
%#inbounds
%#realonly
global ADD
global MULT
ADD  = -2;
MULT = -1;
ensemble_complet   = {};
contrainte_compare = {};
info               = [];
matrice_monome     = [];
ignorance          = [];
ensemble_elements  = [];
vecteur1           = [];
vecteur2           = [];
nombreElement      = 0;
f                  = 1;
j                  = 1;
r                  = 1;
%% separates the objects, removes ADD and MULT
for g = 1 : length(I)
    if ~isempty(I(g).elements)
        ensemble_elements{f} = I(g).elements;
        vecteur2(f)          = I(g).masses;
        vecteur1(f)          = 1;
        f                    = f + 1;
    end
end
info(1).elements = ensemble_elements;
info(2).elements = ensemble_elements;
info(1).masses   = vecteur1;
info(2).masses   = vecteur2;
[matrice_monome,ignorance,nombreElement]          = separation(info,1);
[contraire_complet, contraire]                    = faire_contraire(info);
[matrice_monome_contraire,ignorance,nombreElement] = separation(contraire,1);
%% creates the intersection matrix
matrice_intersection = intersection_matrice(matrice_monome_contraire,1);
matrice_intersection = somme_produit_complet(matrice_intersection);
matrice_intersection = dedouble(matrice_intersection,2);
%% takes the contraint in sum of products, however, if there's none, do
%% nothing and put it all to '1'
entre = 0;
s     = 1;
for r = 1 : length(contrainte)
    if ~isempty(contrainte) & ~isempty(contrainte{r}) & isequal(entre,0)
        for g = 1 : length(contrainte)
            if ~isequal(contrainte{g},{})
                [contrainte_compare{s}, ignorance, nombre] = ...
                                          separation(contrainte{g},3);
                s                                          = s + 1;
            end
        end
        %% remove contraints on the intersection matrix
```

```
        [m,n] = size(matrice_intersection);
        for g = 2 : n
            ensemble_complet                            = [];
            matrice_intersection_trafique =  matrice_intersection(:,g);
            matrice_intersection_trafique(2,2).elements = [];
            ensemble_complet             = ensemble(matrice_intersection_trafique);
            ensemble_complet             = tri(ensemble_complet,1);
            ensemble_complet              = dedouble(ensemble_complet,1);
            for t = 1 : length(contrainte_compare)
                ensemble_complet = enlever_contrainte(ensemble_complet,...
                                                contrainte_compare{t});
            end
            resultat(j).masses = 0;
            for t = 1 : length(ensemble_complet)
                if ~isempty(ensemble_complet(t).elements)
                    resultat(j).masses = resultat(j).masses + ...
                                    ensemble_complet(t).masses;
                end
            end
            resultat(j).elements = matrice_monome(g,1).elements;
            j = j + 1;
        end
        entre = 1;
    %% if there's no constraints, put it all to '1',
    elseif isequal(length(contrainte),r) & isequal(entre,0)
        [m,n] = size(matrice_monome);
        for g = 1 : m
            resultat(j).elements = matrice_monome(g,1).elements;
            resultat(j).masses   = 1;
            j                    = j + 1;
        end
    end
end
plausibilite_complet = dedouble(resultat,1);
```

15.7.17 File : produit_somme_complet.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function that converts input in product of sums
%
% ensemble_complet:     matrix in sum of products
%
% ensemble_produit:     matrix in product of sums
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [ensemble_produit] = produit_somme_complet(ensemble_complet);
global ENLEVE
ENLEVE = {};
ensemble_elements = {}; ensemble_produit  = {};
vecteur           = []; matrice           = [];
p                 = 1;  y                 = 1;
%% go through all sets, puts them in product of sums
for g = 1 : length(ensemble_complet)
    if ~isequal(ensemble_complet(g).elements, ENLEVE)
```

```
ensemble_elements = ensemble_complet(g).elements;
if (length(ensemble_elements) >= 2)
    i                           = 1;
    ensemble_produit(p).elements = {};
    changer                     = 0;
    if length(ensemble_elements) >= 3
        vecteur1 = ensemble_elements{1};
        vecteur2 = ensemble_elements{2};
        if ~(length(vecteur1) > 1 & length(vecteur2) > 1)
            ensemble_produit(p).elements = ensemble_complet(g).elements;
            ensemble_produit(p).masses   = ensemble_complet(g).masses;
            p                            = p + 1;
        else
            changer = 1 ;
        end
    else
        changer = 1;
    end
    if isequal(changer, 1)
        for k = 1 : length(ensemble_elements) - 1
            if (k < 2)
                if (k + 1) > length(ensemble_elements)
                    x = length(ensemble_elements);
                else
                    x = k + 1;
                end
                for w = k : x
                    vecteur = ensemble_elements{w};
                    j       = 1;
                    for f = 1 : length(vecteur)
                        if isequal(length(vecteur),1)
                            ensembleN{j} = [vecteur];
                        else
                            ensembleN{j} = [vecteur(f)];
                            j            = j  + 1;
                        end
                    end
                    if isequal(i,1)
                        matrice(1,2).elements = ensembleN;
                        matrice(1,2).masses   = 0;
                        ensembleN             = {};
                        i                     = 2;
                    elseif isequal(i,2)
                        matrice(2,1).elements = ensembleN;
                        matrice(2,1).masses   = 0;
                        ensembleN             = {};
                        i                     = 1;
                     end
                end
             elseif (k >= 2) & (length(ensemble_elements) > 2)
                w = k + 1;
                j = 1;
                vecteur = ensemble_elements{w};
```

```
                    for f = 1 : length(vecteur)
                        if isequal(length(vecteur),1)
                           ensembleN{j} = [vecteur];
                        else
                           ensembleN{j} = [vecteur(f)];
                           j            = j  + 1;
                        end
                    end
                    matrice(1,2).elements = ensemble_produit(p).elements;
                    matrice(1,2).masses   = 0;
                    matrice(2,1).elements = ensembleN;
                    matrice(2,1).masses   = 0;
                    ensembleN             = {};
                end
                resultat = union_matrice(matrice);
                [s,t]    = size(resultat);
                for r = 1 : s
                    for d = 1 : t
                        masse = resultat(r,d).masses;
                        if isequal(masse, 0)
                            ensemble_produit(p).elements = ...
                                                resultat(r,d).elements;
                            ensemble_produit(p).masses   = ...
                                                ensemble_complet(g).masses;
                        end
                    end
                end
            end
            p = p + 1;
        end
    elseif isequal(length(ensemble_elements),1)
        for k = 1 : length(ensemble_elements)
            vecteur = ensemble_elements{k};
            j       = 1;
            for f = 1 : length(vecteur)
                if isequal(length(vecteur),1)
                    ensembleN{j} = [vecteur];
                else
                    ensembleN{j} = [vecteur(f)];
                    j            = j  + 1;
                end
            end
        end
        ensemble_produit(p).elements = ensembleN;
        ensembleN                    = {};
        ensemble_produit(p).masses   = ensemble_complet(g).masses;
        p                            = p + 1;
    elseif ~isequal(ensemble_elements, [])
        ensemble_produit(p).elements = ensemble_complet(g).elements;
        ensemble_produit(p).masses   = ensemble_complet(g).masses;
        p                            = p + 1;
    end
end
```

```
end
```

15.7.18 File : separation.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  separates products in input data
%
% info:         information from sources (initial data)
% sorte:        type of separation
%
% retour:       separated data (products)
% ignorance:    total ignorance
% nombreElement:number of vectors in sets of each information
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [retour,ignorance_nouveau,nombreElement] = separation(info,sorte)
global ADD
global MULT
global SOURCE
ADD    = -2;
MULT   = -1;
SOURCE = 2;
nouvelle_info     = []; %struc elements: set of vector
ensemble_monome   = []; %cell (1,1) of matrix is empty
matrice_monome    = []; %cell (1,1) of matrix is empty
retour            = [];
ignorance_nouveau = [];
%% takes each elements of each sources and separates the products
[m,n] = size(info);
if ~isequal(sorte,3)
    for g = 1 : n
        nombreElement(g) = length(info(g).elements);
    end
else
    nombreElement(1) = 1;
end
%% case dynamic or two sources
if isequal(sorte,1)
    %% variables
    ligne         = 1;
    colonne       = 2;
    ignorance     = [];
    %% go through each sources
    for g = 1 : n
        ensemble      = info(g).elements;
        vecteur_masse = info(g).masses;
        if isequal(g,SOURCE)
            colonne = 1;
            ligne   = 2;
        end
        %% go through each set of elements
        for h = 1 : length(ensemble)
            vecteur         = ensemble{h};
            nouveau_vecteur = [];
```

```
            nouvel_ensemble = {};
            k               = 1;
             %% go through each element of the vector
            %%     to separate the products and sums
            for j = 1 : length(vecteur)
                if ~isequal(vecteur(j), ADD)
                    if ~isequal(nouveau_vecteur, []) & ~isequal(vecteur(j), MULT)
                        nouveau_vecteur = [nouveau_vecteur, vecteur(j)];
                        if isequal(j,length(vecteur))
                           nouvel_ensemble{k} = nouveau_vecteur;
                           ignorance          = [ignorance, nouveau_vecteur];
                        end
                    elseif ~isequal(vecteur(j), MULT)
                        nouveau_vecteur = [vecteur(j)];
                        if isequal(j,length(vecteur))
                           nouvel_ensemble{k} = nouveau_vecteur;
                           ignorance          = [ignorance, nouveau_vecteur];
                        end
                    end
                else
                    nouvel_ensemble{k} = nouveau_vecteur;
                    ignorance          = [ignorance, nouveau_vecteur];
                    nouveau_vecteur    = [];
                    k                  = k + 1;
                end
            end
            nouvelle_info(g,h).elements = nouvel_ensemble;
            nouvelle_info(g,h).masses   = vecteur_masse(h);
            if isequal(g,1)
                matrice_monome(ligne,colonne).elements = nouvel_ensemble;
                matrice_monome(ligne,colonne).masses   = vecteur_masse(h);
                colonne                                = colonne + 1;
            elseif isequal(g,2)
                matrice_monome(ligne,colonne).elements = nouvel_ensemble;
                matrice_monome(ligne,colonne).masses   = vecteur_masse(h);
                ligne                                  = ligne + 1;
            end
        end
    end
    ignorance = unique(ignorance);
    for r = 1 : length(ignorance)
        ignorance_nouveau{r} = ignorance(r);
    end
    retour = matrice_monome;
%% case static
elseif isequal(sorte,2)
    %% variables
    f               = 1;
    %% go through each sources
    for g = 1 : n
        ensemble      = info(g).elements;
        vecteur_masse = info(g).masses;
        %% go through each set of elements
```

```
        for h = 1 : length(ensemble)
            vecteur         = ensemble{h};
            nouveau_vecteur = [];
            nouvel_ensemble = {};
            k               = 1;
             %% go through each element of the vector
            %%     to separate the products and sums
            for j = 1 : length(vecteur)
                if ~isequal(vecteur(j), ADD)
                    if ~isequal(nouveau_vecteur, []) & ~isequal(vecteur(j), MULT)
                        nouveau_vecteur = [nouveau_vecteur, vecteur(j)];
                        if isequal(j,length(vecteur))
                           nouvel_ensemble{k} = nouveau_vecteur;
                        end
                    elseif ~isequal(vecteur(j), MULT)
                        nouveau_vecteur = [vecteur(j)];
                        if isequal(j,length(vecteur))
                           nouvel_ensemble{k} = nouveau_vecteur;
                        end
                    end
                else
                    nouvel_ensemble{k} = nouveau_vecteur;
                    nouveau_vecteur    = [];
                    k                  = k + 1;
                end
            end
            ensemble_monome(f).elements = nouvel_ensemble;
            ensemble_monome(f).masses  = vecteur_masse(h);
            f                          = f + 1;
        end
    end
    ignorance = [];
    retour    = ensemble_monome;
 %% case contraint
elseif isequal(sorte,3)
    for g = 1 : length(info)
        vecteur         = info{g};
        nouveau_vecteur = [];
        nouvel_ensemble = {};
        k               = 1;
        for h = 1 : length(vecteur)
            if ~isequal(vecteur(h), ADD)
                 if ~isequal(nouveau_vecteur, []) & ~isequal(vecteur(h), MULT)
                     nouveau_vecteur = [nouveau_vecteur, vecteur(h)];
                     if isequal(h,length(vecteur))
                        nouvel_ensemble{k} = nouveau_vecteur;
                     end
                 elseif ~isequal(vecteur(h), MULT)
                     nouveau_vecteur = [vecteur(h)];
                     if isequal(h,length(vecteur))
                        nouvel_ensemble{k} = nouveau_vecteur;
                     end
                 end
```

```
            else
                 nouvel_ensemble{k} = nouveau_vecteur;
                 nouveau_vecteur    = [];
                 k                  = k + 1;
            end
        end
        nouvelle_contrainte{g} = nouvel_ensemble;
    end
    ignorance = [];
    retour    = nouvelle_contrainte;
end
```

15.7.19 File : separation_unique.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  separates products in input data, one info. at a time
%
% info:         information from sources (initial data)
% sorte:        type of separation
%
% matrice_monome:   separated data (products)
% ignorance:        total ignorance
% nombreElement:    number of vectors in sets of each information
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [matrice_monome,ignorance,nombreElement] = ...
                                        separation_unique(info,matrice_monome)
%#inbounds
%#realonly
global ADD
global MULT
global SOURCE
ADD    = -2;
MULT   = -1;
SOURCE = 2;
nouvelle_info  = []; %struc elements: set of vector
ignorance      = [];
if isequal(matrice_monome, [])
    ligne   = 1;
    colonne = 1;
else
    [m,n]   = size(matrice_monome);
    ligne   = m + 1;
    colonne = 1;
end
%% takes each elements of each sources and separates the products
[m,n] = size(info);
for g = 1 : n
    nombreElement(g) = length(info(g).elements);
end
%% go through each sources
for g = 1 : n
    ensemble      = info(g).elements;
    vecteur_masse = info(g).masses;
```

```
    %% go through each set of elements
    for h = 1 : length(ensemble)
        vecteur         = ensemble{h};
        nouveau_vecteur = [];
        nouvel_ensemble = {};
        k               = 1;
        %% go through each elements of the vector
        %% separates the products and sums
        for j = 1 : length(vecteur)
            if ~isequal(vecteur(j), ADD)
                if ~isequal(nouveau_vecteur, []) & ~isequal(vecteur(j), MULT)
                    nouveau_vecteur = [nouveau_vecteur, vecteur(j)];
                    if isequal(j,length(vecteur))
                       nouvel_ensemble{k} = nouveau_vecteur;
                       ignorance          = [ignorance, nouveau_vecteur];
                    end
                elseif ~isequal(vecteur(j), MULT)
                    nouveau_vecteur = [vecteur(j)];
                    if isequal(j,length(vecteur))
                       nouvel_ensemble{k} = nouveau_vecteur;
                       ignorance          = [ignorance, nouveau_vecteur];
                    end
                end
            else
                nouvel_ensemble{k} = nouveau_vecteur;
                ignorance          = [ignorance, nouveau_vecteur];
                nouveau_vecteur    = [];
                k                  = k + 1;
            end
        end
        nouvelle_info(g,h).elements           = nouvel_ensemble;
        nouvelle_info(g,h).masses             = vecteur_masse(h);
        matrice_monome(ligne,colonne).elements = nouvel_ensemble;
        matrice_monome(ligne,colonne).masses   = vecteur_masse(h);
        colonne                                = colonne + 1;
    end
end
ignorance = unique(ignorance);
```

15.7.20 File : somme_produit_complet.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description: function that converts input in sum of products
%
% matrice_contraire:     matrix in product of sums
%
% matrice_complet:     matrix in sum of products
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [matrice_complet] = somme_produit_complet(matrice_contraire);
%#inbounds
%#realonly
ensemble_elements = {};
vecteur           = [];
```

```
matrice          = [];
matrice_complet  = [];
p                = 1;
ensembleN        = {};
[m,n]            = size(matrice_contraire);

if ~isempty(matrice_contraire(1,1).elements)
 u = 1;
 v = 1;
else
  u = 2;
  v = 2;
end
%% go through the sets and puts them in sum of product
for g = u : m
    for t = v : n
        ensemble_elements = matrice_contraire(g,t).elements;
        if ~isequal(ensemble_elements, {})
            matrice_complet(g,t).elements = {};
            matrice_complet(g,t).masses   = 0;
            ensembleN                     = {};
            if isequal(length(ensemble_elements), 1)
                vecteur      = ensemble_elements{1};
                j            = 1;
                ensembleN{j} = [];
                for f = 1 : length(vecteur)
                    ensembleN{j} = [vecteur(f)];
                    j            = j + 1;
                end
                matrice_complet(g,t).elements = ensembleN;
                matrice_complet(g,t).masses   = matrice_contraire(g,t).masses;
            elseif length(ensemble_elements) >= 2
                matrice_complet(g,t).elements = [];
                changer                       = 0;
                if length(ensemble_elements) >= 3
                    vecteur1 = ensemble_elements{1};
                    vecteur2 = ensemble_elements{2};
                    %file produit_somme_complet.m needed an '~' for the IF
                    %here to work as it should be.
                    if (length(vecteur1) > 1 & length(vecteur2) > 1)
                        matrice_complet(g,t).elements = ...
                                              matrice_contraire(g,t).elements;
                        matrice_complet(g,t).masses   = ...
                                              matrice_contraire(g,t).masses;
                    else
                        changer = 1 ;
                    end
                else
                    changer = 1;
                end
                  if isequal(changer,1);
                    matrice_complet(g,t).elements = {};
                    i                             = 1;
```

```
 for k = 1 : length(ensemble_elements) - 1
if (k < 2)
    if (k + 1) > length(ensemble_elements)
        x = length(ensemble_elements);
    else
        x = k + 1;
    end
     for w = k : x
        vecteur = ensemble_elements{w};
        j       = 1;
        for f = 1 : length(vecteur)
            if isequal(length(vecteur),1)
                ensembleN{j} = [vecteur];
            else
                ensembleN{j} = [vecteur(f)];
                j            = j  + 1;
            end
        end

        if isequal(i,1)
            matrice(1,2).elements = ensembleN;
            matrice(1,2).masses   = 0;
            ensembleN             = {};
            i                     = 2;
        elseif isequal(i,2)
            matrice(2,1).elements = ensembleN;
            matrice(2,1).masses   = 0;
            ensembleN             = {};
            i                     = 1;
         end
    end
elseif (k >= 2) & (length(ensemble_elements) > 2)
    w = k + 1;
    j = 1;
    vecteur = ensemble_elements{w};
    for f = 1 : length(vecteur)
        if isequal(length(vecteur),1)
           ensembleN{j} = [vecteur];
        else
           ensembleN{j} = [vecteur(f)];
           j            = j  + 1;
        end
    end
    matrice(1,2).elements = matrice_complet(g,t).elements;
    matrice(1,2).masses   = 0;

    matrice(2,1).elements = ensembleN;
    matrice(2,1).masses   = 0;
    ensembleN             = {};
end
      matrice              = ordre_grandeur(matrice,2);
    resultat              = union_matrice(matrice);
```

```
                        matrice(2,1).elements = {};
                        matrice(1,2).elements = {};

                        [s,b] = size(resultat);
                        for r = 1 : s
                        for d = 1 : b
                            masse = resultat(r,d).masses;
                            if isequal(masse, 0)
                                matrice_complet(g,t).elements = ...
                                                   resultat(r,d).elements;
                                matrice_complet(g,t).masses   = ...
                                                  matrice_contraire(g,t).masses;
                            end
                        end
                        end
                    end
                end
            elseif ~isequal(ensemble_elements, [])
                matrice_complet(g,t).elements = matrice_contraire(g,t).elements;
                matrice_complet(g,t).masses   = matrice_contraire(g,t).masses;
            end
        end
    end
end
if (g >= 2) & (t >= 2)
    matrice_complet = ordre_grandeur(matrice_complet,2);
end
```

15.7.21 File : tri.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that sorts the elements
%
% matrice:      matrix to sort, can be a set
% sorte:        type of input [matrix | set]
%
% retour:       matrix, or set, once the elements are sorted
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [retour] = tri(matrice,sorte)
%#inbounds
%#realonly
%% case matrix
if isequal(sorte,1)
    [m,n]         = size(matrice);
    ensemble_temp = [];
    if m > 1
        u = 2;
        v = 2;
    else
        u = 1;
        v = 1;
    end
    %% go through each elements of the matrix, sort them
```

```
    for h = u : m
        for g = v : n
            ensemble = matrice(h,g).elements;
            for f = 1 : length(ensemble)
                for k = f + 1 : length(ensemble)

                    %% if they are the same length, look at each number, at
                    %% order them
                    if isequal(length(ensemble{f}),length(ensemble{k}))
                        if (ensemble{f} > ensemble{k})
                            ensemble_temp = ensemble{f};
                            ensemble{f}   = ensemble{k};
                            ensemble{k}   = ensemble_temp;
                        end
                    else
                         %% ifnot the same length, put at first, the smaller
                        if length(ensemble{f}) > length(ensemble{k})
                            ensemble_temp = ensemble{f};
                            ensemble{f}   = ensemble{k};
                            ensemble{k}   = ensemble_temp;
                        end
                    end
                end
            end
            matriceOut(h,g).elements = ensemble;
            matriceOut(h,g).masses   = matrice(h,g).masses;
        end
    end
    retour = matriceOut;
%% case set
else
    ensemble_temp = [];
    %% go through each elements of the set, sort them
    for h = 1 : length(matrice)
        ensemble_tri = matrice{h};
        for f = 1 : length(ensemble_tri)
            for k = f + 1 : length(ensemble_tri)
                if isequal(length(ensemble_tri{f}),length(ensemble_tri{k}))
                    if (ensemble_tri{f} > ensemble_tri{k})
                        ensemble_temp   = ensemble_tri{f};
                        ensemble_tri{f} = ensemble_tri{k};
                        ensemble_tri{k} = ensemble_temp;
                    end
                else
                    if length(ensemble_tri{f}) > length(ensemble_tri{k})
                        ensemble_temp   = ensemble_tri{f};
                        ensemble_tri{f} = ensemble_tri{k};
                        ensemble_tri{k} = ensemble_temp;
                    end
                end
            end
        end
        ensembleOut{h} = ensemble_tri;
```

```
    end
    retour = ensembleOut;
end
```

15.7.22 File : union_matrice.m

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Description:  function that computes the union matrix and masses
%
% matrice_monome: objects and masses once separated, on the 1st line/column
%
% matrice_union : returns the result of the union and masses
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [matrice_union] = union_matrice(matrice_monome)
%#inbounds
%#realonly
matrice_union = [];
[m,n]         = size(matrice_monome);
ensembleN     = {};
ensembleM     = {};
vecteurN      = [];
vecteurM      = [];
ensemble1     = [];
ensemble2     = [];
%% go through the 1st line and column, fill the union matrix
for g = 2 : n
    ensembleN = matrice_monome(1,g).elements;
    if ~isequal(ensembleN,{})
        for h = 2 : m
            ensembleM = matrice_monome(h,1).elements;
            if ~isequal(ensembleM,{})
                if isequal(ensembleN, ensembleM)
                    matrice_union(h,g).elements = ensembleN;
                    matrice_union(h,g).masses   = matrice_monome(1,g).masses *...
                                                  matrice_monome(h,1).masses;
                else
                     %% put the identical ones (from same line) togheter
                    [ensembleM,ensembleN] = bon_ordre(ensembleM,ensembleN);
                    %% verifies which one is the higher
                    if length(ensembleM) >= length(ensembleN)
                        ensemble1 = ensembleN;
                        ensemble2 = ensembleM;
                    else
                        ensemble1 = ensembleM;
                        ensemble2 = ensembleN;
                    end
                end
                 %% fill the union matrix
                nouvel_ensemble = {};
                j               = 1;
                for t = 1 : length(ensemble1)
                    for s = 1 : length(ensemble2)
                        if t <= length(ensemble2)
```

```
                        if isequal(ensemble2{s},ensemble1{t})
                            nouvel_ensemble{j} = [ensemble1{t}];
                        else
                            vecteur = [ensemble2{s},ensemble1{t}];
                            nouvel_ensemble{j} = unique(vecteur);
                        end
                    else
                        if isequal(ensemble1{length(ensemble2)},ensemble1{t})
                            nouvel_ensemble{j} = [ensemble1{t}];
                        else
                            vecteur = ...
                                 [ensemble1{length(ensemble2)},ensemble1{t}];
                            nouvel_ensemble{j} = unique(vecteur);
                        end
                    end
                    j = j + 1;
                end
            end
            matrice_union(h,g).elements = nouvel_ensemble;
            matrice_union(h,g).masses   = matrice_monome(1,g).masses *...
                                         matrice_monome(h,1).masses;
        end
    end
    end
end
matrice_union = ordre_grandeur(matrice_union,2);
```

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was translated into Spanish, German, Portuguese, Italian, Dutch, Arabic, Esperanto, Swedish, Farsi, Chinese. He was so attracted by contradictions that, in 1980s, he set up the "Paradoxism" avant-garde movement in literature, philosophy, art, even science, which made many advocates in the world, and it's based on excessive use of antitheses, antinomies, paradoxes in creation - making an interesting connection between mathematics, engineering, philosophy, and literature [`http://www.geocities.com/charlestle/paradoxism.html`] and led him to coining the neutrosophic logic, a logic generalizing the intuitionistic fuzzy logic that is able to deal with paradoxes. In mathematics there are several entries named Smarandache Functions, Sequences, Constants, and especially Paradoxes in international journals and encyclopedias. He organized the "First International Conference on Neutrosophics" at the University of New Mexico, Dec. 1-3, 2001 [`http://www.gallup.unm.edu/~smarandache/FirstNeutConf.htm`]. Small contributions he had in physics and psychology too. Much of his work is held in "The Florentin Smarandache Papers" Special Collections at the Arizona State Univ., Tempe, and Texas State Univ., Austin (USA), also in the National Archives (Rm. Vâlcea) and Romanian Literary Museum (Bucharest), and in the Musée de Bergerac (France). In 2003, he organized with Jean Dezert, the first special session devoted to plausible and paradoxical reasoning for information fusion at the Fusion 2003 Int. conf. on Information Fusion in Cairns, Australia.
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This second volume dedicated to Dezert-Smarandache Theory (DSmT) in Information Fusion brings in new fusion quantitative rules (such as the PCR1-6, where PCR5 for two sources does the most mathematically exact redistribution of conflicting masses to the non-empty sets in the fusion literature), qualitative fusion rules, and the Belief Conditioning Rule (BCR) which is different from the classical conditioning rule used by the fusion community working with the Mathematical Theory of Evidence.

Other fusion rules are constructed based on T-norm and T-conorm (hence using fuzzy logic and fuzzy set in information fusion), or more general fusion rules based on N-norm and N-conorm (hence using neutrosophic logic and neutrosophic set in information fusion), and an attempt to unify the fusion rules and fusion theories.

The known fusion rules are extended from the power set to the hyper-power set and comparison between rules are made on many examples.

One defines the degree of intersection of two sets, degree of union of two sets, and degree of inclusion of two sets which all help in improving the all existing fusion rules as well as the credibility, plausibility, and communality functions.

Also applications of DSmT are showing up to multitarget tracking in clutter based on generalized data association, or target type tracking, to robot's map reconstruction, sonar imagery and radar target classification.

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